# Sequential Estimation of Dynamic Discrete Games: A Comment*

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Aguirregabiria and Mira (2007), henceforth AM (2007), study pseudo maximum likelihood estimators of dynamic games and propose an iterative nested pseudo maximum likelihood method.

This comment revisits the asymptotic properties of the sequential method. We illustrate that the method may not be consistent. We provide an example in which the sequential method converges to a fixed number distinct from the true parameter value with probability approaching one.

***Example.*** Consider a repeated game with $t = 1, 2, ..., \infty$. Every period $t$ two firms, indexed by $i = 1, 2$, simultaneously decide whether to be active or not. Firm $i$'s period payoff is equal to: $\varepsilon_i^1$ if firm $i$ is active and firm $3 - i$ is not active; $\theta + \varepsilon_i^1$ if both firms are active; and $\varepsilon_i^2$ if firm $i$ is not active. The true parameter $\theta_0$ is contained in the interior of a compact interval $\Theta$ with $\Theta = [-1, -10]$. The tuple of random variables $\left(\varepsilon_i^1, \varepsilon_i^2\right)$ are such that the difference $\varepsilon_i = \varepsilon_i^1 - \varepsilon_i^2$ is drawn independently every period from the distribution function $F_\alpha$ and observed privately by firm $i$ prior to making the choice with

$$F_\alpha\left(\varepsilon_i\right) = \begin{cases} 1 - \alpha + 2\alpha\left[\Phi\left(\frac{\varepsilon_i - 1 + \alpha}{\sigma}\right) - \frac{1}{2}\right] & [1 - \alpha, \infty); \\ \varepsilon_i & [\alpha, 1 - \alpha); \\ 2\alpha\Phi\left(\frac{\varepsilon_i - \alpha}{\sigma}\right) & [-\infty, \alpha), \end{cases} \quad (1)$$

where $\Phi$ denotes the standard normal cumulative distribution function, $\sigma = 2\alpha/\sqrt{2\pi}$ and $\alpha(\theta_0) > 0$ small.[1] There are no publicly observed state variables, and firms strategies are a function of the privately observed payoff shock only. Firms play a Markov equilibrium. The example satisfies assumptions (1)-(4) in AM (2007). By construction $F_\alpha$ approaches the uniform distribution in the limit when $\alpha$ vanishes. We assume that $\alpha(\theta_0)$ is chosen sufficiently small (as a function of the level of $\theta_0$) in order to allow us to focus on the uniform distribution part of $F_\alpha$. The assumption $\alpha > 0$ ensures that $\varepsilon_i$ is distributed on the real line.

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[1] The cdf $F_\alpha$ arises when the joint density of $\left(\varepsilon^1, \varepsilon^2\right)$ takes the form $f_\alpha\left(\varepsilon^1 - \varepsilon^2\right) \cdot \phi\left(\varepsilon^2\right)$ where $\phi(.)$ denotes the standard normal pdf and $f_\alpha(x)$ is a pdf that equals $\frac{2\alpha}{\sigma}\phi\left(\frac{x-\alpha}{\sigma}\right)$ for $x \in [-\infty, \alpha)$ ;1 for $x \in [\alpha, 1 - \alpha)$; and $\frac{2\alpha}{\sigma}\phi\left(\frac{x-1+\alpha}{\sigma}\right)$ for $x \in [1 - \alpha, \infty)$.

***Equilibrium.*** Let $P^i$ denote the probability that firm $i$ is active and $\mathbf{P} = (P^1, P^2)$. Firm $i$ is active if and only if $(\theta + \varepsilon_i^1) \cdot P^{3-i} + \varepsilon_i^1 \cdot \left(1 - P^{3-i}\right) > \varepsilon_i^2$, which yields $(\theta) \cdot P^{3-i} > \varepsilon_i^2 - \varepsilon_i^1$ and gives the following expression for firm i's probability of being active:

$$\begin{aligned} P^i &= \Psi(P^{3-i}, \theta) \\ &= 1 - F_\alpha\left(-\theta \cdot P^{3-i}\right) \end{aligned} \tag{2}$$

and we denote $\mathbf{\Psi}(\mathbf{P}, \theta) = (\Psi(P^2, \theta), \Psi(P^1, \theta))$. An equilibrium solves $\mathbf{P} = \mathbf{\Psi}(\mathbf{P}, \theta)$. The symmetric equilibrium for $\alpha$ small is given by $P^i = \frac{1}{1-\theta}$ for $i = 1, 2$.

