MATRICIAL TOPOLOGICAL RANKS FOR TWO ALGEBRAS OF BOUNDED HOLOMORPHIC FUNCTIONS

by

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Abstract. — Let N and D be two matrices over the algebra H^∞ of bounded analytic functions in the disk, or its real counterpart $H^\infty_{\mathbb{R}}$. Suppose that N and D have the same number n of columns. In a generalisation of the notion of topological stable rank 2, it is shown that N and D can be approximated (in the operator norm) by two matrices $\widetilde{N}$ and $\widetilde{D}$, so that the Aryabhatta-Bezout equation $X\widetilde{N} + Y\widetilde{D} = I_n$ admits a solution. This has particular interesting consequences in systems theory. Moreover, in case that N is a square matrix, X can be chosen to be invertible in the case of the algebra H^∞, but not always in the case of $H^\infty_{\mathbb{R}}$.

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Let R be a commutative unital ring with unit element e and let $R^{m \times n}$ be the set of matrices over R with m rows and n columns. The identity matrix of size $n \times n$ will be denoted by I_n. If $M = (a_{i,j}) \in X^{m \times n}$ is a matrix over a normed ring[1] $(X, \|\cdot\|)$ then

$$\|M\|_{op} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{i,j}\|^2}$$

denotes its matrix norm. This norm is equivalent to the operator norm given by the action of M on R^n, where R^n is normed by the function

$$(x_1, \cdots, x_n) \mapsto \sqrt{\sum_{j=1}^{n} \|x_j\|^2}.$$

[1]Here we assume that $\|e\| = 1$ and that $\|xy\| \le \|x\| \, \|y\|$.

An n-tuple $a := (a_1, \ldots, a_n) \in R^n$ is said to be *invertible* or *unimodular*, (for short $a \in U_n(R)$), if there exists a solution $(x_1, \ldots, x_n) \in R^n$ of the Bezout equation $\sum_{j=1}^n a_j x_j = 1$. Let $b \in R$. Then the $(n+1)$-tuple $(a,b) \in U_{n+1}(R)$ is said to be *reducible* if there exists $x \in R^n$ such that $a + bx \in U_n(R)$. The smallest integer n for which every invertible $(n+1)$-tuple is reducible, is called the *Bass stable rank* of R and is denoted by bsr R. If R is an normed ring, then the least integer n for which $U_n(R)$ is dense in R^n is called the *topological stable rank* of R and is denoted by tsr R. It is well known that bsr $R \leq$ tsr R.

In this paper we will mainly consider the algebras $R = H^\infty$ of bounded holomorphic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its real counterpart, $H_\mathbb{R}^\infty$, of all functions in H^∞ that are real on $]-1, 1[$; or in other words

$$H_\mathbb{R}^\infty = \{f \in H^\infty : \overline{f(\overline{z})} = f(z)\}.$$

Our object of study is the Aryabhatta-Bezout equation $XN + YD = I_n$ for matrices N and D over H^∞ and $H_\mathbb{R}^\infty$. We assume that N and D have the same number, n, of columns. Obviously this equation admits a solution if and only the matrix

$$M = \begin{bmatrix} N \\ D \end{bmatrix}$$

has a left inverse. Note that this implies that the number of rows of M is at least n.

The scalar valued case is of course related to the famous corona theorem for these algebras. Using Gelfand theory and the topologically equivalent formulation of the corona-theorem, which tells us that $\mathbb{D}$ is dense in the spectrum $\mathcal{M}(H^\infty)$ of H^∞, it follows that $XN + YD = I_n$ admits a solution in H^∞ if and only if on $\mathbb{D}$,

$$(0.1) \qquad\qquad\qquad M^*M \geq \delta I_n$$

for some number $\delta > 0$. Here M^* is the conjugate transpose of the matrix M, and the symbol $L \geq 0$ means that the self-adjoint matrix L is positive semi-definite. Note that condition (0.1) is equivalent to the assumption that for every character $\chi \in \mathcal{M}(H^\infty)$ the rank of the matrices $\mathcal{M}(\chi)$ over $\mathbb{C}$ are maximal; that is n here (see e.g. [**9**, p. 340,334]).

