PROPERTIES OF A SUBALGEBRA OF $H^\infty(\mathbb{D})$ AND STABILIZATION

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Abstract. Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. Let $\mathbb{T}$ denote the unit circle, and let $S \subset \mathbb{T}$. We denote by $A_S(\mathbb{D})$ the set of all functions $f : \mathbb{D} \cup S \to \mathbb{C}$ that are holomorphic in $\mathbb{D}$ and are bounded and continuous in $\mathbb{D} \cup S$. Equipped with the supremum norm, $A_S(\mathbb{D})$ is a Banach algebra, and it lies between the extreme cases of the disk algebra $A(\mathbb{D})$ and the Hardy space $H^\infty(\mathbb{D})$. We show that $A_S(\mathbb{D})$ has the following properties:

P1. The corona theorem holds for $A_S(\mathbb{D})$.
P2. The integral domain $A_S(\mathbb{D})$ is not a Bézout domain, but it is a Hermite ring.
P3. The stable rank of $A_S(\mathbb{D})$ is 1.
P4. The Banach algebra $A_S(\mathbb{D})$ has topological stable rank 2.

The classes $A_S(\mathbb{D})$ serve as appropriate transfer function classes for infinite dimensional systems that are not exponentially stable, but stable only in some weaker sense. Consequences of the above properties to stabilizing controller synthesis using a coprime factorization approach are discussed.

1. Notation

We will use the following standard notation:

(1) $\mathbb{C}$ denotes the complex plane, and $\mathbb{C}^*$ the extended complex plane.
(2) $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.
(3) $A(\mathbb{D}) = \{f : \overline{\mathbb{D}} \to \mathbb{C} \mid f$ is holomorphic in $\mathbb{D}$ and f is continuous and bounded on $\overline{\mathbb{D}}\}$.
(4) $H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f$ is holomorphic and bounded in $\mathbb{D}\}$.

2. Introduction

In the factorization approach to control system analysis and synthesis, one starts with a frequency domain description of the system in terms of its transfer function, and expresses the transfer function as a ratio of two stable transfer functions. Many important control problems can then be formulated and solved with this approach. The book by Vidyasagar [32] is a classical reference and the recent papers by Quadrat [18], [19], [20] give a modern comprehensive treatment of the factorization approach.

In order to use a factorization approach for solving control problems, we would like to factor the unstable transfer function as a ratio of transfer functions from a certain stable subclass. In the case of infinite-dimensional systems, there are several different notions of internal stability: exponential stability, strong stability, weak stability and so on. In control design, the properties demanded from the class of stable transfer functions depends on the type of systems being considered. So it is natural to expect a wide range of function classes for stable transfer functions in the case of infinite dimensional systems. Among the classical

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algebras considered for the purposes of systems theory, we mention the disk algebra $A(\mathbb{D})$, the Callier-Desoer class (see [1]), the Hardy space $H^\infty(\mathbb{D})$ and the Nevanlinna class (see [5]).

In this article we introduce a new class of transfer functions appropriate for infinite-dimensional systems that do not have an exponentially stable generator, but which is stable in some weaker sense. For convenience we work with the unit disk, since given any holomorphic function g defined in the right half plane, we obtain a new function $g_\circ$ defined in the disk $\mathbb{D}$ by composing g with the fractional linear transformation $\mu : \mathbb{C}^* \to \mathbb{C}^*$ given by

$$z \mapsto \mu(z) = s = \frac{1-z}{1+z}, \tag{1}$$

that takes $\mathbb{D}$ to the open right half-plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$, and $\mathbb{T}$ to the imaginary axis $i\mathbb{R}$ with the point at ∞. The map μ is one-to-one, onto and holomorphic.

In this article, we consider a family of function classes $A_S(\mathbb{D})$ (lying between the extremal classes of the disk algebra $A(\mathbb{D})$ and the Hardy space $H^\infty(\mathbb{D})$) comprising functions that are holomorphic in the open unit disk, and bounded and continuous on the open unit disk together with a subset S of the unit circle.

Before we begin proving the properties of $A_S(\mathbb{D})$ mentioned in the abstract, we give some system theoretic background which will provide the motivation for studying these function classes $A_S(\mathbb{D})$. We refer the reader to Staffans [27] for background material on the theory of infinite-dimensional well posed linear systems. In control theory, one is interested in *stable* systems. We recall that the generator A of a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on a Hilbert space X is said to be *exponentially stable* if there exist positive constants M and ϵ such that

$$\forall t \geq 0, \quad \|e^{tA}\| \leq Me^{-\epsilon t}. \tag{2}$$

A necessary condition for exponential stability is that the spectrum of A, $\sigma(A)$, lies in the open left half plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < -d\}$ for some positive d. As there is no spectrum on $i\mathbb{R}$, the corresponding transfer function g is typically bounded and holomorphic in an open right half plane containing the origin, and has a continuous extension to $i\mathbb{R}$ including the point at infinity. Correspondingly on the unit disk, $g_\circ$ is an element in the disk algebra $A(\mathbb{D})$. Thus in the factorization approach to stabilization, the disk algebra $A(\mathbb{D})$ is used as the class of transfer functions when one has exponentially stable generators.

It turns out that exponential stability is a rather strong condition, and there are many other weaker notions of stability that do arise often in applications. One such important concept is that of *strong stability*, where one demands only that A has the following property: for all $x \in X$, $e^{tA}x \to 0$ (in X) as $t \to \infty$. Thus, as opposed to exponential stability, the rate of convergence to zero may not be uniform, but depends on what initial condition we have. An elementary example is the following: $X = \ell^2$,

$$A = \begin{bmatrix} -1 & & & \\ & -\frac{1}{2} & & \\ & & -\frac{1}{3} & \\ & & & \ddots \end{bmatrix} \text{ and } e^{tA} = \begin{bmatrix} e^{-t} & & & \\ & e^{-\frac{1}{2}t} & & \\ & & e^{-\frac{1}{3}t} & \\ & & & \ddots \end{bmatrix}.$$

Then it can be shown that A is strongly stable. However, clearly A is not exponentially stable since the spectrum of the operator A is the set of its eigenvalues $-1, -\frac{1}{2}, -\frac{1}{3}, \dots,$ together with 0, and so $\sigma(A) \cap i\mathbb{R} \neq \emptyset$. Typically, the transfer function g corresponding to this A has a loss of continuity at the point $0 \in i\mathbb{R}$, and then $g_\circ$ is in $H^\infty(\mathbb{D})$, but does not have a continuous extension to $\mathbb{T}$ (indeed the point $1 \in \mathbb{T}$ corresponds to the point $0 \in i\mathbb{R}$ under μ). But $H^\infty(\mathbb{D})$ is too large a space to consider, since although $g_\circ$ may not be continuously extendible to 1, it does have a continuous extension to the whole unit circle $\mathbb{T}$ except the point at 1. More generally with other notions of stability weaker than exponential stability, the spectrum of A typically intersects the extended imaginary axis in some closed subset. We then expect that the corresponding $g_\circ \in H^\infty(\mathbb{D})$ is not an element of $A(\mathbb{D})$, but nevertheless continuous at all points on $\mathbb{T}$ except the points of the closed set $\mathbb{T} \cap \mu^{-1}(\sigma(A))$, and hence an element of $A_{\mathbb{T}\cap\mu^{-1}(\sigma(A))}(\mathbb{D})$. See Figure 1.

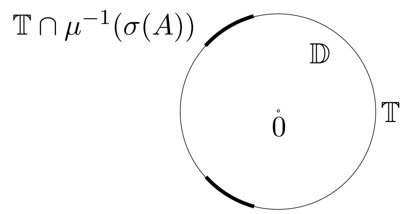

FIGURE 1. Nonexponentially stable A.

Motivated by the above considerations, for systems that have generators A that are not exponentially stable, but are stable only in some weaker sense, we introduce the following class of functions.

Definition. Let S be a subset of $\mathbb{T}$. Let

$$A_S(\mathbb{D}) = \{f : \mathbb{D} \cup S \to \mathbb{C} \mid f \text{ is holomorphic in } \mathbb{D} \text{ and } f \text{ is continuous and bounded on } \mathbb{D} \cup S\},$$

equipped with the supremum norm $\|\cdot\|_\infty$: if $f \in A_S(\mathbb{D})$, then $\|f\|_\infty := \sup_{z\in\mathbb{D}\cup S} |f(z)|$.

We note that if $S = \mathbb{T}$, then $A_{\mathbb{T}}(\mathbb{D})$ is the usual disk algebra $A(\mathbb{D})$, while if $S = \emptyset$, then one obtains the Hardy space $H^\infty(\mathbb{D})$. If S_1, S_2 are two subsets of $\mathbb{T}$ such that $S_1 \subset S_2$, then we have $A_{S_2}(\mathbb{D}) \subset A_{S_1}(\mathbb{D})$. In this manner, we obtain the family of function algebras, $\mathscr{F} = \{A_S(\mathbb{D}) \mid S \subset \mathbb{T}\}$, partially ordered with respect to set inclusion. Thus we classify transfer functions by points on the imaginary axis (equivalently on $\mathbb{T}$, when passing over to the disk) to which there exists a continuous extension.

The spaces $A_S(\mathbb{D})$ considered here have been studied earlier from a pure mathematics point of view, for instance, see Détraz [8] and Stray [28].

Just as with the extremal cases of the disk algebra $A(\mathbb{D})$ and the Hardy space $H^\infty(\mathbb{D})$, which are Banach algebras, it turns out that each function class $A_S(\mathbb{D})$ is a Banach algebra, and we prove this below, after we recall the notion of a Banach algebra.

Definitions. A *complex algebra* is a vector space R over $\mathbb{C}$ in which an associative and distributive multiplication is defined, that is, $x(yz) = (xy)z$, $(x + y)z = xz + yz$, $x(y + z) = xy + xz$ for all $x, y, z \in R$, and which is related to scalar multiplication so that $\alpha(xy) = x(\alpha y) = (\alpha x)y$ for all $x, y \in R$ and all scalars α.