The equilibrium is the unique symmetric equilibrium but it is not a stable equilibrium in the sense that the fixed point on the best response mapping is not asymptotically stable. The equilibrium is evolutionary stable in the sense of Smith (1982). The instability property in the best response mapping shall play a central role in establishing inconsistency of the NPL method but does not appear a reasonable equilibrium refinement concept for the incomplete information Markov game. The reason is that another firms' strategy is not observable and it is not clear how firms would learn from opponents' behavior to justify the best response mapping as a refinement concept. In order to 'learn' from opponents' play, a firm would have to calculate long-run averages to infer strategies. But any such long-run average calculation would violate the Markov assumption.

***NPL Method.*** Let $\widetilde{\mathbf{P}}_M = (\widetilde{P}_M^1, \widetilde{P}_M^2)$ denote the sample frequency estimator of the choice probabilities. The pseudo log-likelihood for any tuple $(P^1, P^2)$ is proportional to

$$\begin{aligned} Q_M(\theta, \mathbf{P}) \propto\ & \widetilde{P}_M^1 \ln\left(1 - F_\alpha\left(-\theta \cdot P^2\right)\right) + \left(1 - \widetilde{P}_M^1\right) \ln F_\alpha\left(-\theta \cdot P^2\right) \\ & + \widetilde{P}_M^2 \ln\left(1 - F_\alpha\left(-\theta \cdot P^1\right)\right) + \left(1 - \widetilde{P}_M^2\right) \ln F_\alpha\left(-\theta \cdot P^1\right) \end{aligned} \tag{3}$$

AM (2007) define the NPL method on page 18, equations (29) and (30), as a sequence of estimators $\left\{\widehat{\theta}_M^K\right\}$, where the K-stage solves

$$\widehat{\theta}_M^K = \arg\max_{\theta \in \Theta} Q_M\left(\theta, \widehat{\mathbf{P}}_{K-1}\right), \tag{4}$$

and the probabilities $\left\{\widehat{P}_K\right\}$ are obtained recursively as

$$\widehat{\mathbf{P}}_K = \mathbf{\Psi}\left(\widehat{\mathbf{P}}_{K-1}, \widehat{\theta}_M^K(\widehat{\mathbf{P}}_{K-1})\right). \tag{5}$$

We shall examine the limit of the sequential method. Notice that function (5) is distinct from the best response function (2) as $\theta$ is not fixed but a function of the choice probabilities.

AM (2007) introduce the NPL fixed point as a pair $\left(\widehat{\theta}, \widehat{\mathbf{P}}\right)$ that satisfies the following two conditions:

$$\widehat{\theta} = \arg\max_{\theta \in \Theta} Q_M\left(\theta, \widehat{\mathbf{P}}\right) \text{ and } \widehat{\mathbf{P}} = \mathbf{\Psi}\left(\widehat{\mathbf{P}}, \widehat{\theta}\right). \tag{6}$$

The NPL estimator is defined as the NPL fixed point with the highest value of the pseudo likelihood among all the NPL fixed points. AM (2007) establish the consistency of the NPL estimator. We shall illustrate that the sequential method may not converge to the correct fixed point. We shall illustrate that the sequential NPL method can lead to inconsistent estimates.

***NPL Limit.*** We examine the limit $\widehat{\theta}_M^{\infty}$ of the NPL sequence $\widehat{\theta}_M^K$. An estimator $\theta_M$ is consistent if $\theta_M \xrightarrow{P} \theta_0$ that is $\lim_{M \longrightarrow \infty} \Pr(|\theta_M - \theta_0| \geq \mu) = 0$. The proof of the following result is given in the Appendix.

**Limit Result.**

*(i)* $\widetilde{\mathbf{P}}_M \xrightarrow{P} \mathbf{P}_0$.