The following question arises: if condition (0.1) is not satisfied, then does the Aryabhatta-Bezout equation admit a solution for certain data that are arbitrarily close to N and D?

In the case of a large class of algebras of smooth functions on the closed disk, a positive answer was given in [**5**]; for example if R is a normed ring whose norm dominates the supremum norm and for which the polynomials are dense.

Here we give a positive answer for H^∞ and $H_{\mathbb{R}}^\infty$. In fact, we will prove a general result for normed rings with topological stable rank at most 2, and the results for H^∞ and $H_{\mathbb{R}}^\infty$ are then corollaries. We will also discuss the problem whether the matrix X itself can be chosen to be invertible.

Finally we note that these results have important consequences in the theory of stabilisation of systems in control theory. Roughly speaking, the above results imply that when one has a plant which has a transfer function in the field of fractions of any of the rings above, then it can be replaced by a *stabilizable* plant within an arbitrary degree of accuracy in the product topology. We elaborate on this below. It is known that not every system whose transfer function is a matrix with entries in the field of fractions over H^∞ or $H_{\mathbb{R}}^\infty$ admits a coprime factorisation; see [**3**]. However, our main results rescue this undesirable situation in the following sense. Suppose that we are given a system transfer function G which does not have a coprime factorisation and suppose that G has a factorisation[(2)] $G = ND^{-1}$, where N and D are matrices with entries from $H_{\mathbb{R}}^\infty$ (or H^∞). Then our main results imply that G can be replaced by a new system $\widetilde{G}$ possessing the *coprime* factorisation $\widetilde{G} = \widetilde{N}\widetilde{D}^{-1}$, and the new system $\widetilde{G}$ can be chosen to be arbitrarily close to G, in the sense that $\|\widetilde{N} - N\|_{op}$ and $\|\widetilde{D} - D\|_{op}$ can be made arbitrarily small. For further details, we refer the reader to [**5**].

1. The matricial stable rank for normed rings with tsr$= 2$

In order to prove our main result in this section, namely Theorem 1.3 below, we will use two technical lemmas, which we prove first.

A more general version of the following appears in [**1**]; see also [**4**].

Lemma 1.1. — *Let R be a commutative unital ring with bsr $R = 2$. Suppose that (f, g, h) is an invertible triple in R. Then there exists an invertible matrix $V \in R^{3 \times 3}$ such that $\begin{bmatrix} f & g & h \end{bmatrix} V = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.*

Proof. — Let $k, l \in R$ be so that $(f + kh, f + lh)$ is an invertible pair. Then there exist $\alpha, \beta \in R$ such that $\alpha(f + kh) + \beta(g + lh) = 1 - h$. Now let

$$V = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ k & l & k\alpha + l\beta + 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(f + kh) & -(g + lh) & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then V does the job. $\qquad\square$

[(2)] Here $D \in R^{n \times n}$ is invertible as an element of the matrix ring $\mathbb{F}_R^{n \times n}$, $\mathbb{F}_R$ being the field of fractions of the integral domain R under consideration, and D^{-1} then has each entry in $\mathbb{F}_R$. Thus in the case of $H_{\mathbb{R}}^\infty$ or H^∞, D should have a determinant that is not the zero function, and D^{-1} then has each entry which is a ratio of two functions in $H_{\mathbb{R}}^\infty$ or H^∞, respectively.

The next lemma says that in a normed ring R with topological stable rank at most 2, one can perform an approximate Gauss elimination with invertible matrices.