A *Banach algebra* is a complex algebra R which is also a Banach space under a norm satisfying $\|xy\| \le \|x\|\|y\|$ for all $x, y \in R$.

Theorem 2.1. *Let $S \subset \mathbb{T}$. $A_S(\mathbb{D})$ is a Banach algebra.*

Proof. The completeness can be shown as follows. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then for each $z \in \mathbb{D} \cup S$, the sequence $(f_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, and so by the completeness of $\mathbb{C}$, it has a limit, say $f(z)$. These pointwise limits give rise to a complex valued function f defined on $\mathbb{D} \cup S$. We claim that f belongs to $A_S(\mathbb{D})$. f is the uniform limit of the f_n's on $\mathbb{D}\cup S$. In particular, in each compact subset of $\mathbb{D}$, the sequence $(f_n)_{n\in\mathbb{N}}$ of holomorphic functions converges uniformly to f, and so f is holomorphic in $\mathbb{D}$ (see Theorem 9.12.1 on page 229 of Dieudonné [9]). Continuity and boundedness on $\mathbb{D}\cup S$ follows from the fact that the convergence is uniform. □

$A_S(\mathbb{D})$ is commutative, and has as the identity element the constant function taking value 1 everywhere on $\mathbb{D} \cup S$, and this identity element has norm 1.

We now give examples to show that these Banach algebras $A_S(\mathbb{D})$ arise quite naturally when considering transfer functions of infinite-dimensional linear systems.

The first example shows that the class $A_S(\mathbb{D})$ is particularly useful when considering systems that have generators A that are not exponentially stable, but are stable only in some weaker sense. Indeed, it is typical that the spectrum of A has accumulation points on the extended imaginary axis when A is strongly stable, and so one can expect a loss of continuity at these points on the extended imaginary axis for the transfer function.

Example. Let $\ell_2(\mathbb{N})$ denote the Hilbert space of square summable sequences, and let the standard orthonormal basis for $\ell_2(\mathbb{N})$ be denoted by $\{e_n \mid n \in \mathbb{N}\}$. Consider the system

$$\begin{array}{rcl} x'(t) & = & A_0x(t) + Bu(t) \\ y(t) & = & B^*x(t) \end{array}$$

on $\ell_2(\mathbb{N})$, where $A_0 : D(A_0)\ (\subset \ell_2(\mathbb{N})) \to \ell_2(\mathbb{N})$ and $B \in \mathscr{L}(\mathbb{C}, \ell_2(\mathbb{N}))$ are given by

$$A_0 = \left[\begin{array}{cc|cc|cc|c} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ \hline & & 0 & 2 & & & \\ & & -2 & 0 & & & \\ \hline & & & & 0 & 3 & \\ & & & & -3 & 0 & \\ \hline & & & & & & \ddots \end{array}\right], \quad \text{and} \quad B = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{3} \\ 0 \\ \vdots \end{bmatrix}, \tag{3}$$

where $D(A_0) = \{x \in \ell_2(\mathbb{N}) \mid \sum_{n=1}^{\infty}(|n\langle x, e_{2n-1}\rangle|^2 + |n\langle x, e_{2n}\rangle|^2) < \infty\}$. The following properties were shown in Curtain and Sasane [3]:

(1) A_0 is a Riesz spectral operator with the eigenvalues $\pm ni$, $n \in \mathbb{N}$, and the corresponding (orthogonal) Riesz basis of eigenvectors $\frac{1}{\sqrt{2}}(e_n \pm ie_{n+1})$,
(2) A_0 is the generator of a strongly continuous contraction semigroup on $\ell_2(\mathbb{N})$,
(3) A_0 has compact resolvent,
(4) (A_0, B) is approximately controllable, and (A_0^*, B^*) is approximately observable,
(5) $A_0 - BB^*$ and $A_0^* - BB^*$ generate strongly stable semigroups on $\ell_2(\mathbb{N})$,
(6) $g(s) = B^*(sI - A_0 + BB^*)^{-1}B$ has a Hankel operator that is bounded, but not compact.

From Hartman's theorem (see for instance Corollary 4.10 on page 46 of Partington [17]), we see that the transfer function $g(s) = B^*(sI - A_0 + BB^*)^{-1}B$ cannot be continuous at infinity. Hence the corresponding function on the disk $g_\circ$ does not belong to the disk algebra $A(\mathbb{D})$.

As $\|BB^*\| \leq \frac{1}{4}$, from Theorem 3.6 on page 209 of Kato [13], it follows that $\sigma(A_0 - BB^*) \subset \cup_{m\in\mathbb{Z}}\{s \in \mathbb{C} \mid |s - mi| \leq \frac{1}{4}\}$. Thus with $S := (\cup_{m\in\mathbb{Z}}\mu^{-1}\{s \in \mathbb{C} \mid |s - (m + \frac{1}{2})i| < \frac{1}{4}\}) \cap \mathbb{T}$, we have that $g_\circ \in A_S(\mathbb{D})$. $\Diamond$

Example. Consider a well posed linear system with the generating operators A, B, C and transfer function G, such that $0 \in \mathbb{C} \setminus \sigma(A)$. Then the *reciprocal system* of the well posed linear system, introduced by Curtain (see for example [2]), is the linear system with the bounded generating operators A^{-1}, $A^{-1}B$, $-CA^{-1}$, $g(0)$ and transfer function $g^-(s) = g(0) - CA^{-1}(sI - A^{-1})^{-1}A^{-1}B = g(\frac{1}{s})$. Reciprocal systems are useful in the analysis of control systems, since the operators A^{-1}, $A^{-1}B$, $-CA^{-1}$, $g(0)$ are all bounded: indeed, one can pass from the original system to its reciprocal, solve the transformed control problem for it, and then return back to the original system (see for example, [4]).

We note that if A is exponentially stable, then g^- is bounded and holomorphic in the open right half-plane $\mathbb{C}_+$ and continuous (and even holomorphic) in a neighbourhood ∞. Hence the corresponding function $g_\circ^-$ on the unit disk belongs to the space $A_S(\mathbb{D})$, where S is a suitably small arc around the point $\{-1\}$. $\Diamond$

We would like to develop a factorization approach to stabilizing controller design for the algebra $A_S(\mathbb{D})$. Then starting from a given plant in $A_S(\mathbb{D})$, we would also be able to construct a stabilizing controller in the *same* class $A_S(\mathbb{D})$. The properties that play an important role in the factorization approach (see [32], [18], [19], [20]) are listed below, and it is known that the disk algebra $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$ have these useful properties:

- P1. The corona theorem.
- P2. The Hermite property.
- P3. Stable rank $= 1$.
- P4. Topological stable rank $= 2$.

We prove that the properties P1, P2, P3 and P4 also hold for the infinitely many intermediate spaces $A_S(\mathbb{D})$, where S is an arbitrary subset of the unit circle. That the property P1 is true for $A_S(\mathbb{D})$, was already known to be true (see [28] and [8]), but we give new bounds on the solution in Section 3.

The properties P1, P2, P3, P4 are proved in Sections 3, 4, 5, 6, respectively. Applications of these properties to coprime factorization and stabilization are given in Section 7.

3. The corona theorem

In Section 7 we will give a test for coprimeness over $A_S(\mathbb{D})$ of a matrix pair (N, D) in Theorem 7.1. This test for coprimeness is obtained by using a necessary and sufficient condition for the Bézout identity to hold in the algebra $A_S(\mathbb{D})$, which is given in Theorem 3.3, called the corona theorem for $A_S(\mathbb{D})$.

The first part of Theorem 3.3, that is, the statement in Theorem 3.3 up to (12), is not new, and can be found in the Corollary on page 514 of Stray [28] and in Corollary 2 on page 835 of Détraz [8]. Nevertheless, for the sake of completeness, a proof of Theorem 3.3 is given here. As in [28], we reprove Theorem 3.3 using Carleson's corona theorem and an approximation result, but we do this for arbitrary subsets of $\mathbb{T}$ (as opposed to [28], where only closed subsets of $\mathbb{T}$ were considered) and also show the existence of solutions with bounds (see (14) and the remark following Theorem 3.3). This proof was shown by Rosay [25].

Theorem 3.3 is a generalization of Carleson's corona theorem for $H^\infty(\mathbb{D})$, and the proof of Theorem 3.3 given here uses the full strength of Carleson's theorem. So we do not obtain a new proof of the Carleson corona theorem when $S = \mathbb{T}$!

The classical Carleson's corona theorem is the following, and for a simplified proof of this theorem, we refer the reader to Narasimhan and Nievergelt [15].

Theorem 3.1. *Let $f_1, \dots, f_n \in H^\infty(\mathbb{D})$. There exists a $\delta > 0$ such that*

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} |f_i(z)| \geq \delta, \tag{4}$$

iff there exist $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1. \tag{5}$$

There exists a constant $C_\emptyset(n, \delta)$ such that for all $f_1, \dots, f_n \in H^\infty(\mathbb{D})$ satisfying (4) and

$$\forall z \in \mathbb{D}, \quad |f_i(z)| \leq 1, \quad i \in \{1, \dots, n\}, \tag{6}$$

there exist $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ satisfying (5) with the bounds

$$\forall z \in \mathbb{D}, \quad |g_i(z)| \leq C_\emptyset(n, \delta), \quad i \in \{1, \dots, n\}. \tag{7}$$

This theorem also happens to be true with the disk algebra $A(\mathbb{D})$ instead of $H^\infty(\mathbb{D})$. A nonconstructive proof of the corona theorem for $A(\mathbb{D})$, relying on Zorn's lemma, using the elementary theory of Banach algebras can be found in Rudin [26] (Theorem 18.18, page 365), which gives the result without the existence of a universal bound for the solution. Theorem 3.3 below, applied to the case $S = \mathbb{T}$, also yields the existence of such a universal constant for $A(\mathbb{D})$.