*(ii) Suppose* $\widetilde{\mathbf{P}}_M$ *is the starting value of the NPL choice probability sequence. Then* $\widehat{\theta}_M^{\infty} \xrightarrow{P} -1$ for any $\theta_0 \in (-1, -10)$.

The limit estimator $\widehat{\theta}_M^{\infty}$ converges with probability one to the number $-1$ for any value of the true parameter $\theta_0 \in (-1, -10)$ even when the choice probability sequence is initialized at the consistent frequency estimator. The example shows that properties of the estimator $\widehat{\theta}_M^K$ rely on the order in which the limes is taken. When $M$ is held fixed and the limes $K \longrightarrow \infty$ is considered, then the sequential method converges to a number distinct from the true value.

***Illustration.*** The following figure illustrates the NPL difference equation graphically. To simplify the illustration, we depict the NPL difference equation in terms of choice probability ratios, $\widehat{p}_K = \widehat{P}_K^2 / \widehat{P}_K^1$. The NPL sequence for $\widehat{p}_K$ is formally stated in equations (8) and (9) in the Appendix. The illustration assumes a true parameter value of $\theta_0 = -2$. The equilibrium choice probabilities are then $P^1 = P^2 = 1/3$ and $\widetilde{P}_M^1 \approx \widetilde{P}_M^2 \approx 1/3$ for large $M$.

[Figure 1 about here]

The NPL difference equation has three fixed points. The middle fixed point of $p = 1$ yields the true parameter value $\theta_0 = -2$. This fixed point is unstable as the slope of the difference equation is larger than one at $p = 1$. So, the NPL sequence attains the fixed point $p = 1$ only if it starts at the true value $p = 1$. For any starting value with $p \neq 1$ the NPL sequence moves away from that point.

There are two additional fixed points of the NPL sequence with approximate values for $(p, \theta)$ of $(3.73, -1)$, $(1/3.73, -1)$ respectively. These fixed points are stable and notice that both fixed points imply an approximate parameter value of $\theta \approx -1$. Any starting point $p > 1$ converges to the fixed point $(3.73, -1)$ with equilibrium choice probabilities of $P_\infty^1 \approx 0.21, P_\infty^2 \approx 0.79$. Any starting point $p < 1$ converges to the third point equalling $(1/3.73, -1)$ with equilibrium choice probabilities of $P_\infty^1 \approx 0.79, P_\infty^2 \approx 0.21$.

The NPL method always converges. A researcher reaches a stable fixed point with probability approaching one as $M$ increases. The stable fixed points have approximately the same likelihood values and the same parameter value estimates. Hence, the researcher may incorrectly conclude that the NPL estimate of $\theta$ is unique. Observe that the probability that the NPL method converges to the true parameter value approaches zero as $M$ increases as only starting values that lie on the 45 degree line, where $p = 1$, yield consistent estimates. With the frequency estimator the probability that $\widehat{p} = 1$ approaches zero as $M$ increases.

Note that the instability of the fixed point at $p = 1$ stems from $\theta_0$ being smaller than $-1$. For $\theta_0 \in (-1, 0)$ the NPL difference equation will only have one stable fixed point and the NPL method will converge to the true parameter with probability approaching one.

AM (2007) explain that in case of multiple fixed points the researcher may initiate the sequence at different starting values for the choice probabilities $\widehat{\mathbf{P}}_0$ and choose the sequence that maximizes the pseudo maximum likelihood in the limit. This suggestion works in the example only if the econometrician guesses correctly that the choice probability estimates lie on the 45 degree line. The 45 degree line emerges in this simple example because of the assumed symmetry. Guessing the relationship between choice probabilities correctly may be more difficult in richer settings. For instance introducing a slight asymmetry in payoffs in the current example would require the researcher to find the solution to a cubic equation.[2] Yet, and this is already observed in AM (2007), consistent estimates of $\theta$ emerge only if all the NPL fixed points are calculated and compared. Computationally the task of finding all fixed points is demanding. Importantly, this task is not achieved by the NPL method when the fixed point on the best response mapping is not asymptotically stable.

The inconsistency of the NPL method appears not to be an artefact of the chosen static example. A Monte Carlo study in Pesendorfer and Schmidt-Dengler (2008) illustrates that the same problem may emerge in richer settings. In a rich and realistic dynamic entry game the NPL method converged, but did not converge to the true value in three of five dynamic entry equilibria.