Lemma 1.2. — *Let R be a normed commutative ring with identity having topological stable rank at most 2. Suppose that $n \geq 2$ and $m \geq 1$. If $M \in R^{(n+m)\times n}$, then given $\epsilon > 0$, there exist invertible matrices $U \in R^{(n+m)\times(n+m)}$, $V \in R^{n\times n}$ and a matrix $M_{n-1} \in R^{(n-1+m)\times(n-1)}$ such that $\|M - UAV\|_{op} < \epsilon$, where*

$$
A = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & M_{n-1} & \\ 0 & & & \end{array} \right].
$$

Proof. — Let $m_{i,j}$ denote the entry in the ith row and jth column of M. Since tsr $R \leq 2$, we can find $\widetilde{m}_{1,1}, \widetilde{m}_{2,1} \in R$ such that $(\widetilde{m}_{1,1}, \widetilde{m}_{2,1})$ is an invertible pair and

$$\|m_{1,1} - \widetilde{m}_{1,1}\| < c\epsilon \quad \text{and} \quad \|m_{2,1} - \widetilde{m}_{2,1}\| < c\epsilon,$$

where c is a constant which will be specified below. Let $\widetilde{M}$ be the matrix obtained from M by replacing the entries $m_{1,1}$ and $m_{2,1}$ by $\widetilde{m}_{1,1}$ and $\widetilde{m}_{2,1}$ respectively. We choose c such that $\|M - \widetilde{M}\|_{op} < \epsilon$. (For example, if $\mathbb{C}^n$ is equipped with the usual Euclidean norm, then $c < \frac{1}{\sqrt{2}}$ will do.) Since bsr $R \leq$ tsr $R \leq 2$, it follows from Lemma 1.1 that there exists an invertible matrix $U_* \in R^{3\times 3}$ such that with

$$
\underbrace{\left[\begin{array}{c|c} U_* & 0 \\ \hline 0 & I_{n+m-3} \end{array} \right]}_{=:U_1 \in R^{(n+m)\times(n+m)}} \widetilde{M} = \left[\begin{array}{cccc} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ a_1 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_{n+m-3} & * & \cdots & * \end{array} \right].
$$

Define the invertible matrix $U_2 \in R^{(n+m)\times(n+m)}$ by

$$
U_2 := \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \vdots & & & \vdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline -a_1 & 0 & 0 & 1 & & & \\ -a_2 & 0 & 0 & 0 & 1 & & \\ \vdots & \vdots & & & & \ddots & \\ -a_{n+m-3} & 0 & 0 & 0 & & & 1 \end{array} \right].
$$

Thus

$$U_2 U_1 \widetilde{M} = \left[\begin{array}{c|ccc} 1 & b_1 & \cdots & b_{n-1} \\ \hline 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{array} \right].$$

Let $V \in R^{n \times n}$ denote the invertible matrix defined via

$$V^{-1} = \left[\begin{array}{cccc} 1 & -b_1 & \cdots & -b_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array} \right].$$

Then with $U \in R^{(n+m) \times (n+m)}$ defined by $U^{-1} = U_2 U_1$, we have

$$U^{-1} \widetilde{M} V^{-1} = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & M_{n-1} & \\ 0 & & & \end{array} \right].$$

Finally, we have

$$\left\| M - U \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & M_{n-1} & \\ 0 & & & \end{array} \right] V \right\|_{op} = \| M - \widetilde{M} \|_{op} < \epsilon.$$

This completes the proof. $\qquad\qquad\square$

We now prove the main result from this section.

Theorem 1.3. — *Let R be a normed commutative ring with identity having topological stable rank at most 2. Suppose that $n \geq 2$ and $m \geq 1$. If $N \in R^{n \times n}$ and $D \in R^{m \times n}$, then given $\epsilon > 0$, there exist $\widetilde{N} \in R^{n \times n}$ and $\widetilde{D} \in R^{m \times n}$ such that*

$$\| D - \widetilde{D} \|_{op} + \| N - \widetilde{N} \|_{op} < \epsilon,$$

and $X \widetilde{N} + Y \widetilde{D} = I_n$. Moreover, if the Bass stable rank of R is 1, then X can be chosen to be invertible.