Theorem 3.3 gives the same results as the above two cases, in the more general case when S is between the two extreme cases: $\emptyset \subset S \subset \mathbb{T}$. This can be proved using Carleson's corona theorem for $H^\infty(\mathbb{D})$ and the following approximation result.

Lemma 3.2. *Let $S \subset \mathbb{T}$. If $f_1, \dots, f_n \in A_S(\mathbb{D})$, then given any $\epsilon_1 > 0$ and any $\epsilon_2 > 0$, there exists an open set Ω containing $\mathbb{D} \cup S$ (which depends on ϵ_1 and ϵ_2 in general), and there exist holomorphic functions $f_i^{\mathrm{e}} : \Omega \to \mathbb{C}$, $i \in \{1, \dots, n\}$ such that*

$$\forall z \in \mathbb{D} \cup S, \quad \forall i \in \{1, \dots, n\}, \quad |f_i(z) - f_i^{\mathrm{e}}(z)| < \epsilon_1, \text{ and} \tag{8}$$

$$\forall z \in \Omega \setminus \mathbb{D}, \quad \exists z_* \in S \text{ such that } \forall i \in \{1, \dots, n\}, \quad |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| < \epsilon_2. \tag{9}$$

Proof. Let $i \in \{1, \dots, n\}$. From Theorem 1 of Range [21] (see also Davie et al. [7]), it follows that there exists an open simply connected set Ω_i' containing $\mathbb{D} \cup S$, and an holomorphic $f_i^{\mathrm{e}} : \Omega_i' \to \mathbb{C}$ such that

$$\forall z \in \mathbb{D}, \quad |f_i(z) - f_i^{\mathrm{e}}(z)| < \epsilon_1. \tag{10}$$

Let $\Omega' = \cap_{i=1}^{n} \Omega_i'$, and replace f_i^{e}'s by their restrictions to Ω'. By using the continuity of f_i on $\mathbb{D} \cup S$, and also that of f_i^{e}, (10) yields (8).

Let $i \in \{1, \dots, n\}$. For each $z_* \in S$, there exists an $r_{z_*}^i > 0$ such that the open ball with center z_* and radius $r_{z_*}^i$ is contained in Ω', that is, $B(z_*, r_{z_*}^i) \subset \Omega'$, and moreover, for all $z \in B(z_*, r_{z_*}^i)$, $|f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| < \epsilon_2$. Define $r_{z_*} = \min\{r_{z_*}^1, \dots, r_{z_*}^n\}$, and let $\Omega = \mathbb{D} \cup (\cup_{z_* \in S} B(z_*, r_{z_*}))$. Then Ω is an open simply connected set containing $\mathbb{D} \cup S$, and the restriction of f_i^{e}'s to Ω satisfy (8) and (9). $\square$

The following result is the corona theorem for the algebra $A_S(\mathbb{D})$.

Theorem 3.3. *Let $S \subset \mathbb{T}$ and $f_1, \dots, f_n \in A_S(\mathbb{D})$. There exists a $\delta > 0$ such that*

$$\forall z \in \mathbb{D} \cup S, \quad \sum_{i=1}^{n} |f_i(z)| \geq \delta, \tag{11}$$

iff there exist $g_1, \dots, g_n \in A_S(\mathbb{D})$ such that

$$\forall z \in \mathbb{D} \cup S, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1. \tag{12}$$

There exists a constant $C_S(n, \delta)$ such that for all $f_1, \dots, f_n \in A_S(\mathbb{D})$ satisfying (11) and

$$\forall z \in \mathbb{D} \cup S, \quad |f_i(z)| \leq 1, \quad i \in \{1, \dots, n\} \tag{13}$$

then there exist $g_1, \dots, g_n \in A_S(\mathbb{D})$ satisfying (12) with the bounds

$$\forall z \in \mathbb{D} \cup S, \quad |g_i(z)| \leq C_S(n, \delta), \quad i \in \{1, \dots, n\}. \tag{14}$$

Proof. The necessity of the condition (11) for (12) to hold is obvious, and we prove the sufficiency. Assume that (13) holds, as this can always be ensured by multiplication by a suitable constant (and replacing the δ). Let

$$M_\delta = \frac{1}{\frac{\delta}{4n} + \frac{\delta}{2n} + 1}, \quad \epsilon = \min \left\{ \frac{1}{2n M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta\right)}, \frac{\delta}{2n} \right\}, \tag{15}$$

where $C_\emptyset(\cdot, \cdot)$ denotes a universal constant in Carleson's Theorem 3.1 above.

Then from Lemma 3.2, there exists an open simply connected neighbourhood Ω of $\mathbb{D} \cup S$ and holomorphic functions $f_i^{\mathrm{e}} : \Omega \to \mathbb{C}$, $i \in \{1, \dots, n\}$, such that

$$\forall z \in \mathbb{D} \cup S, \quad \forall i \in \{1, \dots, n\}, \quad |f_i(z) - f_i^{\mathrm{e}}(z)| < \epsilon, \text{ and} \tag{16}$$

$$\forall z \in \Omega \setminus \mathbb{D}, \quad \exists z_* \in S \text{ such that } \forall i \in \{1, \dots, n\}, \quad |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| < \frac{\delta}{4n}. \tag{17}$$

Then for all $z \in \mathbb{D} \cup S$,

$$\begin{aligned}
\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^{n} |f_i(z) - (f_i(z) - f_i^{\mathrm{e}}(z))| \geq \sum_{i=1}^{n} (|f_i(z)| - |f_i(z) - f_i^{\mathrm{e}}(z)|) \\
&\geq \delta - n \cdot \frac{\delta}{2n} \quad \text{(using (11), (16) and (15))} \\
&= \frac{\delta}{2} \qquad (18) \\
&> \frac{\delta}{4}. \qquad (19)
\end{aligned}$$

Furthermore, for $z \in \Omega \setminus \mathbb{D}$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z_*) - (f_i^{\mathrm{e}}(z_*) - f_i^{\mathrm{e}}(z))| \quad \text{(where } z_* \text{ is as in (17))} \\
&\geq \sum_{i=1}^{n} (|f_i^{\mathrm{e}}(z_*)| - |f_i^{\mathrm{e}}(z_*) - f_i^{\mathrm{e}}(z)|) \geq \frac{\delta}{2} - n \cdot \frac{\delta}{4n} \quad \text{(using (18) and (17))} \\
&= \frac{\delta}{4}. \qquad (20)
\end{aligned}$$

From (19) and (20), we obtain

$$\forall z \in \Omega, \quad \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| \geq \frac{\delta}{4}. \tag{21}$$

For all $z \in \mathbb{D} \cup S$,

$$\begin{aligned} |f_i^{\mathrm{e}}(z)| &\leq |f_i^{\mathrm{e}}(z) - f_i(z)| + |f_i(z)| < \epsilon + 1 \\ &\leq \frac{\delta}{2n} + 1 \\ &< \frac{\delta}{4n} + \frac{\delta}{2n} + 1 = \frac{1}{M_\delta}. \end{aligned} \tag{22}$$

Furthermore, for all $z \in \Omega \setminus \mathbb{D}$,

$$\begin{aligned} |f_i^{\mathrm{e}}(z)| &\leq |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| + |f_i^{\mathrm{e}}(z_*)| \quad \text{(where } z_* \text{ is as in (17))} \\ &< \frac{\delta}{4n} + \frac{\delta}{2n} + 1 \quad \text{(using (17) and (22))} \\ &= \frac{1}{M_\delta}. \end{aligned}$$

Hence for all $z \in \Omega$,

$$|M_\delta f_i^{\mathrm{e}}(z)| \leq 1. \tag{23}$$

By the Riemann mapping theorem (see for instance Theorem 14.8 on page 283 of Rudin [26]), there exists a one-to-one holomorphic map φ from Ω onto $\mathbb{D}$. Thus $\varphi^{-1} : \mathbb{D} \to \Omega$ is also holomorphic. For each $i \in \{1, \dots, n\}$, the maps $M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1} \in H^\infty(\mathbb{D})$ and moreover, from (21) and (23) we obtain for all $z \in \mathbb{D}$, $\sum_{i=1}^{n} |(M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1})(z)| \geq \frac{\delta}{4} M_\delta$, and $|(M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1})(z)| \leq 1$. Thus by Carleson's corona theorem (Theorem 3.1), it follows that there exist $\tilde{g}_1, \dots, \tilde{g}_n \in H^\infty(\mathbb{D})$ such that for all $z \in \mathbb{D}$, $\sum_{i=1}^{n} (M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1})(z) \tilde{g}_i(z) = 1$, and moreover we can choose the $\tilde{g}_i$'s such that for all $z \in \mathbb{D}$, $|\tilde{g}_i(z)| \leq C_\emptyset\left(n, \frac{\delta}{4} M_\delta\right)$, for all $i \in \{1, \dots, n\}$. Now define $g_i^{\mathrm{e}} = M_\delta \tilde{g}_i \circ \varphi$, $i \in \{1, \dots, n\}$. Then we have that each g_i^{e} is holomorphic in Ω, and

$$\forall z \in \Omega, \quad \sum_{i=1}^{n} f_i^{\mathrm{e}}(z) g_i^{\mathrm{e}}(z) = 1 \quad \text{and} \quad |g_i^{\mathrm{e}}(z)| \leq M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta\right). \tag{24}$$

Let $h : \mathbb{D} \cup S \to \mathbb{C}$ be defined by

$$h(z) = \sum_{i=1}^{n} f_i(z) g_i^{\mathrm{e}}(z), \quad z \in \mathbb{D} \cup S. \tag{25}$$

Then $h \in A_S(\mathbb{D})$. Furthermore, for all $\mathbb{D} \cup S$,

$$\begin{aligned} |h(z)| &= \left| \sum_{i=1}^{n} f_i(z) g_i^{\mathrm{e}}(z) \right| = \left| 1 - \sum_{i=1}^{n} (f_i^{\mathrm{e}}(z) - f_i(z)) g_i^{\mathrm{e}}(z) \right| \geq 1 - \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z) - f_i(z)| |g_i^{\mathrm{e}}(z)| \\ &\geq 1 - n \cdot \frac{1}{2n M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta\right)} \cdot M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta\right) \quad \text{(using (16), (15) and (24))} \\ &= \frac{1}{2}. \end{aligned} \tag{26}$$