## APPENDIX

**Proof of Limit Result.** (i) This follows immediately as the sample frequency estimator is consistent. (ii) We begin by describing the expression for the NPL difference equation. In the description we initially impose the condition that along the NPL sequence

$$P^1, P^2 \in (\alpha, 1-\alpha). \tag{A}$$

Later-on, we establish that the condition (A) indeed holds at each point along the NPL sequence $\widehat{\mathbf{P}}_K$. Observe that condition (A) eventually holds at the starting values, that is for any $\mu > 0$ there exists an $\overline{M}$ such that, for all $M > \overline{M}$, $\Pr(\widetilde{P}_M^1, \widetilde{P}_M^2 \in (\alpha, 1-\alpha)) > 1 - \mu$. This follows immediately from part (i) as $P_0 \in (\alpha, 1-\alpha)$.

The necessary first order condition in problem (4) when $P^1, P^2$ satisfy property (A) yields $\partial Q_M / \partial \theta =$

$$\widetilde{P}_M^1 P^2 / \left(1 + \theta \cdot P^2\right) + (1 - \widetilde{P}_M^1)/\theta + \widetilde{P}_M^2 P^1 \left(1 + \theta \cdot P^1\right) + (1 - \widetilde{P}_M^2)/\theta = 0 \tag{7}$$

[2] Details of such an example are available from the authors upon request.

which gives rise to a quadratic equation in $\theta$.[3] Substituting the solution $\widehat{\theta}_M(\mathbf{P})$ into equation (5) yields the following difference equation characterizing the NPL method

$$\left(P_K^1, P_K^2\right) = (1 + h_M\left(P_{K-1}^2/P_{K-1}^1\right), 1 + h_M\left(P_{K-1}^2/P_{K-1}^1\right) \cdot \frac{P_{K-1}^1}{P_{K-1}^2}) \quad (8)$$

where $h_M(p) = -\frac{2-\widetilde{P}_M^1}{4} - \frac{2-\widetilde{P}_M^2}{4} \cdot p + \frac{1}{4}\sqrt{[2-\widetilde{P}_M^1 - (2-\widetilde{P}_M^2)\cdot p]^2 + 4\cdot \widetilde{P}_M^1 \widetilde{P}_M^2 \cdot p}$

We wish to study the limit of the NPL sequence (8). Notice that the right hand side in equation (8) is determined by the probability ratios $p_{K-1} = P_{K-1}^2/P_{K-1}^1$ and does not depend on the probability levels. Restating the sequence in terms of the probability ratios yields a one dimensional difference equation which is easier to analyze:

$$p_K = g_M(p_{K-1}) = \frac{1 + h_M(p_{K-1})/p_{K-1}}{1 + h_M(p_{K-1})}. \quad (9)$$

When $\widetilde{P}_M^1, \widetilde{P}_M^2 < 1/2$ the function $g_M$ in (9) has exactly three fixed points:

$$\begin{aligned} p &= 1, \\ p_M^* &= [2-\widetilde{P}_M^1 - \widetilde{P}_M^2 + \sqrt{(2-\widetilde{P}_M^1-\widetilde{P}_M^2)^2 - 4\widetilde{P}_M^1\widetilde{P}_M^2}]/(2\widetilde{P}_M^1), \\ \text{and } p_M^{**} &= [2-\widetilde{P}_M^1 - \widetilde{P}_M^2 - \sqrt{(2-\widetilde{P}_M^1-\widetilde{P}_M^2)^2 - 4\widetilde{P}_M^1\widetilde{P}_M^2}]/(2\widetilde{P}_M^1) \end{aligned}$$

with $p_M^{**} < 1 < p_M^*$. Part (i) and the assumption $\theta_0 < -1$ imply that for any $\mu > 0$ there exists an $\overline{M}$ such that, for all $M > \overline{M}$, $\Pr(\widetilde{P}_M^1, \widetilde{P}_M^2 < 1/2) > 1-\mu$, and the described fixed points arise with probability 1 as $M \to \infty$. The first fixed point implies equal choice probabilities of $\frac{\widetilde{P}_M^1+\widetilde{P}_M^2}{2}$ which yields $\theta$ close to $\theta_0$. The second and third fixed point yield choice probabilities of $(\frac{1}{1+p_M^*}, \frac{p_M^*}{1+p_M^*})$,$(\frac{1}{1+p_M^{**}}, \frac{p_M^{**}}{1+p_M^{**}})$ respectively with $\theta = -1$.