Proof. — We prove the result by induction on n. Consider first the case when $n = 2$. Let

$$M := \left[\begin{array}{c} N \\ D \end{array} \right] \in R^{(2+m) \times 2}.$$

By Lemma 1.2, there exist invertible matrices $U \in R^{(2+m) \times (2+m)}$ and $V \in R^{2 \times 2}$, and a matrix

$$L := \begin{bmatrix} 1 & 0 \\ 0 & a_1 \\ \vdots & \vdots \\ 0 & a_{1+m} \end{bmatrix} \in R^{(2+m) \times 2}$$

such that $\|M - ULV\|_{op} < \frac{\epsilon}{2}$. Since tsr $R \leq 2$, there exist $\widetilde{a}_1, \widetilde{a}_2 \in R$ such that the pair $(\widetilde{a}_1, \widetilde{a}_2)$ is invertible and

$$\|a_1 - \widetilde{a}_1\| < c\epsilon \quad \text{and} \quad \|a_2 - \widetilde{a}_2\| < c\epsilon,$$

where c will be specified below. Let $\widetilde{L}$ be given by

$$\widetilde{L} = \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{a}_1 \\ 0 & \widetilde{a}_2 \\ 0 & a_3 \\ \vdots & \vdots \\ 0 & a_{1+m} \end{bmatrix} \in R^{(2+m) \times 2}.$$

We choose c above such that $\|L - \widetilde{L}\|_{op} < \frac{\epsilon/2}{\|U\|_{op} \|V\|_{op}}$. If $b_1, b_2 \in R$ are such that $b_1 \widetilde{a}_1 + b_2 \widetilde{a}_2 = 1$, then we see that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_1 & b_2 & 0 & \cdots & 0 \end{bmatrix} \widetilde{L} = I_2,$$

and so $\widetilde{L}$ is left invertible. Let $\widetilde{M} := U\widetilde{L}V \in R^{(2+m) \times 2}$. Then $\widetilde{M}$ is left-invertible. Moreover,

$$\begin{aligned} \|M - \widetilde{M}\|_{op} &= \|M - ULV + ULV - U\widetilde{L}V\|_{op} \\ &\leq \|M - ULV\|_{op} + \|U(L - \widetilde{L})V\|_{op} \\ &\leq \frac{\epsilon}{2} + \|U\|_{op} \frac{\epsilon/2}{\|U\|_{op} \|V\|_{op}} \|V\|_{op} = \epsilon. \end{aligned}$$

We partition $\widetilde{M}$ in conformity with that of M, and define $\widetilde{N} \in R^{2 \times 2}$ and $\widetilde{D} \in R^{m \times 2}$ via

$$\widetilde{M} = \begin{bmatrix} \widetilde{N} \\ \widetilde{D} \end{bmatrix} \in R^{(2+m) \times 2}.$$

This completes the proof when $n = 2$.

Now we suppose that the result is true for some $n \geq 2$, and prove it for $n+1$, that is, when $M \in R^{(n+1+m) \times (n+1)}$. By Lemma 1.2, there exist invertible

matrices $U \in R^{(n+1+m)\times(n+1+m)}$ and $V \in R^{(n+1)\times(n+1)}$, and a matrix

$$L := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & M_n & \\ 0 & & & \end{bmatrix} \in R^{(n+1+m)\times(n+1)}$$

where $M_n \in R^{(n+m)\times n}$ and $\|M - ULV\|_{op} < \frac{\epsilon}{2}$. It follows from the induction hypothesis that there exists a left invertible matrix $\widetilde{M_n} \in R^{(n+m)\times n}$ such that

$$\|M_n - \widetilde{M_n}\|_{op} < \frac{\epsilon/2}{\|U\|_{op}\|V\|_{op}}.$$

Define

$$\widetilde{L} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \widetilde{M_n} & \\ 0 & & & \end{bmatrix} \in R^{(n+1+m)\times(n+1)},$$

and $\widetilde{M} = U\widetilde{L}V \in R^{(n+1+m)\times(n+1)}$. We have

$$\|L - \widetilde{L}\|_{op} = \|M_n - \widetilde{M_n}\|_{op} < \frac{\epsilon/2}{\|U\|_{op}\|V\|_{op}},$$

and $\widetilde{M}$ is left invertible. Finally,

$$\|M - \widetilde{M}\|_{op} = \|M - ULV + ULV - U\widetilde{L}V\|_{op} \le \|M - ULV\|_{op} + \|U(L - \widetilde{L})V\|_{op} < \epsilon.$$

By the principle of induction, the result follows for all $n \ge 2$.