Now define $g_i : \mathbb{D} \cup S \to \mathbb{C}$, $i \in \{1, \dots, n\}$ by $g_i(z) = \frac{g_i^{\mathrm{e}}(z)}{h(z)}$, $z \in \mathbb{D} \cup S$. Then the g_i's belong to $A_S(\mathbb{D})$, and from (25) we obtain for all $z \in \mathbb{D} \cup S$, $\sum_{i=1}^n f_i(z) g_i(z) = 1$. Moreover, for all $i \in \{1, \dots, n\}$,

$$\forall z \in \mathbb{D} \cup S, \quad |g_i(z)| \leq 2 M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta \right) =: C_S(n, \delta). \tag{27}$$

This completes the proof of the theorem. ☐

Remark. In Garnett [11] (see page 327), the following bound was given for the universal constant $C_\emptyset(n, \delta)$ in (7):

$$C_\emptyset(n, \delta) \leq C \cdot \left(\frac{n^{\frac{3}{2}}}{\delta^2} + \frac{n^2}{\delta^4} \right), \tag{28}$$

where C is a constant not dependent on n and δ. In Theorem 3.3, for $C_S(n, \delta)$ in (14), the following bound was obtained (see (27) in the proof):

$$C_S(n, \delta) \leq C' \cdot \left(n^{\frac{3}{2}} \left(\frac{3}{n} + \frac{4}{\delta} \right)^2 + n^2 \left(\frac{3}{n} + \frac{4}{\delta} \right)^4 \right). \tag{29}$$

For a fixed n, the right hand sides of (28) and (29) are of the same order in δ for $\delta \downarrow 0$.

Before we derive consequences of Theorem 3.3, we recall the following terminology from the elementary theory of Banach algebras.

Definitions. Let R be a commutative Banach algebra with identity 1_R. A *complex homomorphism* is a nonzero homomorphism $\varphi : R \to \mathbb{C}$ such that $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(\alpha x) = \alpha \varphi(x)$, $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in R$ and all scalars α. It can be verified that for every complex homomorphism φ, there holds that $\varphi(1_R) = 1$, and that φ is a continuous linear functional with norm at most equal to 1: $\|\varphi\| = \sup_{\|x\| \leq 1} |\varphi(x)| \leq 1$. Let $\mathfrak{M}(R)$ denote the set of complex homomorphisms of R. Then $\mathfrak{M}(R)$ is a subset of R^*, the set of all bounded linear functionals from R to $\mathbb{C}$, and in fact it is contained in the unit ball of R^*. R^* can be equipped with the weak-star topology. Recall that a set $G \subset R^*$ is open in the weak-star topology iff for every $g \in G$, there are finitely many points $x_1, \dots, x_n \in X$ and positive reals $\epsilon_1, \dots, \epsilon_n$ such that $\cap_{i=1}^n \{ f \in R^* \mid |f(x_i) - g(x_i)| < \epsilon_i \} \subset G$. $\mathfrak{M}(R)$ equipped with the induced weak-star topology from R^* is a topological space, and this topology on $\mathfrak{M}(R)$ is called the *Gelfand topology.*

A subset I of R is called an *ideal* if I is a subspace of R (as a vector space), and $xy \in I$ for all $x \in R$ and $y \in I$. A *maximal ideal* is a proper ideal (that is, $\neq R$) which is not contained in any larger proper ideal.

There is a one-to-one correspondence between homomorphisms of R onto $\mathbb{C}$ and maximal ideals M in R. The correspondence is defined by $M = \ker(\varphi)$. Owing to this correspondence, the set $\mathfrak{M}(R)$ of all complex homomorphisms of R is called the *space of maximal ideals of* R.

With each element $x \in R$, we associate a complex-valued function $\widehat{x}$ on $\mathfrak{M}(R)$ as follows:

$$\widehat{x}(\varphi) = \varphi(x), \quad \varphi \in \mathfrak{M}(R).$$

$\widehat{x}$ is called the *Gelfand transform* of x.

We now recall the following known result.

Lemma 3.4. *Let R be a commutative complex Banach algebra with identity 1_R and let $\mathfrak{M}(R)$ be the space of maximal ideals of R, and let $\mathfrak{M}_0 \subset \mathfrak{M}(R)$. Then the following are equivalent:*

1. *$\mathfrak{M}_0$ is dense (in the Gelfand topology) in $\mathfrak{M}(R)$.*
2. *Let $x_1, \dots, x_n \in R$. There exist $y_1, \dots, y_n \in R$ such that $x_1y_1 + \dots + x_ny_n = 1_R$ iff there exists a $\delta > 0$ such that for all $\varphi \in \mathfrak{M}_0$, $|\widehat{x}_1(\varphi)| + \dots + |\widehat{x}_n(\varphi)| \geq \delta$.*
3. *Let $\Lambda \in R^{n \times m}$. Then there is a $V \in R^{m \times n}$ such that $V\Lambda = I$ iff there exists a $\delta > 0$ such that for all $\varphi \in \mathfrak{M}_0$, $\widehat{\Lambda}(\varphi)^* \widehat{\Lambda}(\varphi) \geq \delta I$.*

Proof. See for instance pages 201-203 of Duren [10]. □

In 2 and 3, we can also write 'if' instead of 'iff', as the converse can be shown to be true for any $\mathfrak{M}_0 \subset \mathfrak{M}(R)$.

Let $S \subset \mathbb{T}$. Then for each $z_0 \in \mathbb{D} \cup S$, the evaluation map $f \mapsto f(z_0)$ is a complex homomorphism from $A_S(\mathbb{D})$ onto $\mathbb{C}$. With this identification of the set $\mathbb{D} \cup S$ as a subset of $\mathfrak{M}(A_S(\mathbb{D}))$, we now obtain the following theorem.

Corollary 3.5. *Let $S \subset \mathbb{T}$. Then the following hold:*

1. *$\mathbb{D} \cup S$ is dense (in the Gelfand topology) in $\mathfrak{M}(A_S(\mathbb{D}))$.*
2. *Let $\Lambda \in A_S(\mathbb{D})^{n \times m}$. Then there is a $V \in A_S(\mathbb{D})^{m \times n}$ such that $V\Lambda = I$ iff there exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $\Lambda(z)^* \Lambda(z) \geq \delta I$.*

Proof. This follows from Theorem 3.3 and Lemma 3.4. □

Note that 1 in the above Corollary 3.5 says that the "corona" $\mathfrak{M}(A_S(\mathbb{D})) \setminus \overline{\mathbb{D} \cup S}$ is empty.

In Section 7, we will apply the result given in item 2 of Corollary 3.5 in order to characterize matrix coprime pairs in $A_S(\mathbb{D})$.

4. The Hermite property

In Section 7, we will consider unstable transfer functions that can be expressed as a quotient of two elements from $A_S(\mathbb{D})$. We first remark that $A_S(\mathbb{D})$ is an integral domain (that is, it is a commutative ring with an identity element in which the product of two nonzero elements is zero iff at least one of the elements is zero), so that we can consider its field of fractions. We will show that not every transfer function obtained as a ratio of elements of $A_S(\mathbb{D})$ has a coprime factorization in $A_S(\mathbb{D})$ in Section 7, by using the result in Theorem 4.1 below, which says that $A_S(\mathbb{D})$ is not a Bézout domain.

Definition. R is said to be a *Bézout domain* if every finitely generated ideal in R is principal.

The fact that $A_S(\mathbb{D})$ is a Bézout domain is unlike the situation with the ring $H(\mathbb{D})$ of holomorphic functions (see Theorem 15.15 of Rudin [26]), but is similar to the extremal cases of $A_\emptyset(\mathbb{D}) = H^\infty(\mathbb{D})$ (see von Renteln [22]) and of $A_\mathbb{T}(\mathbb{D}) = A(\mathbb{D})$ (see Vidyasagar et al. [31]).

Theorem 4.1. *Let $S \subset \mathbb{T}$. $A_S(\mathbb{D})$ is not a Bézout domain.*

Proof. In Logemann [14], it was shown that if R is subring of $H^\infty(\mathbb{C}_+)$ that contains the Laplace transform of functions in $L^1((0,\infty);\mathbb{C})$, then R contains a finitely generated ideal which is not principal. (In fact, on page 249 of [14], an explicit construction of such a finitely generated, non-principal ideal is given in terms of Blaschke products.) The disk algebra $A(\mathbb{D})$ contains the Laplace transforms of integrable functions (composed with μ^{-1}, where μ denotes the Moebius function given by (1)) (see for example §A.6.2 on page 636 of Curtain and Zwart

[6]), and $A \subset A_S(\mathbb{D})$. Consequently, $A_S(\mathbb{D})$ contains finitely generated ideals that are not principal. Hence $A_S(\mathbb{D})$ is not a Bézout domain. □

In Section 7, we will show that Theorem 4.1 has the consequence that not every transfer function has a coprime factorization. However, we will also show that if a transfer function does have a right (or left) coprime factorization then it also has a left (respectively, right) coprime factorization. This is a consequence of Theorem 4.3, which we prove next. We begin by giving a few preliminaries.

Definitions. Let R be a ring. A square matrix $U \in R^{m\times m}$ is said to be *unimodular* if it is invertible in $R^{m\times m}$. Let $X \in R^{m\times n}$ with $m < n$. X is said to be *complementable* if there exists a unimodular matrix $U \in R^{n\times n}$ that contains X as a submatrix. A row $\left[\begin{array}{ccc} x_1 & \dots & x_n \end{array}\right] \in R^{1\times n}$ is called a *unimodular row* if the ideal generated by $x_1, \dots, x_n$ is equal to the ring R. A ring R is called *Hermite* if every unimodular row is complementable.