Which of the described fixed points is attained as the NPL limit is determined by the shape of the function $g_M$ and the starting values. Next, we observe four properties of $g_M$ which are then used to determine the limit of the NPL sequence. Then, we briefly sketch the proof of these properties.[4]

Properties of $g_M$:
(1) $g_M(p) > 1$ if and only if $p > 1$, and $g_M(p) = 1$ if and only if $p = 1$.
(2) $g_M$ has a non-negative derivative for $p \geq 1$.
(3) The derivative $\partial g_M(p)/\partial p$ evaluated at $p = 1$ equals $-1 + 2/(\widetilde{P}_M^1 + \widetilde{P}_M^2)$ and, from part (i), $\left.\frac{\partial g_M(p)}{\partial p}\right|_{p=1} \xrightarrow{P} -\theta_0$.

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[3] with solution $\widehat{\theta}_M(\mathbf{P}) = -\frac{2-\widetilde{P}_M^1}{4P^2} - \frac{2-\widetilde{P}_M^2}{4P^1} + \sqrt{\left[\frac{2-\widetilde{P}_M^1}{4P^2} - \frac{2-\widetilde{P}_M^2}{4P^1}\right]^2 + \frac{\widetilde{P}_M^1\widetilde{P}_M^2}{4P^1P^2}}$.

[4] A complete proof of the properties can be obtained from the authors.

(4) For any $\mu > 0$ there exists an $\overline{M}$ such that, for all $M > \overline{M}$, $\Pr(lim_{p\to\infty}$ $g_M(p) < \infty) \geq 1-\mu$.

Property (A) implies that $1 + h_M(p_{K-1})/p_{K-1}, 1 + h_M(p_{K-1}) \in (\alpha, 1-\alpha)$, and property (1) follows immediately from inspection of equation (9). Without loss of generality we may relabel firms identities and by property (1) we may restrict attention to the case $p \geq 1$ and to fixed points $p = 1$ and $p = p_M^*$. We will do so for the remainder of this proof. Properties (2) and (3) can be seen by taking the derivative. Property (4) can be established by using l'Hospital's rule as $lim_{p\to\infty} h_M(p) = lim_{p\to\infty}[(h_M(p)/p)/(1/p)] = lim_{p\to\infty} \frac{\partial(h_M(p)/p)}{\partial p} / \frac{\partial(1/p)}{\partial p}$.

The derivative property (3) combined with monotonicity property (2) imply that the fixed point $p = 1$ is unstable and fixed points $p_M^*$ (and $p_M^{**}$) are stable. To see that observe that for any $\mu > 0$ there exists an $\overline{M}$ such that, for all $M > \overline{M}$, with probability $1-\mu$ the monotone function $g_M$ intersects the 45 degree line at $p = 1$ from below as the slope is strictly larger than one at $p = 1$. In turn this implies that the function $g_M$ intersects the 45 degree line at fixed points $p_M^*$ (and $p_M^{**}$) from above. Now, as the function $g_M$ is monotone for $p \geq 1$ (and finite at $\infty$) it must hold that the slope of the function $g_M$ at fixed points $p_M^*$ (and $p_M^{**}$) is between zero and one (and strictly less then one from property 4) which establishes (local) stability.

We can now determine the limit of the NPL sequence. For any $\mu > 0$ there exists an $\overline{M}$ such that, for all $M > \overline{M}$, with probability $1-\mu$ equation (9) converges to fixed point $p_M^*$ whenever the starting value exceeds one (and it converges to fixed point two, $p_M^{**}$, whenever the starting value is less than one). To see that, notice that for starting values in the interval $(1, p_M^*)$ the difference equation (9) will increase towards fixed point $p_M^*$ as the function $g_M$ is monotone increasing and above the 45 degree line. On the other hand for starting values in the interval $(p_M^*, \infty)$, the difference equation (9) will decrease towards fixed point $p_M^*$ as the function $g_M$ is monotone increasing and below the 45 degree line.