In the equation $X\widetilde{N} + Y\widetilde{D} = I_n$, we now prove that X can be chosen to be invertible. In order to do this, we will use a result of Vasershtein [**8**] that relates the Bass stable rank of the ring of matrices $R^{n\times n}$ with entries from a commutative ring R with that of the Bass stable rank of R itself:

$$\text{bsr } R^{n\times n} = \left\lceil \frac{\text{bsr } R - 1}{n} \right\rceil + 1,$$

where for a real number r, $\lceil r \rceil$ denotes the least integer larger than or equal to r. Thus, in our case if bsr $R = 1$, it follows that bsr $R^{n\times n} = 1$ for all n. We consider the cases $n > m$, $n = m$ and $m > n$ separately.

If $n = m$, the result follows immediately by applying the fact that bsr $R^{n\times n} = 1$ to the invertible pair $(\widetilde{N}, \widetilde{D})$ in the ring $R^{n\times n}$.

If $n > m$, then consider the invertible pair (N_1, D_1) in the ring $R^{n\times n}$, where $N_1 := \widetilde{N}$ and D_1 is the matrix obtained from $\widetilde{D}$ by appending $n - m$ rows of zeros to $\widetilde{D}$. There exist an invertible matrix $X_1 \in R^{n\times n}$ and a matrix $Y_1 \in R^{n\times n}$ such that $X_1 N_1 + Y_1 D_1 = I_n$. So if Y is the matrix obtained by deleting the last $n - m$ columns of Y_1, we see that $X_1\widetilde{N} + Y\widetilde{D} = I_n$.

If $m > n$, then partition $\widetilde{D}$ into blocks of n rows, and append extra rows of zeros (if necessary) to obtain the matrix

$$D_* = \begin{bmatrix} \widetilde{D}_1 \\ \vdots \\ \widetilde{D}_k \end{bmatrix},$$

where each $\widetilde{D}_j \in R^{n \times n}$, $j = 1, \ldots, k$, and the first m rows of D_* match those of $\widetilde{D}$, and the last $nk - m$ rows consist of zeros. In fact, $k = \lceil \frac{m}{n} \rceil$. Since the pair $(\widetilde{N}, \widetilde{D})$ is invertible, then also the pair $(\widetilde{N}, D_*)$ is invertible. Thus it follows that also the tuple $(\widetilde{N}, \widetilde{D}_1, \cdots, \widetilde{D}_k)$ is invertible in the ring $R^{n \times n}$, and so there exist matrices $X, Y_1, \ldots, Y_k \in R^{n \times n}$ such that

$$X\widetilde{N} + Y_1\widetilde{D}_1 + \cdots + Y_k\widetilde{D}_k = I_n.$$

Now we consider the invertible pair $(\widetilde{N}, Y_1\widetilde{D}_1 + \cdots + Y_k\widetilde{D}_k)$ in the ring $R^{n \times n}$. As bsr $R^{n \times n} = 1$, there exist an invertible $U \in R^{n \times n}$ and a matrix Y such that

$$U\widetilde{N} + Y(Y_1\widetilde{D}_1 + \cdots + Y_k\widetilde{D}_k) = I_n,$$

that is,

$$U\widetilde{N} + \begin{bmatrix} YY_1 \mid \ldots \mid YY_k \end{bmatrix} D_* = I_n.$$

Hence $U\widetilde{N} + Y_*\widetilde{D} = I_n$, where Y_* is the matrix consisting of the first m columns of $\begin{bmatrix} YY_1 \mid \ldots \mid YY_k \end{bmatrix}$. This completes the proof. $\qquad \square$

The first assertion of the Theorem 1.3 is true whenever N and D are *rectangular* matrices with the same number n of columns, provided the sum of the number of rows of N and D is *strictly* bigger than n. In fact we can also choose X such that one has one-sided invertibility. We prove this below.