Let $S \subset \mathbb{T}$. If $f_1, \dots, f_n \in A_S(\mathbb{D})$, then $\| \left[\begin{array}{ccc} f_1 & \dots & f_n \end{array}\right] \|_\infty := \sup_{z\in\mathbb{D}\cup S} (\sum_{i=1}^n |f_i(z)|^2)^{\frac{1}{2}}$. If $P \in A_S(\mathbb{D})^{p\times m}$, then $\|P\|_\infty = \sup_{z\in\mathbb{D}\cup S} \|P(z)\|_{\mathscr{L}(\mathbb{C}^m,\mathbb{C}^p)}$. The case $m = p$ is of particular interest. Indeed, $A_S(\mathbb{D})^{m\times m}$ equipped with the norm $\|\cdot\|_\infty$ is a Banach algebra with the unit I. The set of invertible elements in $A_S(\mathbb{D})^{m\times m}$ is denoted by $\mathscr{G}(A_S(\mathbb{D})^{m\times m})$.

In order to prove Theorem 4.3, we will need the following key result.

Theorem 4.2. *If $f_1, \dots, f_n \in H^\infty(\mathbb{D})$ and there exists a $\delta > 0$ such that*

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^n |f_i(z)| \geq \delta, \tag{30}$$

then

$$\exists \Lambda = \left[\begin{array}{c} f \\ F \end{array}\right] \in \mathscr{G}(H^\infty(\mathbb{D})^{n\times n}), \text{ where } f = \left[\begin{array}{ccc} f_1 & \dots & f_n \end{array}\right], \text{ and } F \in H^\infty(\mathbb{D})^{(n-1)\times n}. \tag{31}$$

If $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ are such that for all $z \in \mathbb{D}$, $\sum_{i=1}^n f_i(z)g_i(z) = 1$, then there exists a $F \in H^\infty(\mathbb{D})^{(n-1)\times n}$ such that Λ satisfying (31) is such that $\|\Lambda^{-1}\|_\infty \leq \|g\|_\infty(1 + \|f\|_\infty) + 1$, where $g := \left[\begin{array}{ccc} g_1 & \dots & g_n \end{array}\right]$.

Proof. By Carleson's corona theorem, we know that under the condition (30), there exist $g_1, \dots, g_n$ in $H^\infty(\mathbb{D})$ such that for all $z \in \mathbb{D}$, $\sum_{i=1}^n f_i(z)g_i(z) = 1$. Then the result follows from Tolokonnikov's lemma (see for example, Appendix 3, §10 on page 293 of Nikolski [16]). □

We are now ready to prove the following theorem. This result was known in the case of $A_\emptyset = H^\infty(\mathbb{D})$ (this follows from Tolokonnikov's lemma; see §10 in Appendix 3 of Nikolski [16]), and also in the case of the disk algebra $A = A_\mathbb{T}$ (see Corollary 71 and Example 72 on pages 346-347 of Vidyasagar [32]).

Theorem 4.3. *Let $S \subset \mathbb{T}$. $A_S(\mathbb{D})$ is a Hermite ring.*

Proof. Let $f_1, \dots, f_n \in A_S(\mathbb{D})$ be such that the ideal generated by $f_1, \dots, f_n$ is the full ring $A_S(\mathbb{D})$. Then there exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $\sum_{i=1}^n |f_i(z)| \geq \delta > 0$. Without loss of generality, we can also assume that for all $z \in \mathbb{D}\cup S$, $(\sum_{i=1}^n |f_i(z)|^2)^{\frac{1}{2}} \leq \frac{1}{2}$. Indeed, the f_i's and δ can be scaled without altering the hypothesis that the ideal generated by $f_1, \dots, f_n$

is the full ring $A_S(\mathbb{D})$. Let

$$M(\delta,n)=2\sqrt{n}C_\emptyset\left(n,\frac{\delta}{4}\right)+1,\quad \epsilon_1=\min\left\{\frac{\delta}{2n},\frac{1}{2M(\delta,n)},\frac{1}{4\sqrt{n}}\right\}\text{ and }\epsilon_2=\min\left\{\frac{\delta}{4n},\frac{1}{4\sqrt{n}}\right\}.$$

Then from Lemma 3.2, there exists an open connected neighbourhood Ω of $\mathbb{D}\cup S$ and holomorphic functions $f_i^{\mathrm{e}}:\Omega\to\mathbb{C}$, $i\in\{1,\dots,n\}$, such that

$$\forall z\in\mathbb{D}\cup S,\quad \forall i\in\{1,\dots,n\},\quad |f_i(z)-f_i^{\mathrm{e}}(z)|<\epsilon_1,\text{ and} \tag{32}$$

$$\forall z\in\Omega\setminus\mathbb{D},\quad \exists z_*\in S\text{ such that }\forall i\in\{1,\dots,n\},\quad |f_i^{\mathrm{e}}(z)-f_i^{\mathrm{e}}(z_*)|<\epsilon_2. \tag{33}$$

Then for all $z\in\mathbb{D}\cup S$,

$$\begin{aligned}\sum_{i=1}^n|f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^n|f_i(z)-(f_i(z)-f_i^{\mathrm{e}}(z))|\geq\sum_{i=1}^n(|f_i(z)|-|f_i(z)-f_i^{\mathrm{e}}(z)|)\\ &> \delta-n\cdot\frac{\delta}{2n}=\frac{\delta}{2}>\frac{\delta}{4},\end{aligned}$$

and for all $z\in\Omega\setminus\mathbb{D}$, we have

$$\begin{aligned}\sum_{i=1}^n|f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^n|f_i^{\mathrm{e}}(z_*)-(f_i^{\mathrm{e}}(z_*)-f_i^{\mathrm{e}}(z))|\quad\text{(where }z_*\text{ is as in (33))}\\ &\geq \sum_{i=1}^n(|f_i^{\mathrm{e}}(z_*)|-|f_i^{\mathrm{e}}(z_*)-f_i^{\mathrm{e}}(z)|)>\frac{\delta}{2}-n\cdot\frac{\delta}{4n}=\frac{\delta}{4}.\end{aligned}$$

Consequently,

$$\forall z\in\Omega,\quad \sum_{i=1}^n|f_i^{\mathrm{e}}(z)|>\frac{\delta}{4}>0. \tag{34}$$

Furthermore for all $z\in\mathbb{D}\cup S$,

$$|f_i^{\mathrm{e}}(z)|\leq|f_i(z)|+|f_i^{\mathrm{e}}(z)-f_i(z)|\leq|f_i(z)|+\epsilon_1<|f_i(z)|+\epsilon_1+\epsilon_2, \tag{35}$$

and so for all $z\in\mathbb{D}\cup S$,

$$\begin{aligned}(\sum_{i=1}^n|f_i^{\mathrm{e}}(z)|^2)^{\frac12} &\leq (\sum_{i=1}^n|f_i(z)|^2)^{\frac12}+(\epsilon_1+\epsilon_2)\sqrt{n}\leq\sup_{z\in\mathbb{D}\cup S}(\sum_{i=1}^n|f_i(z)|^2)^{\frac12}+(\epsilon_1+\epsilon_2)\sqrt{n}\\ &\leq \frac12+(\epsilon_1+\epsilon_2)\sqrt{n}. \end{aligned}\tag{36}$$

On the other hand, if $z\in\Omega\setminus\mathbb{D}$, and if z_* is as in (33), then

$$|f_i^{\mathrm{e}}(z)|\leq|f_i(z_*)|+|f_i(z_*)-f_i^{\mathrm{e}}(z_*)|+|f_i^{\mathrm{e}}(z_*)-f_i^{\mathrm{e}}(z)|\leq|f_i(z_*)|+\epsilon_1+\epsilon_2, \tag{37}$$

and so for all $z\in\Omega\setminus\mathbb{D}$,

$$\begin{aligned}(\sum_{i=1}^n|f_i^{\mathrm{e}}(z)|^2)^{\frac12} &\leq (\sum_{i=1}^n|f_i(z_*)|^2)^{\frac12}+(\epsilon_1+\epsilon_2)\sqrt{n}\leq\sup_{z\in\mathbb{D}\cup S}(\sum_{i=1}^n|f_i(z)|^2)^{\frac12}+(\epsilon_1+\epsilon_2)\sqrt{n}\\ &\leq \frac12+(\epsilon_1+\epsilon_2)\sqrt{n}. \end{aligned}\tag{38}$$

From (36) and (38), it follows that

$$\sup_{z\in\Omega}\Big(\sum_{i=1}^{n}|f_i^{\mathrm{e}}(z)|^2\Big)^{\frac{1}{2}} \le \frac{1}{2}+(\epsilon_1+\epsilon_2)\sqrt{n} \le \frac{1}{2}+\Big(\frac{1}{4\sqrt{n}}+\frac{1}{4\sqrt{n}}\Big)\sqrt{n} \le 1. \tag{39}$$

By the Riemann mapping theorem, there exists a one-to-one holomorphic map φ from Ω onto $\mathbb{D}$. For each $i \in \{1,\dots,n\}$, the maps $f_i^{\mathrm{e}}\circ\varphi^{-1} \in H^\infty(\mathbb{D})$ satisfy

$$\forall z\in\mathbb{D},\quad \sum_{i=1}^{n}|(f_i^{\mathrm{e}}\circ\varphi^{-1})(z)| > \frac{\delta}{4} > 0 \quad \text{(using (34)), and}$$

$$\forall z\in\mathbb{D},\quad |(f_i^{\mathrm{e}}\circ\varphi^{-1})(z)| \le \sup_{z\in\Omega}\Big(\sum_{i=1}^{n}|f_i^{\mathrm{e}}(z)|^2\Big)^{\frac{1}{2}} \le 1 \quad \text{(using (39))}.$$

So by Carleson's corona theorem, it follows that there exist $g_1,\dots,g_n \in H^\infty(\mathbb{D})$ such that

$$\forall z\in\mathbb{D},\quad \sum_{i=1}^{n}(f_i^{\mathrm{e}}\circ\varphi^{-1})(z)g_i(z) = 1,$$

and for each $i\in\{1,\dots,n\}$, for all $z\in\mathbb{D}$, $|g_i(z)| \le C_\emptyset\left(n,\frac{\delta}{4}\right)$.

Let $f^{\mathrm{e}} := \begin{bmatrix} f_1^{\mathrm{e}} & \dots & f_n^{\mathrm{e}}\end{bmatrix}$. By Theorem 4.2, there exists $\Lambda\in\mathscr{G}(H^\infty(\mathbb{D})^{n\times n})$ such that

$$\Lambda = \begin{bmatrix} f^{\mathrm{e}}\circ\varphi^{-1} \\ F\end{bmatrix},$$

where $F\in H^\infty(\mathbb{D})^{(n-1)\times n}$, and if $g := \begin{bmatrix} g_1 & \dots & g_n\end{bmatrix}$, then

$$\|\Lambda^{-1}\|_\infty \le \|g\|_\infty(1+\|f^{\mathrm{e}}\circ\varphi^{-1}\|_\infty)+1 < \sqrt{n}C_\emptyset\left(n,\frac{\delta}{4}\right)(1+1)+1 = M(\delta,n).$$

For $z\in\mathbb{D}$, $\Lambda(z)\Lambda^{-1}(z)=I$, and so for $z\in\Omega$, $\begin{bmatrix} f^{\mathrm{e}}(z) \\ (F\circ\varphi)(z)\end{bmatrix}(\Lambda^{-1}\circ\varphi)(z) = I$. In particular,

$$\forall z\in\mathbb{D}\cup S,\quad \begin{bmatrix} f(z) \\ (F\circ\varphi)(z)\end{bmatrix}(\Lambda^{-1}\circ\varphi)(z) = I - \begin{bmatrix} f^{\mathrm{e}}(z)-f(z) \\ 0\end{bmatrix}(\Lambda^{-1}\circ\varphi)(z).$$

As

$$\left\|\begin{bmatrix} f^{\mathrm{e}}-f \\ F\circ\varphi\end{bmatrix}(\Lambda^{-1}\circ\varphi)\right\|_\infty \le \left\|\begin{bmatrix} f^{\mathrm{e}}-f \\ 0\end{bmatrix}\right\|_\infty \|\Lambda^{-1}\circ\varphi\|_\infty \le \epsilon_1 M(\delta,n) \le \frac{1}{2},$$

it follows that (see for example, Theorem 18.3 on page 357 of Rudin [26])

$$I - \begin{bmatrix} f^{\mathrm{e}}-f \\ 0\end{bmatrix}(\Lambda^{-1}\circ\varphi) \in \mathscr{G}(A_S(\mathbb{D})^{n\times n}).$$

We have

$$F\circ\varphi\in A_S(\mathbb{D})^{(n-1)\times n} \quad\text{and}\quad (\Lambda^{-1}\circ\varphi)\left(I-\begin{bmatrix} f-f^{\mathrm{e}} \\ 0\end{bmatrix}(\Lambda^{-1}\circ\varphi)\right)^{-1} \in \mathscr{G}(A_S(\mathbb{D})^{n\times n}),$$

and $\begin{bmatrix} f(z) \\ (F\circ\varphi)(z)\end{bmatrix}(\Lambda^{-1}\circ\varphi)(z)\left(I-\begin{bmatrix} f(z)-f^{\mathrm{e}}(z) \\ 0\end{bmatrix}(\Lambda^{-1}\circ\varphi)(z)\right)^{-1} = I$, $z\in\mathbb{D}\cup S$. $\square$

In Section 7, we use this Hermite property of $A_S(\mathbb{D})$ to show that if an unstable transfer function has either a left or a right coprime factorization, then it has both.

5. Stable rank

In this section we prove that just as with $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$, the stable rank of each $A_S(\mathbb{D})$ equals 1. In Section 7, we will apply this result to conclude that stabilizability is equivalent to strong stabilizability for transfer functions that are obtained as a ratio of elements from $A_S(\mathbb{D})$. This means that if a plant is stabilizable (which means that there exists a controller, possibly *unstable*, that stabilizes the closed loop interconnection), then in fact it can be stabilized by a *stable* controller.

We begin by recalling the definition of stable rank.

Definitions. Let $n \in \mathbb{N}$. Then the set of unimodular rows in $R^{1\times n}$ is denoted by $\mathscr{U}_n(R)$. A row $\begin{bmatrix} a_1 & \dots & a_{n+1} \end{bmatrix} \in \mathscr{U}_{n+1}(R)$ is said to be *stable* if there exists a row $\begin{bmatrix} b_1 & \dots & b_n \end{bmatrix} \in R^{1\times n}$ such that $\begin{bmatrix} a_1 + a_{n+1}b_1 & \dots & a_n + a_{n+1}b_n \end{bmatrix} \in \mathscr{U}_n(R)$.

If there exists an $n \in \mathbb{N}$ such that every vector of $\mathscr{U}_{n+1}(R)$ is stable, then the *stable rank* of R, is the smallest $n \in \mathbb{N}$ such that every vector of $\mathscr{U}_{n+1}(R)$ is stable.

If there does not exist an $n \in \mathbb{N}$ such that every vector of $\mathscr{U}_{n+1}(R)$ is stable, then the *stable rank* of R, is defined to be $+\infty$.

Sergei Treil showed that the stable rank of $H^\infty(\mathbb{D})$ is equal to 1 (Theorem 1 in [30]):

Theorem 5.1. *Let $f_1, f_2 \in H^\infty(\mathbb{D})$. If there exists a $\delta > 0$ such that $|f_1(z)| + |f_2(z)| \geq \delta$, $|f_1(z)| \leq 1$, and $|f_2(z)| \leq 1$, $z \in \mathbb{D}$, then there exists a $g \in H^\infty(\mathbb{D})$ such that $h := f_1 + f_2 g \in \mathscr{G}(H^\infty(\mathbb{D}))$, and moreover for all $z \in \mathbb{D}$, $|g(z)| \leq D_\emptyset(\delta)$ and $|h(z)^{-1}| \leq D_\emptyset(\delta)$, where $D_\emptyset(\delta)$ denotes a constant depending only on δ.*

Also, the stable rank of the disc algebra is equal to 1, and this was proved in Theorem 1 of Jones et al. [12]. We prove below that in fact the stable rank of each $A_S(\mathbb{D})$ is equal to 1.

Theorem 5.2. *Let $S \subset \mathbb{T}$. The stable rank of $A_S(\mathbb{D})$ is equal to* 1.

Proof. Let $\begin{bmatrix} f_1 & f_2 \end{bmatrix} \in A_S(\mathbb{D})^{1\times 2}$ be a unimodular row. Without loss of generality, we may assume that for all $z \in \mathbb{D} \cup S$, $|f_1(z)| \leq \frac{1}{2}$ and $|f_2(z)| \leq \frac{1}{2}$. Then there exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $|f_1(z)| + |f_2(z)| \geq \delta$. Let

$$\epsilon_1 = \min\left\{\frac{\delta}{4}, \frac{1}{4}, \frac{1}{2\left(1 + D_\emptyset\left(\frac{\delta}{4}\right)\right)^2}\right\} \quad \text{and} \quad \epsilon_2 = \min\left\{\frac{\delta}{4}, \frac{1}{4}\right\}.$$

Proceeding as in the proof of Theorem 4.3, using Lemma 3.2, we obtain the existence of an open connected set Ω containing $\mathbb{D} \cup S$ and holomorphic functions f_1^{e}, f_2^{e} defined on Ω that satisfy

$$\begin{aligned}
&\forall z \in \mathbb{D} \cup S, \quad |f_1(z) - f_1^{\mathrm{e}}(z)| < \epsilon_1 \text{ and } |f_2(z) - f_2^{\mathrm{e}}(z)| < \epsilon_1,\\
&\forall z \in \Omega \setminus \mathbb{D}, \quad \exists z_* \in S \text{ such that } |f_1^{\mathrm{e}}(z) - f_1^{\mathrm{e}}(z_*)| < \epsilon_2 \text{ and } |f_2^{\mathrm{e}}(z) - f_2^{\mathrm{e}}(z_*)| < \epsilon_2,\\
&\forall z \in \Omega, \quad |f_1^{\mathrm{e}}(z)| + |f_2^{\mathrm{e}}(z)| > \frac{\delta}{4} \quad \text{(see (34))}\\
&\forall z \in \Omega, \quad |f_1^{\mathrm{e}}(z)| \leq 1 \text{ and } |f_2^{\mathrm{e}}(z)| \leq 1, \quad \text{(see (35) and (37))}.
\end{aligned}$$

If $\varphi : \Omega \to \mathbb{D}$ denotes a one-to-one holomorphic map from Ω onto $\mathbb{D}$, then from Treil's theorem (Theorem 5.1), it follows that there exists a $g \in H^\infty(\mathbb{D})$ such that

$$h := f_1^{\mathrm{e}} \circ \varphi^{-1} + (f_2^{\mathrm{e}} \circ \varphi^{-1}) \cdot g \in \mathscr{G}(H^\infty(\mathbb{D})), \text{ with } |g(z)| \leq D_\emptyset\left(\frac{\delta}{4}\right) \text{ and } |h(z)^{-1}| \leq D_\emptyset\left(\frac{\delta}{4}\right).$$

So for all $z \in \mathbb{D}$, $|h(z)| \leq |(f_1^{\mathrm{e}} \circ \varphi^{-1})(z)| + |(f_2^{\mathrm{e}} \circ \varphi^{-1})(z)||g(z)| \leq 1 + D_\emptyset(\frac{\delta}{4})$. For all $z \in \Omega$, $(f_1^{\mathrm{e}}(z) + f_2^{\mathrm{e}}(z)(g \circ \varphi)(z))\frac{1}{(h\circ\varphi)(z)} = 1$. In particular, for all $z \in \mathbb{D} \cup S$,

$$\begin{aligned}&(f_1(z) + f_2(z)(g \circ \varphi)(z))\frac{1}{(h\circ\varphi)(z)}\\ &= 1 - ((f_1^{\mathrm{e}}(z) - f_1(z)) + (f_2^{\mathrm{e}}(z) - f_2(z))(g \circ \varphi)(z))\frac{1}{(h\circ\varphi)(z)} =: \Phi(z).\end{aligned}$$

Then for all $z \in \mathbb{D} \cup S$,

$$\begin{aligned}|\Phi(z)| &= \left|1 - ((f_1^{\mathrm{e}}(z) - f_1(z)) + (f_2^{\mathrm{e}}(z) - f_2(z))(g \circ \varphi)(z))\frac{1}{(h\circ\varphi)(z)}\right|\\ &\geq 1 - \epsilon_1\left(1 + D_\emptyset\left(\frac{\delta}{4}\right)\right)^2 \geq \frac{1}{2}.\end{aligned}$$

Hence $\Phi \in \mathscr{G}(A_S(\mathbb{D}))$, and so $f_1 + f_2 \cdot (g \circ \varphi) \in \mathscr{G}(A_S(\mathbb{D}))$. As $g \circ \varphi \in A_S(\mathbb{D})$, this completes the proof. $\square$

6. Topological stable rank

In this section we prove that just as with $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$, the topological stable rank of each $A_S(\mathbb{D})$ is equal to 2. In Section 7, we will apply this theorem to show that every unstabilizable plant is as close as we want to a stabilizable plant.

First we recall the notion of topological stable rank.

Definition. Let R be a Banach algebra. If there exists an $n \in \mathbb{N}$ such that $\mathscr{U}_n(R)$ is dense in $R^{1\times n}$ in the product topology, then the *topological stable rank* of R is the smallest $n \in \mathbb{N}$ such that $\mathscr{U}_n(R)$ is dense in $R^{1\times n}$.

If there does not exist an $n \in \mathbb{N}$ such that $\mathscr{U}_n(R)$ is dense in $R^{1\times n}$ in the product topology, then the *topological stable rank* of R is defined to be $+\infty$.

We recall the following two known results.

Theorem 6.1. *(Suárez [29]) The topological stable rank of $H^\infty(\mathbb{D})$ is equal to 2.*

Theorem 6.2. *The following hold:*

1. *The topological stable rank of $A(\mathbb{D})$ is equal to 2.*
2. $\overline{\mathscr{G}(A(\mathbb{D}))} = \{0\} \cup \{f \in A(\mathbb{D}) \mid \mathscr{Z}(f) \subset \mathbb{T}\}$*, where the notation $\mathscr{Z}(f)$ is used to denote the set of zeros of $f \in A(\mathbb{D})$: $\mathscr{Z}(f) = \{z \in \overline{\mathbb{D}} \mid f(z) = 0\}$.*

Proof. Item 1 was established in Rieffel [23]. The claim in item 2, giving the characterization of $\overline{\mathscr{G}(A(\mathbb{D}))}$, was shown in the example on page 154 following the proof of Proposition 1 in Robertson [24]. $\square$

Using the Theorems 6.1 and 6.2 above, we prove that the topological stable rank of $A_S(\mathbb{D})$ is equal to 2, for arbitrary $S \subset \mathbb{T}$.

Theorem 6.3. *Let $S \subset \mathbb{T}$. The topological stable rank of $A_S(\mathbb{D})$ is equal to 2.*

Proof. Let $\begin{bmatrix} f_1 & f_2 \end{bmatrix} \in A_S(\mathbb{D})^{1\times 2}$. Let $\epsilon > 0$. Using Lemma 3.2, we obtain the existence of an open connected set Ω containing $\mathbb{D} \cup S$ and holomorphic functions f_1^{e}, f_2^{e} defined on Ω that satisfy for all $z \in \mathbb{D} \cup S$, $|f_1(z) - f_1^{\mathrm{e}}(z)| < \frac{\epsilon}{2}$ and $|f_2(z) - f_2^{\mathrm{e}}(z)| < \frac{\epsilon}{2}$. Let $\varphi : \Omega \to \mathbb{D}$ denote a one-to-one holomorphic map from Ω onto $\mathbb{D}$. Then $\left[\begin{array}{cc} f_1^{\rm e}\circ\varphi^{-1} & f_2^{\rm e}\circ\varphi^{-1}\end{array}\right] \in H^\infty(\mathbb{D})^{1\times 2}$, and since the topological stable rank of $H^\infty(\mathbb{D})$ is equal to 2, it follows that there exist g_1, g_2 in $H^\infty(\mathbb{D})$ such that $\left[\begin{array}{cc} g_1 & g_2\end{array}\right] \in \mathscr{U}_2(H^\infty(\mathbb{D}))$, and for all $z \in \mathbb{D}$, $|(f_1^{\rm e}\circ\varphi^{-1})(z) - g_1(z)| < \frac{\epsilon}{2}$ and $|(f_2^{\rm e}\circ\varphi^{-1})(z) - g_2(z)| < \frac{\epsilon}{2}$. As $\left[\begin{array}{cc} g_1 & g_2\end{array}\right] \in \mathscr{U}_2(H^\infty(\mathbb{D}))$, it follows that there exists a $\delta > 0$ such that for all $z \in \mathbb{D}$, $|g_1(z)|+|g_2(z)| \geq \delta$. Hence for all $z \in \mathbb{D}\cup S$, $|(g_1\circ\varphi)(z)|+|(g_2\circ\varphi)(z)| \geq \delta > 0$, and by Theorem 3.3, it follows that $\left[\begin{array}{cc} g_1\circ\varphi & g_2\circ\varphi\end{array}\right] \in \mathscr{U}_2(A_S(\mathbb{D}))$. Moreover, for all $z \in \mathbb{D}\cup S$, $|f_1(z) - (g_1\circ\varphi)(z)| < \epsilon$ and $|f_2(z) - (g_2\circ\varphi)(z)| < \epsilon$. So it follows that the topological stable rank of $A_S(\mathbb{D})$ is at most equal to 2.

Next we show that the topological stable rank cannot be 1, that is, $\mathscr{G}(A_S(\mathbb{D}))$ is not dense in $A_S(\mathbb{D})$. In order to do this, we first mention that since the topological stable rank of $A(\mathbb{D})$ is equal to 2, $\mathscr{G}(A(\mathbb{D}))$ is not dense in $A(\mathbb{D})$. Indeed from item 2 in Theorem 6.2 above, it follows that if $f \in A(\mathbb{D})$ is not identically zero, and has a zero in $\mathbb{D}$, then f does not lie in the closure of $\mathscr{G}(A(\mathbb{D}))$. Consequently, the polynomial function p in $A(\mathbb{D})$, defined by $p(z) = z$, $z \in \overline{\mathbb{D}}$, does not belong to $\overline{\mathscr{G}(A(\mathbb{D}))}$. Clearly $p \in A_S(\mathbb{D})$. We now prove that $p \notin \overline{\mathscr{G}(A_S(\mathbb{D}))}$. Assume, on the contrary, that $p \in \overline{\mathscr{G}(A_S(\mathbb{D}))}$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{G}(A_S(\mathbb{D}))$ that converges to p uniformly on $\mathbb{D}\cup S$. Let $r \in (0,1)$, and define $q(z) = rz$, $z \in \overline{\mathbb{D}}$, and for each $n \in \mathbb{N}$, $g_n(z) = f_n(rz)$, $z \in \overline{\mathbb{D}}$. Then q and the g_n's all belong to A and the sequence $(g_n)_{n\in\mathbb{N}}$ converges to q in A. As the f_n's belong to $\mathscr{G}(A_S(\mathbb{D}))$, from item 2 of Corollary 3.5, it follows that for each $n \in \mathbb{N}$, there exists a $\delta_n > 0$ such that for all $z \in \mathbb{D}\cup S$, $|f_n(z)| \geq \delta_n$. Consequently for all $z \in \overline{\mathbb{D}}$, $|g_n(z)| \geq \delta_n > 0$, and by again from item 2 of Corollary 3.5 (now with $S = \mathbb{T}$!), it follows that $g_n \in \mathscr{G}(A)$. But q is not identically zero, and $q(0) = 0$. This contradicts the fact that any nonzero element of A having a zero in $\mathbb{D}$ does not belong to $\overline{\mathscr{G}(A)}$. This completes the proof. □

7. Coprime factorization and stabilization

Finally, in this section we proceed to give consequences for systems theory of the results established in the previous sections. The outline is as follows.

(1) Using the corona theorem for $A_S(\mathbb{D})$, we give an necessary and sufficient condition for a matrix pair to be right coprime.
(2) We consider unstable transfer functions which we write as a ratio of elements from $A_S(\mathbb{D})$. Not all such unstable transfer functions will have a coprime factorization. However, using the Hermite property of $A_S(\mathbb{D})$ we get the fact that a transfer function has a doubly coprime factorization iff it has a right (or a left) coprime factorization. Thus, using the result from Vidyasagar [32], we get a parameterization of all stabilizing controllers, analogous to the famous Youla parameterization.
(3) Using the fact that the stable rank of $A_S(\mathbb{D})$ is equal to 1, we prove that plants which are stabilizable are in fact strongly stabilizable, that is, the stabilizing controller can be chosen to be stable.
(4) Finally, we use the property that the topological stable rank of $A_S(\mathbb{D})$ is 2 to show that any transfer function is as close as we like to a transfer function that is stabilizable.

We begin by applying the result given in 2 of Corollary 3.5 in order to characterize matrix coprime pairs in $A_S(\mathbb{D})$.

Definitions. Let $S \subset \mathbb{T}$. Matrices with entries in $A_S(\mathbb{D})$ will be denoted by $\mathrm{Mat}(A_S(\mathbb{D}))$. If $N, D \in \mathrm{Mat}(A_S(\mathbb{D}))$, then the pair (N, D) is called *right coprime* (with respect to $A_S(\mathbb{D})$) if there exist $X, Y \in \mathrm{Mat}(A_S(\mathbb{D}))$ such that the matrix Bézout identity holds: $XN + YD = I$. A *left coprime* pair of matrices is defined analogously.

The following result gives a test for coprimeness of a matrix pair.

Theorem 7.1. *Let $S \subset \mathbb{T}$. Let $N \in A_S(\mathbb{D})^{m\times p}$ and $D \in A_S(\mathbb{D})^{p\times p}$. The pair (N, D) is right coprime iff there exits a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $N(z)^*N(z) + D(z)^*D(z) \geq \delta I$.*

Proof. This follows from Corollary 3.5 (see also Lemma 34 on page 340 of Vidyasagar [32]). □

We now consider unstable transfer functions that can be expressed as a quotient of two elements from $A_S(\mathbb{D})$. Having shown that $A_S(\mathbb{D})$ is an integral domain in Theorem **??**, we can consider its field of fractions. We recall this notion below.

Definitions. If R is an integral domain, then a *fraction* is a symbol $\frac{N}{D}$, where $N, D \in R$ and $D \neq 0$. Define the relation $\sim$ on the set of all fractions as follows: $\frac{N_1}{D_1} \sim \frac{N_2}{D_2}$ if $N_1 D_2 = N_2 D_1$. The relation $\sim$ is an equivalence relation on the set of all fractions. The equivalence class of $\frac{N}{D}$ is denoted by $[\frac{N}{D}]$. The *field of fractions*, denoted by $\mathbb{F}(R)$, is the set $\mathbb{F}(R) = \{[\frac{N}{D}] \mid N, D \in R \text{ and } D \neq 0\}$, of equivalence classes of the relation $\sim$, with addition and multiplication defined as follows: $[\frac{N_1}{D_1}] + [\frac{N_2}{D_2}] = [\frac{N_1 D_2 + N_2 D_1}{D_1 D_2}]$ and $[\frac{N_1}{D_1}][\frac{N_2}{D_2}] = [\frac{N_1 N_2}{D_1 D_2}]$. $\mathbb{F}(R)$ is then a field with these operations. Let $S \subset \mathbb{T}$.

Matrices with entries in $\mathbb{F}(A_S(\mathbb{D}))$ will be denoted by $\mathrm{Mat}(\mathbb{F}(A_S(\mathbb{D})))$. If $P \in \mathrm{Mat}(\mathbb{F}(A_S(\mathbb{D})))$, then P is said to have a *right coprime factorization* if there exists a pair (N, D) with $N, D \in \mathrm{Mat}(A_S(\mathbb{D}))$ such that D is a square matrix, $\det(D) \neq 0$, $P = ND^{-1}$, and (N, D) is right coprime. A *left coprime factorization* is defined analogously. A transfer function having a right coprime factorization and a left coprime factorization is said to have a *doubly coprime factorization.*

Using the result from Theorem 4.1 which says that $A_S(\mathbb{D})$ is not a Bézout domain, we obtain the following result, which says that not every element from $\mathbb{F}(A_S(\mathbb{D}))$ possesses a coprime factorization.

Corollary 7.2. *Let $S \subset \mathbb{T}$. There exist $P \in \mathbb{F}(A_S(\mathbb{D}))$ that do not have a coprime factorization.*

Proof. This is a consequence of Lemma 7 on page 332 of Vidyasagar [32] and Theorem 4.1. □

Thus, given an arbitrary $P \in \mathrm{Mat}(\mathbb{F}(A_S(\mathbb{D})))$, the existence of a right coprime factorization for P is not automatic. However, if P does have a right coprime factorization, then *all* right coprime factorizations of P can be characterized, and we give this characterization in the next result. A similar characterization can also be obtained for left coprime factorizations.

Theorem 7.3. *Let $S \subset \mathbb{T}$. If $P \in \mathrm{Mat}(\mathbb{F}(A_S(\mathbb{D})))$ has a right coprime factorization (N, D), then (N', D') is a right coprime factorization of P iff there exists a unimodular matrix U such that $N' = NU$ and $D' = DU$.*

Proof. This follows from Lemma 2 on page 331 of Vidyasagar [32]. □

Coprime factorization plays an important role in stabilizing a plant using a factorization approach, where by 'stabilization', we mean the following.

Definitions. Let $S \subset \mathbb{T}$. Let $P, C \in \text{Mat}(\mathbb{F}(A_S(\mathbb{D})))$. The pair (P, C) is said to be *stable* if

$$\mathscr{H}(P,C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+PC)^{-1} \\ C(I+PC)^{-1} & (I+PC)^{-1} \end{bmatrix} \tag{40}$$

is well defined, and belongs to $\text{Mat}(A_S(\mathbb{D}))$. We define

$$\mathscr{S}(P) = \{C \in \text{Mat}(\mathbb{F}(A_S(\mathbb{D}))) \mid (P, C) \text{ is a stable pair}\}.$$

$P \in \mathbb{F}(A_S(\mathbb{D}))^{p \times m}$ is said to be *stabilizable* if $\mathscr{S}(P) \neq \emptyset$.

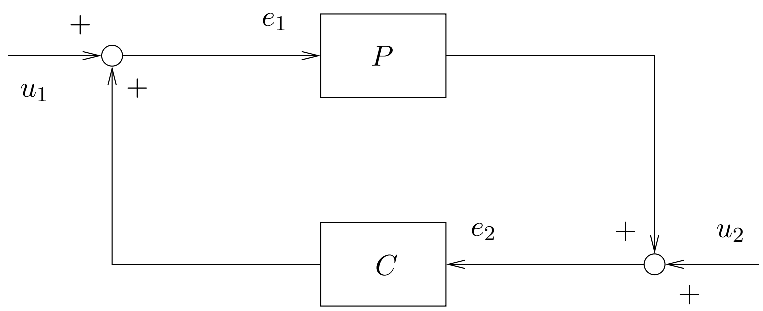

FIGURE 2. Closed loop interconnection of the plant P and the controller C.

As shown in Figure 2, $\mathscr{H}(P,C)$ in (40) is the transfer function of

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The stabilization problem for a plant is solved completely once a transfer function has a doubly coprime factorization.

Theorem 7.4. *Let $S \subset \mathbb{T}$. Let $P \in \text{Mat}(\mathbb{F}(A_S(\mathbb{D})))$ have a right coprime factorization (N_r, D_r) and a left coprime factorization (D_l, N_l). Let $X_r, Y_r, X_l, Y_l \in \text{Mat}(A_S(\mathbb{D}))$ be such that $X_r N_r + Y_r D_r = I$ and $N_l X_l + D_l Y_l = I$. Then*

$$\begin{aligned} \mathscr{S}(P) &= \{(Y_r - QN_l)^{-1}(X_r + QD_l) \mid Q \in \text{Mat}(A_S(\mathbb{D})) \text{ and } \det(Y_r - QN_l) \neq 0\} \\ &= \{(X_l + D_r Q)(Y_l - N_r Q)^{-1} \mid Q \in \text{Mat}(A_S(\mathbb{D})) \text{ and } \det(Y_l - N_r Q) \neq 0\}. \end{aligned}$$

Proof. This follows from Theorem 12 on page 364 of Vidyasagar [32]. □

We know that not every $P \in \text{Mat}(\mathbb{F}(A_S(\mathbb{D})))$ has a coprime factorization. Thus in light of Theorem 7.4, the natural question then arises: if P has a right (or a left) coprime factorization, then does it have a left (respectively right) coprime factorization? It turns out that $P \in \text{Mat}(\mathbb{F}(A_S(\mathbb{D})))$ has one iff it has the other, which we prove below in Corollary 7.5. This is a consequence of Theorem 4.3.

Corollary 7.5. *Let $S \subset \mathbb{T}$ and suppose that $P \in \text{Mat}(\mathbb{F}(A_S(\mathbb{D})))$. Then:*

(1) *If P has a right coprime factorization, then P has a left coprime factorization.*
(2) *If P has a left coprime factorization, then P has a right coprime factorization.*

Proof. This follows from Theorem 4.3 and Theorem 66 on page 347 of Vidyasagar [32]. □

Thus the above result says that if P possesses either a left or a right coprime factorization, then it possesses a doubly coprime factorization.

Next, using the fact that the stable rank of $A_S(\mathbb{D})$ is equal to 1, we show the equivalence of stabilizability and strong stabilizability.

Definition. Let $S \subset \mathbb{T}$. $P \in \mathbb{F}(A_S(\mathbb{D}))^{p\times m}$ is said to be *strongly stabilizable* if $\mathscr{S}(P) \cap A_S(\mathbb{D})^{m\times p} \neq \emptyset$.

We have the following result.

Theorem 7.6. *Let $S \subset \mathbb{T}$ and suppose that $P \in \operatorname{Mat}(\mathbb{F}(A_S(\mathbb{D})))$. The following are equivalent:*

(1) *P is stabilizable.*
(2) *P is strongly stabilizable.*

Proof. This follows for instance from Corollary 6.6 on page 2280 of Quadrat [20] and Theorem 5.2. □

Finally, using the fact that the topological stable rank of $A_S(\mathbb{D})$ is equal to 2, we show that every unstabilizable SISO plant defined by a transfer function $P \in \mathbb{F}(A_S(\mathbb{D}))$ is 'as close as we want it to be' to a stabilizable plant, in the following sense.

Theorem 7.7. *Let $S \subset \mathbb{T}$ and suppose that $P = \frac{N}{D} \in \mathbb{F}(A_S(\mathbb{D}))$, with $N, D \in A_S(\mathbb{D})$ and $D \neq 0$. Given any $\epsilon > 0$, there exist $N_\epsilon \in A_S(\mathbb{D})$ and $D_\epsilon \in A_S(\mathbb{D}) \setminus \{0\}$ such that $\|N - N_\epsilon\|_\infty < \epsilon$ and $\|D - D_\epsilon\|_\infty < \epsilon$, and moreover (N_ϵ, D_ϵ) are coprime.*

Proof. This follows from Proposition 7.4 on page 2281 of Quadrat [20] and Theorem 6.3. □

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