Next, we establish property (A). We already know from part (i) that for any $\mu > 0$ there exists an $\overline{M}$ such that, for all $M > \overline{M}$, $\Pr(\widetilde{P}_M^1, \widetilde{P}_M^1 \in (P_0-\alpha, P_0+\alpha)) > 1-\mu$. We need to establish that the updated choice probabilities, based on the updating equation (8), are contained in $(\alpha, 1-\alpha)$ whenever $p \in [\alpha/(1-\alpha), (1-\alpha)/\alpha]$. Without loss of generality we relabel firms' identities so that $P^2 \geq P^1$, and examine the condition for $p \in [1, (1-\alpha)/\alpha]$. We need to show that $\alpha < P_K^1(p)$ and $P_K^2(p) < 1-\alpha$. The second inequality can be established by rewriting the equation $h_M(p)$ conveniently as $h_M(p) = -[(2-\widetilde{P}_M^1)/4 + (2-\widetilde{P}_M^2)\cdot p/4] + \sqrt{[(2-\widetilde{P}_M^1)/4 + (2-\widetilde{P}_M^2)\cdot p/4]^2 - (2-\widetilde{P}_M^1-\widetilde{P}_M^1)\cdot p/2}$. For any $\mu > 0$ there exists an $\overline{M}$ such that, for all $M > \overline{M}$, with probability $1-\mu$ the term in round brackets is strictly positive, and the term under the square root is strictly smaller than the square of the first term in square brackets. Thus, with probability $1-\mu$, the expression $h_M(p)$ is strictly less than zero on $[1, p_M^*]$. Since $P_K^2 = 1 + h_M(p)/p$ this implies that $P_K^2 < 1-\alpha$. An examination of the derivative of $P_K^1(p)$ reveals that it equals $\partial h_M(p)/\partial p$ which is non-positive. Thus, it suffices to establish that $\lim_{p\to\infty} P_K^1(p) > \alpha$ with probability $1-\mu$. Rewriting the inequality yields $\sqrt{[2-\widetilde{P}_M^1+(2-\widetilde{P}_M^2)\cdot p]^2-8(2-\widetilde{P}_M^1-\widetilde{P}_M^1)\cdot p} > -4(1-\alpha)+[2-\widetilde{P}_M^1+(2-\widetilde{P}_M^2)\cdot p]$. The expression under the root is positive (which can be immediately seen from the equivalent representation of the root in (8)). Squaring both the left hand side and right hand side, yields (after cancelling), $p\cdot[\widetilde{P}_M^1-\alpha(2-\widetilde{P}_M^2)] > (1-\alpha)[\widetilde{P}_M^1-2\alpha]$, which indeed holds with probability $1-\mu$ for $p$ sufficiently large.

So far we have shown that for starting values $P^2 \neq P^1$ the NPL sequence converges to the limit $\theta=-1$. To complete the argument, we need to establish that $\lim_{M\longrightarrow\infty} \Pr(\widetilde{P}_M^1=\widetilde{P}_M^2)=0$. Note that the most likely outcome of an $(M,P_0)$ binomial distribution is given by $\overline{k}=\lfloor (M+1)P_0\rfloor$, where $\lfloor x\rfloor$ is the smallest integer less or equal to $x$. Using this notation we find that an upper bound on the probability $\Pr(\widetilde{P}_M^1=\widetilde{P}_M^2)=\sum_{k=0}^{M}\binom{M}{k}(P_0)^k(1-P_0)^{M-k}\binom{M}{k}(P_0)^k(1-P_0)^{M-k}$ is given by $\left[\sum_{k=0}^{M}\binom{M}{k}(P_0)^k(1-P_0)^{M-k}\right]\left[\binom{M}{\overline{k}}(P_0)^{\overline{k}}(1-P_0)^{M-\overline{k}}\right]=\Pr(k=\overline{k})$. Robbins (1955) illustrates bounds on $M!$ and shows that $M!=\sqrt{2\pi M}\left(\frac{M}{e}\right)^M e^{r_M}$, where $\frac{1}{12M+1}<r_M<\frac{1}{12M}$. For $M>\max(\frac{1}{P_0},\frac{P_0}{1-P_0})$ we can use these bounds to obtain that $\Pr(k=\overline{k})$ is less than or equal to