We need the following elementary lemma.

Lemma 1.4. — *Let $A \in R^{n \times n}$ and $B \in R^{n \times m}$ be two matrices. Suppose that A is invertible.*

(a) *If A decomposes as $A = \begin{bmatrix} X & Y \end{bmatrix}$, where $X \in R^{n \times m}$, then X is left invertible.*

(b) *The matrix $C = \begin{bmatrix} A & B \end{bmatrix}$ is right invertible.*

Proof. — Let U be the inverse of A.

(a) Write U as $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, where U_1 has m rows. Then $U_1 X = I_m$.

(b) The matrix $\begin{bmatrix} U \\ 0 \end{bmatrix}$ is a right inverse of C. $\qquad \square$

Corollary 1.5. — *Let R be a normed commutative ring with identity having topological stable rank at most 2. Suppose that $n \geq 2$ and m_N, m_D be such that $m_N + m_D > n$. If $N \in R^{m_N \times n}$ and $D \in R^{m_D \times n}$, then given $\epsilon > 0$, there exist $\widetilde{N} \in R^{m_N \times n}$, $\widetilde{D} \in R^{m_D \times n}$, $X \in R^{n \times m_N}$ and $Y \in R^{n \times m_D}$ such that*

$$\|D - \widetilde{D}\|_{op} + \|N - \widetilde{N}\|_{op} < \epsilon,$$

and $X\widetilde{N} + Y\widetilde{D} = I_n$. Moreover, if the Bass stable rank of R is 1, then X can be chosen to be left-invertible if $m_N \leq n$, and to be right-invertible if $m_N \geq n$.

Proof. — First let us consider the case when $m_N < n$. Consider the matrix N_* obtained by appending the first $n - m_N$ rows of D to N. Call the matrix obtained by deleting the first $n - m_N$ rows of D as D_*. By Theorem 1.3 applied to N_* and D_*, we know that there exist matrices $\widetilde{N}_*$, $\widetilde{D}_*$ and X_*, Y_* such that X_* is invertible, $X_* \widetilde{N}_* + Y_* \widetilde{D}_* = I_n$, and

$$\|N_* - \widetilde{N}_*\|_{op} + \|D_* - \widetilde{D}_*\|_{op} < c\epsilon,$$

where the c will be chosen below. Now define the matrix $\widetilde{N}$ by taking the first m_N rows of $\widetilde{N}_*$. Thus:

$$\widetilde{N}_* = \left[\begin{array}{c} \widetilde{N} \\ L \end{array} \right],$$

for some matrix $L \in R^{(n - m_N) \times n}$. Let the matrix $\widetilde{D}$ be defined by taking the last $n - m_N$ rows of $\widetilde{N}_*$ and appending below these the rows of $\widetilde{D}_*$, that is,

$$\widetilde{D} = \left[\begin{array}{c} L \\ \widetilde{D}_* \end{array} \right].$$

The c is chosen small enough so that $\|N - \widetilde{N}\|_{op} + \|D - \widetilde{D}\|_{op} < \epsilon$. Let X be the matrix obtained by taking only the first m_N columns of X_*, and Y_1 be the matrix obtained from X_* by deleting the first $n - m_N$ columns. Thus:

$$X_* = \left[\begin{array}{cc} X & Y_1 \end{array} \right].$$

Since X_* is invertible, it follows that X is left-invertible. Let Y be the matrix obtained by appending to Y_1 the matrix Y_*, that is,

$$Y = \left[\begin{array}{cc} Y_1 & Y_* \end{array} \right].$$

The equation $X_* \widetilde{N}_* + Y_* \widetilde{D}_* = I_n$ now reads as

$$X\widetilde{N} + Y\widetilde{D} = I_n,$$

with X left-invertible. In fact

$$
\begin{aligned}
X_* \widetilde{N}_* + Y_* \widetilde{D}_* &= \begin{bmatrix} X & Y_1 \end{bmatrix} \begin{bmatrix} \widetilde{N} \\ L \end{bmatrix} + Y_* \widetilde{D}_* \\
&= X \widetilde{N} + \left(Y_1 L + Y_* \widetilde{D}_* \right) \\
&= X \widetilde{N} + \begin{bmatrix} Y_1 & Y_* \end{bmatrix} \begin{bmatrix} L \\ \widetilde{D}_* \end{bmatrix} \\
&= X \widetilde{N} + Y \widetilde{D}_*.
\end{aligned}
$$

Now let us consider the case when $m_N > n$. Consider the matrix N_* obtained by deleting the last $m_N - n$ rows of N. Call the matrix obtained by taking the last $m_N - n$ rows of N and appending them above the matrix D as D_*. By Theorem 1.3 applied to N_* and D_*, we know that there exist matrices $\widetilde{N}_*$, $\widetilde{D}_*$ and X_*, Y_* such that X_* is invertible, $X_* \widetilde{N}_* + Y_* \widetilde{D}_* = I_n$, and

$$
\|N_* - \widetilde{N}_*\|_{op} + \|D_* - \widetilde{D}_*\|_{op} < c\epsilon,
$$

where the c will be chosen below. Now define the matrix $\widetilde{N}$ by appending below the matrix $\widetilde{N}_*$ the first $m_N - n$ rows of the matrix $\widetilde{D}_*$. Let $\widetilde{D}$ be the matrix obtained by deleting the first $m_N - n$ rows of $\widetilde{D}_*$. The c is chosen small enough so that $\|N - \widetilde{N}\|_{op} + \|D - \widetilde{D}\|_{op} < \epsilon$. Let X be the matrix obtained by appending (on the right) to the matrix X_* the first $m_N - n$ columns of Y_*. Since X_* is invertible, it follows that X is right-invertible. Let Y be the matrix obtained from Y_* by deleting its first $m_N - n$ columns. Then the equation $X_* \widetilde{N}_* + Y_* \widetilde{D}_* = I_n$ reads as $X \widetilde{N} + Y \widetilde{D} = I_n$, with X right-invertible. $\qquad\square$

2. The matrical stable rank for H^∞

Due to a result of D. Suarez [6], it is known that if f, g are two functions in H^∞, then they can be uniformly approximated by functions $\widetilde{f}$ and $\widetilde{g}$ so that for some $x, y \in H^\infty$, $x\widetilde{f} + y\widetilde{g} = 1$; that is, the topological stable rank of H^∞ is two.

In other words, if

$$
M := \begin{bmatrix} f \\ g \end{bmatrix},
$$

then M admits an approximation

$$
\widetilde{M} := \begin{bmatrix} \widetilde{f} \\ \widetilde{g} \end{bmatrix}
$$

such that $\widetilde{M}$ is left invertible: $\begin{bmatrix} x & y \end{bmatrix} \widetilde{M} = I_2$.

The following corollary of Theorem 1.3 generalizes the above result to matrices.

Corollary 2.1. — *Suppose that $n \geq 2$ and $m \geq 1$. Let $N \in (H^\infty)^{n \times n}$ and $D \in (H^\infty)^{m \times n}$. Then for every $\epsilon > 0$, there exist matrices $\widetilde{N} \in (H^\infty)^{n \times n}$ and $\widetilde{D} \in (H^\infty)^{m \times n}$ as well as $X \in (H^\infty)^{n \times n}$ and $Y \in (H^\infty)^{n \times m}$ such that*

$$\|D - \widetilde{D}\|_{op} + \|N - \widetilde{N}\|_{op} < \epsilon$$

and $X\widetilde{N} + Y\widetilde{D} = I_n$. Moreover, X can be chosen to be invertible.

Indeed this immediate from Theorem 1.3, since tsr $H^\infty = 2$ [**6**] and bsr $H^\infty = 1$ [**7**]. We may say in short, that H^∞ has the matricial topological rank 2.

We note that in general, a square matrix $M \in (H^\infty)^{n \times n}$ cannot be approximated by an invertible matrix. For example, just take

$$M = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$$

and note that its determinant, z, cannot be uniformly approximated by invertible functions in H^∞.

3. The real Banach algebra $H_{\mathbb{R}}^\infty$

Next we switch to the real algebra $H_{\mathbb{R}}^\infty$ of all bounded analytic functions in $\mathbb{D}$ that are real on $]-1, 1[$.

It is known (see [**2**]) that $\mathrm{bsr}(H_{\mathbb{R}}^\infty) = \mathrm{tsr}(H_{\mathbb{R}}^\infty) = 2$. Also, the scalar valued Bezout equation $\sum_{j=1}^n x_j f_j = 1$ has a solution if and only $\sum_{j=1}^n |f_j| \geq \delta > 0$ in $\mathbb{D}$ (see [**2**], [**10**]). Hence one may say that $\mathbb{D}$ is dense in the spectrum of $H_{\mathbb{R}}^\infty$. Note, however, that the evaluation (or point) functionals, ϕ_a, corresponding to points $a \in \mathbb{D} \backslash \mathbb{R}$ are not in a one-to-one correspondence with the elements in $\mathbb{D} \backslash \mathbb{R}$ (indeed, $\phi_a = \phi_{\overline{a}}$), and that the maximal ideals in $H_{\mathbb{R}}^\infty$ having codimension 2 do not correspond to a unique $\mathbb{C}$-valued $\mathbb{R}$-linear character on $H_{\mathbb{R}}^\infty$. (Here $\mathbb{C}$ is viewed as an algebra over $\mathbb{R}$.) We observe that $f_1, \ldots, f_n \in H_{\mathbb{R}}^\infty$ generate a proper ideal in $H_{\mathbb{R}}^\infty$ if and only if they do in H^∞. Henceforth, the matrix $F \in (H_{\mathbb{R}}^\infty)^{(n+m) \times n}$ is left invertible if and only if for any character $\chi \in \mathcal{M}(H^\infty)$ the matrices $F(\chi)$ have maximal rank, n. Thus condition 0.1 is also necessary and sufficient for the existence of a solution in $H_{\mathbb{R}}^\infty$ to the Aryabhatta-Bezout equation $XN + YD = I_n$.

Theorem 1.3 now specializes as follows. Note that the stable rank of $H_{\mathbb{R}}^\infty$ is not 1, but 2. Thus, in contrast to Corollary 2.1, now the matrix X cannot always be chosen to be invertible. (The standard counterexample for $n = 1$ is $N = z$, $D = 1 - z^2$.)

Corollary 3.1. — *Suppose that $n \geq 2$ and $m \geq 1$. Let $N \in (H_\mathbb{R}^\infty)^{n \times n}$ and $D \in (H_\mathbb{R}^\infty)^{m \times n}$. Then for every $\epsilon > 0$, there exist matrices $\widetilde{N} \in (H_\mathbb{R}^\infty)^{n \times n}$ and $\widetilde{D} \in (H_\mathbb{R}^\infty)^{m \times n}$ as well as $X \in (H_\mathbb{R}^\infty)^{n \times n}$ and $Y \in (H_\mathbb{R}^\infty)^{n \times m}$ such that*

$$\|D - \widetilde{D}\|_{op} + \|N - \widetilde{N}\|_{op} < \epsilon$$

and $X\widetilde{N} + Y\widetilde{D} = I_n$.

Remarks.

– Corollary 3.1 answers a question raised in [**5**].
– In applications in control theory, the linear systems and transfer functions have *real* coefficients, and so in this context it is important to consider *real* algebras, since otherwise the controllers obtained are physically meaningless. Thus from the control-theoretic viewpoint of controller synthesis, one works with the algebra real $H_\mathbb{R}^\infty$ rather than H^∞.

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