$$
\begin{aligned}
&\sqrt{\frac{M}{2\pi\overline{k}\left(M-\overline{k}\right)}}\frac{(MP_0)^{\overline{k}}}{\left(\overline{k}\right)^{\overline{k}}}\frac{(M(1-P_0))^{M-\overline{k}}}{\left(M-\overline{k}\right)^{(M-\overline{k})}}e^{\frac{1}{12M}}\\
\leq\ &\sqrt{\frac{M}{2\pi(MP_0-1)\left(M(1-P_0)-P_0\right)}}\frac{(MP_0)^{\overline{k}}}{(MP_0-1)^{\overline{k}}}\frac{(M(1-P_0))^{M-\overline{k}}}{(M(1-P_0)-P_0)^{(M-\overline{k})}}e^{\frac{1}{12M}}\\
=\ &\sqrt{\frac{1}{2\pi(MP_0-1)\left((1-P_0)-\frac{P_0}{M}\right)}}\frac{1}{\left(1-\frac{1}{P_0M}\right)^{\overline{k}}}\frac{1}{\left(1-\frac{P_0}{M(1-P_0)}\right)^{(M-\overline{k})}}e^{\frac{1}{12M}}\\
\leq\ &\sqrt{\frac{1}{2\pi(MP_0-1)\left((1-P_0)-\frac{P_0}{M}\right)}}\frac{1}{\left(1-\frac{1}{P_0M}\right)^{(M+1)P_0}}\frac{1}{\left(1-\frac{P_0}{M(1-P_0)}\right)^{M}}e^{\frac{1}{12M}}
\end{aligned}
$$

The inequalities follow because $(MP_0-1)<\lfloor (M+1)P_0\rfloor<(M+1)P_0$. The first term in the last expression converges to zero, and the remaining three terms are bounded. It follows that $\lim_{M\to\infty}\Pr(\widetilde{P}_M^1=\widetilde{P}_M^2)=0$. This completes the proof. ■

# References


[1] Aguirregabiria, V., and Mira P. (2007): "Sequential Estimation of Dynamic Discrete Games," Econometrica, 1-53.

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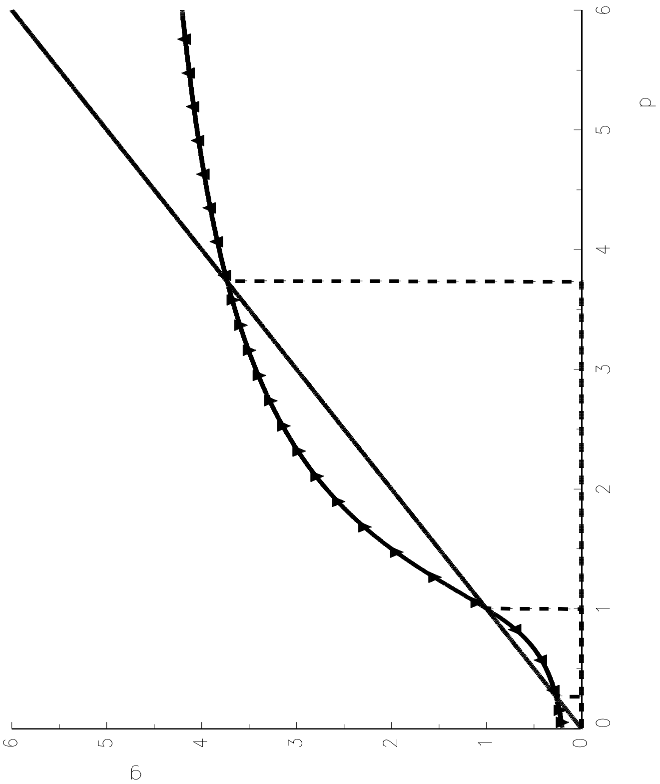


Figure 1: