Institutional Asset Pricing with Relative Performance*

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Abstract

We propose an equilibrium asset pricing model in which institutional investors with heterogeneous beliefs care about relative performance. We find that relative performance has two effects: First, it leads agents to trade more similarly which effectively decreases the difference of opinions. Second, it decreases the impact of dominant agent in the extreme economy which effectively increases the difference of opinions. When the first effect is dominant, which corresponds to a normal economy, the volatility is lower with relative performance than that without relative performance. When the economy is extreme (either good or bad), the second effect is dominant, hence the volatility is higher with relative performance. Moreover, relative performance also has impacts on portfolio choices, stock prices and market price of risks.

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1 Introduction

In the fund management industry, the compensation to the money managers is normally a fixed proportion of asset under management. Thus, managers care not only about the trading profit but also about the fund flows. More importantly, empirical evidence such as in Chevalier and Ellison (1997), Sirri and Tufano (1998), Huang, Wei and Yan (2007), shows the positive and convex relationship between the fund flows and *relative* performance. In this sense, fund managers care about the relative performance among the peer group. In the literature of delegated portfolio management, most people focus on how the relative performance affects risk taking behaviors and the equilibrium implications of asset prices (as discussed below). However, how relative performance could affect trading generated from difference of opinions remains uncertain. In this paper, we analyze this problem within a dynamic general equilibrium model with heterogeneous beliefs.

We consider a continuous time, finite horizon economy with two assets, interpreted as the risky stock and the risk-free bond, respectively. There are two risk averse agents, interpreted as fund managers, who optimally allocate their wealth among two assets to maximize their utility at the final date. Each manager has CRRA utility function over both the final wealth and the relative performance. The relative performance can be considered as the key factor that influence the capital inflows/outflows of a certain fund. Consistent with the empirical evidence, in our paper, we assume the fund flows are increasing and convex in relative performance. We adopt the standard exchange economy with Lucas (1978) type of aggregate dividends, which follow the geometric Brownian motion. The heterogeneous beliefs come from two agents' different opinions about the drift process of the dividend.

Comparing our model to the benchmark case (i.e. an identical model but without relative performance), we find that the portion of two agents on the final dividend is affected by their relative performance. By setting risk aversion parameter as an integer, we solve the equilibrium in closed form. Only those quantities generated by the difference of opinions are affected by relative performance such as the stock volatility, the market price of risk etc. For an arbitrary risk aversion coefficient, our model cannot be solved in analytical form. In order to get analytical solutions, we consider a special case in which the risk aversion coefficient equals to 2.

In the special case, we first analyze the portfolio choices. Specifically, relative performance leads agents to trade more similarly. Moreover, when the relative performance is infinitely strong, both agents submit the same demand. This is the same as the economy with one representative agent who has the average beliefs of the economy. For each agent's demand, relative performance only affects the demand generated by heterogeneous beliefs. Relative performance affects the way that two agents share the final dividend, hence affects their expectations of the final wealth. Note that the expectations are conditional on the current states of the world. When both agents believe the economy is very good; on expectation, pessimistic agent with relative performance holds more shares than she does without relative performance, while the optimistic one holds less than she does without relative performance. Thus, in this case, pessimistic agent has more impact compared to the benchmark case. Perceiving this, optimistic agent tends to hedge more on the heterogeneous beliefs relative to the benchmark case, and pessimistic one hedges less. Note that the signs for the hedging demands on heterogeneous beliefs are negative (positive) for the optimistic (pessimistic) agent. When both agents believe it is in a very bad economy, the opposite is true. In some cases, two agents may disagree with each other on the status of the economy, and both hedge more relative to the benchmark case.

Regarding the market price of risk, we show that when the economy is good, the optimistic investor possesses less wealth with relative performance than she does without relative performance. Therefore, although the optimistic agent still dominates the market, the stock is less overvalued with relative performance. Hence, the market price of risk is higher with relative performance than that without relative performance. When the economy is bad, by the similar logic, the market price of risk is lower than that without relative performance. Moreover, the model also indicates that the market prices of risks are counter-cyclical for both agents.

The stock price is also affected by the relative performance. When the economy is very good (bad), the stock price is lower (higher) with relative performance than that without relative performance. This is the aggregate result of the portfolio choices. When both agents believe the economy is good, relative performance leads the optimistic agent to hedge more on the heterogeneous beliefs and the pessimistic one to hedge less. Given the signs for the hedging demands (negative for the optimistic agent, and positive for the pessimistic agent), the aggregate demand is lower, hence the stock price is lower. When both agents believe that it is in a bad economy, the opposite is true. When two agents disagree with good or bad economy, the stock price could either be higher or lower than that without relative performance.

Relative performance also affects the stock volatility. When the economy is normal (i.e., not extremely good or bad), the volatility is smaller relative to the benchmark case; however, in the extreme economy, it is larger. This is because: on the one hand, relative performance leads agents to trade similarly, which decreases the difference of opinions, hence decreases the stock volatility; On the other hand, relative performance decreases the fraction of wealth held by the dominant agent in the extreme economy,[1] which effectively increases the difference of

[1] It decreases the fraction of wealth held by the optimistic (pessimistic) agent in the good (bad) economy.

opinions. As a result, the volatility is larger with relative performance than that without relative performance in the extreme economy.

Our paper also has an important implication about price impact and survivalship of irrational traders. Milton Friedman argued that irrational traders will not have impact on the long-run asset price since they will not survive. Kogan, Ross, Wang and Westerfield (2006) demonstrate that survival and price impact are two independent concepts. Although irrational traders will not survive in the long run,[2] they will have significant impact on asset prices. Our paper demonstrates that irrational traders will have higher survival probability when they have relative performance concern which will induce them to trade more similarly to rational traders. Meanwhile, irrational traders will have smaller impact on stock price volatility when the economy is not in the extreme case. However in the case of extreme case, only irrational traders can survive longer when they care about relative performance. Hence, their survival will increase the asset price volatility in the extreme case.

Our paper is closely related to the asset pricing literature with heterogeneous beliefs and delegated portfolio management. For asset pricing with heterogeneous beliefs, the general framework is by Basak (2001, 2005) in which two agents disagree with security's process. Other researchers consider the framework in which one agent has the correct belief, and the other has the incorrect one, for example, Kogan, Ross, Wang and Westerfield (2006), Yan (2008). Those papers examines the mis-pricing caused by the one with incorrect belief. Moreover, Scheinkman and Xiong (2003) combine the heterogeneous beliefs and short sales constraints, and show that this can create bubbles. Our paper combines the Basak's framework with the relative performance, and examines the equilibrium asset prices.

Delegated portfolio management literature is growing field of research. This is reasonable since a large fraction of financial asset are held by institutional investors (Allen, 2000). Therefore, it is important for us to consider how the behaviors of institutions affects asset prices. In the literature, most of people consider models with a single representative fund manager. For example, Vayanos (2004), Vayanos and Woolley (2008), and He and Krishnamurthy (2009, 2010) belong to this category. Because there is only one agent, relative performance does not matter.

For the investigation of relative performance, researchers either use the relative performance to some exogenous benchmark, or the relative performance among the peer group, which is the same as our paper. For example, Cuoco and Kaniel (2010), Shang (2008), Basak and Pavlova (2010), consider the relative performance to some passive benchmark, e.g. S&P 500. On

[2] Note that we assume irrational traders always have wrong beliefs.

the other hand, Kapur and Timmermann (2005), Basak and Makarov (2009, 2010), Kaniel and Kondor (2009) consider relative performance to the peer group of managers. All those papers, however, consider only how the relative performance may affect risk taking behaviors of investors. To our best knowledge, this paper is the first to investigate how the relative performance affects the trading behavior generated by difference of opinions.

Some papers study the asset pricing model with asymmetric information in which the agents either know or do not know. For example, Dasgupta and Prat (2006, 2008) show that career concerns can cause more uninformative trading and slow down the information revelation process, and Guerreri and Kondor (2010) show that career concerns can generate some "reputation premium" for the bond return, and hence increase the volatility of bond prices. In some sense, relative performance concerns are close to reputation. Our paper, different from those, considers the case that agents either agree or disagree with their observations (i.e. heterogeneous beliefs).

The rest of the paper is organized as follows. We first introduce the model setup in Section 2, and develop a benchmark case without relative performance in Section 3. Section 4 presents a general model with relative performance. Section 5 shows a special case when the risk aversion coefficient equals two, and analyzes the characteristics of volatility, portfolio choices, stock prices and market prices of risks. In Section 6, we numerically consider more special cases as a robustness check. As an extension of this paper, Section 7 discusses the survivalship of irrational traders when they care about relative performance. Section 8 concludes.

2 Model Setup

In this section, we first present the model setup of the economy with the heterogeneous beliefs and relative performances.

2.1 Economy

We consider a continuous time, finite horizon [0,T] economy with two assets which are risky and risk-free respectively. We interpret the risky asset as a stock which has the following dynamics

$$\frac{dS_t}{S_t} = \mu_{s,t}dt + \sigma_{s,t}dB_t \tag{1}$$

where $\sigma_{s,t} > 0$ and B_t is the standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. Note that the Brownian motion B_t is the *only* source of uncertainty in this economy. The drift $\mu_{s,t}$ and diffusion $\sigma_{s,t}$ are determined in equilibrium. The stock is in positive net supply, and pays the liquidating dividend D_T at time T. We assume D_t follows a geometric Brownian motion

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t \tag{2}$$

where μ_D and σ_D are positive constants. The risk free asset, interpreted as a bond, is in zero net supply, and has a constant return r. For simplicity, we assume $r = 0$.

There are two agents, interpreted as fund managers, in the market who optimally allocate their fund among risky and risk-fee assets. Each manager i invests a fraction, $\theta_{i,t}$, of her investment wealth $W_{i,t}$ on the stock. Hence, $W_{i,t}$ follows

$$dW_{i,t} = \theta_{i,t} W_{i,t} \left(\mu_{s,t} dt + \sigma_{s,t} dB_t\right) \tag{3}$$

We assume that managers have the same initial endowment which means each manager has $W_{i,0} = \frac{S_0}{2}$ initial wealth.

2.2 Relative Performance

Consistent to the standard compensation contracts in the mutual fund industry, fund managers maximize the terminal size of the asset under management which consists of both trading profit and money flows. Since empirical evidence shows the positive relationship between fund flows and the fund relative performance,[3] fund managers who maximize the fund size (trading profit and fund flows) care about their relative performance. For this reason, we assume the objective function of manager i has the general form:

$$v_i(W_{i,T}, R_{i,T}) \tag{4}$$

where v_i is an increasing function in $W_{i,T}$ and $R_{i,T}$. $W_{i,T}$ is the manager i's wealth at time T, and $R_{i,T}$ is the relative return of manager i. By denoting the other manager as manager j, fund manager i's relative performance, $R_{i,T}$, is defined as:

$$R_{i,T} = \frac{W_{i,T}/W_{i,0}}{W_{j,T}/W_{j,0}} \tag{5}$$

where $i = 1$, $j = 2$ or vice versa. From our previous assumption, $W_{1,0} = W_{2,0} = \frac{S_0}{2}$. Then $R_{i,T} = \frac{W_{i,T}}{W_{j,T}}$ depends only on the ratio of their fund size at time T.

The rational justification for the relative performance in the objective function $v_i(W_{i,T}, R_{i,T})$ is similar to Basak and Makarov (2009) who assume manager's investment horizon is until $T^{'}$ ($T^{'} > T$). To be specific, manager i has CRRA preferences over $W_{i,T'}$.

$$v_{i,T'}\left(W_{i,T'}\right) = \frac{\left(W_{i,T'}\right)^{1-\gamma}}{1-\gamma} \tag{6}$$

[3] For example, Chevalier and Ellison (1997).

From the managers' optimization problem from T to $T^{'}$, we can find the indirect utility function at time T[4] which is the objective function for each manager at time T. Define $f_{i,T} := (R_{i,T})^k$, the time-T indirect utility function is:

$$v_{i,T} = \frac{(W_{i,T} f_{i,T})^{1-\gamma}}{1-\gamma} \tag{7}$$

where $k > 1$.[5] $f_{i,T}$ is the flow-performance function, which represents the rate of money flows for manager i due to her relative performance. Moreover, if $f_{i,T} > 1$, the fund experiences an inflow, and if $f_{i,T} < 1$, an outflow. We can see that flow-performance relationship $f_{i,T}$ is increasing and convex ($k > 1$) in the manager i's relative performance over the period $[0, T]$, which is consistent with the empirical results mentioned in the introduction section.

2.3 Heterogeneous Beliefs

Manager i has the probability space $\left(\Omega, \mathcal{F}^i, \left\{\mathcal{F}^i_t\right\}, \mathcal{P}^i\right)$. Following the standard filtering theorem, the dividend process under fund manager i's belief follows

$$\frac{dD_t}{D_t} = \mu_{i,D} dt + \sigma_D dB_{i,t} \tag{8}$$

By Girsanov's theorem, $dB_{i,t} = dB_t + \eta_i dt$ is the Brownian Motion in manager i's probability space , and $\eta_i = \frac{\mu_D - \mu_{i,D}}{\sigma_D}$. For two agents, 1 and 2, equation (8) implies

$$dB_{2,t} = dB_{1,t} + \overline{\mu} dt \tag{9}$$

where

$$\overline{\mu} = \frac{\mu_{1,D} - \mu_{2,D}}{\sigma_D} \tag{10}$$

(10) represents investors' disagreement on the drift of the dividend process, normalized by its diffusion term. $\overline{\mu} > 0$ implies that manager 1 is more optimistic and vice versa. Given the priors of agents, $\overline{\mu}$ is an exogenous parameter.[6] Under the subjective measures of manager 1 and 2, the stock has the dynamics

$$\begin{aligned} dS_t &= S_t[\mu_{s,t} dt + \sigma_{s,t} dB_t] \\ &= S_t[\mu_{i,t} dt + \sigma_{s,t} dB_{i,t}], \text{ for } i = 1, 2 \end{aligned} \tag{11}$$

For two agents, they must agree with the price so we have the relationship between the perceived means:

$$\mu_{1,t} - \mu_{2,t} = \sigma_{s,t} \overline{\mu} \tag{12}$$

[4]For more details, please see the proof of lemma 1 of Basak and Makarov (2009).

[5]The objective function is consistent with the catch-up with Jones utility function.

[6]Details can be found from Basak (2004).

Because the market is complete, there exists a unique state price density process, π_i, for each investor i:

$$\frac{d\pi_{i,t}}{\pi_{i,t}} = -\kappa_{i,t} dB_{i,t} \tag{13}$$

where

$$\kappa_{i,t} = \frac{\mu_{i,t}}{\sigma_{s,t}} \tag{14}$$

is the *perceived* market price of risk (Sharpe ratio) for manager 1 and 2 respectively. We also have $\kappa_1 - \kappa_2 = \overline{\mu}$ which is the measure of the disagreement between agents' perceived market prices of risk.

3 Benchmark Case: No Relative Performance ($k = 0$)

In the section, we analyze a benchmark case model as if there is no relative performance, that is, $k = 0$. When $k = 0$, the indirect utility function, (7), becomes standard CRRA utility function, so that the problem for manager i becomes

$$\max E^i \left[\frac{{W_{i,T}}^{1-\gamma}}{1-\gamma} \right]$$

$$\text{s.t.} \quad dW_{i,t} = \theta_{i,t} W_{i,t} \left(\mu_{i,t} dt + \sigma_{s,t} dB_{i,t} \right)$$

This problem becomes the standard model with heterogeneous beliefs (e.g. Basak 2005).[7] Solving the above problem, we show the optimal consumptions and state prices at time T in the following lemma.

Lemma 1. *When $k = 0$, the final wealth for two agents are:*

$$W^0_{1,T} = \frac{D_T}{1 + \lambda(T)^{\frac{1}{\gamma}}} \; ; \qquad W^0_{2,T} = \frac{\lambda(T)^{\frac{1}{\gamma}} D_T}{1 + \lambda(T)^{\frac{1}{\gamma}}} \tag{15}$$

The state prices at time T are

$$\pi^0_{1,T} = \frac{(1 + \lambda(T)^{\frac{1}{\gamma}})^{\gamma}}{y_1 D_T^{\gamma}}; \quad \pi^0_{2,T} = \frac{(1 + \lambda(T)^{\frac{1}{\gamma}})^{\gamma}}{y_2 D_T^{\gamma} \lambda(T)} \tag{16}$$

The process $\lambda(t)$ is

$$\lambda(t) = \frac{y_1 \pi_{1,t}}{y_2 \pi_{2,t}} \tag{17}$$

where y_i is the Lagrange multiplier for manager i's optimization problem, and $\pi_{i,t}$ is the perceived state price density for manager i, and $i = 1, 2$.

Proof. The proof is in the Appendix A. □

[7] However, in the model, agents only consume at time T which is different to Basak (2005) in which agents consume continuously.

The superscript 0 means no relative performance ($k = 0$). (15) shows that two agents share the final dividend D_T, and the sharing rule depends on $\lambda(T)^{\frac{1}{\gamma}}$. (16) gives the state prices at time T. (17) shows the dynamics of $\lambda(t)$ which is the stochastic weight for the central planner's problem (Basak (2005))[8]. By Ito's Lemma, we can get the dynamics of $\lambda(t)$:

$$\frac{d\lambda(t)}{\lambda(t)} = -\overline{\mu} dB_{1,t} \tag{18}$$

$$d\frac{1}{\lambda(t)} = \frac{1}{\lambda(t)}\overline{\mu} dB_{2,t} \tag{19}$$

Given that the priors of two agents, $\overline{\mu}$ is exogenous, (18) and (19) indicate that $\lambda(t)$ is an exogenous process. Note that there is only one uncertainty B_t in the economy, from (18) one can see that $\lambda(t)$ has a one-to-one relationship with B_t and hence $\lambda(t)$ can represent the status of the economy. Particlularly, $\lambda(t)$ is opposite to the status of economy, for example, when the economy is good, B_t has a large value (i.e. the stock price is high), while $\lambda(t)$ has a rather small value.

Calibrating the equilibrium needs the explicit expression of state price density $\pi_{i,t}$, which can be calculated as $\pi_{i,t} = E_t^i(\pi_{i,T})$ by its martingale property. However, the difficulty for calculating the expectation is the term $(1 + \lambda(T)^{\frac{1}{\gamma}})^{\gamma}$. This can be solved as

$$\sum_{i=0}^{\gamma} \left(\begin{array}{c} \gamma \\ i \end{array} \right) \lambda(T)^{\frac{i}{\gamma}} \tag{20}$$

when γ is an integer. Thus, we assume γ is an integer, and solve the equilibrium in the following proposition.

Proposition 1. *When γ is an integer, the state prices are*

$$\begin{aligned} \pi_{1,t}^0 &= \frac{1}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \left(\begin{array}{c} \gamma \\ i \end{array} \right) \lambda(t)^{\frac{i}{\gamma}} e^{\left[-\frac{\overline{\mu}^2}{2}\frac{i}{\gamma} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) + \frac{1}{2}\left[\frac{i}{\gamma}\overline{\mu} + \gamma\sigma_D\right]^2\right](T-t)} \\ \pi_{2,t}^0 &= \frac{1}{y_2 D_t^{\gamma}} \sum_{i=0}^{\gamma} \left(\begin{array}{c} \gamma \\ i \end{array} \right) \lambda(t)^{\frac{i-\gamma}{\gamma}} e^{\left[\frac{\overline{\mu}^2}{2}\frac{i-\gamma}{\gamma} - \gamma\left(\mu_2 - \frac{\sigma_D^2}{2}\right) + \frac{1}{2}\left[\frac{i-\gamma}{\gamma}\overline{\mu} + \gamma\sigma_D\right]^2\right](T-t)} \end{aligned}$$

market prices of risk are

$$\begin{aligned} \kappa_{1,t}^0 &= \gamma\sigma_D + \delta_{1,t}^0 \overline{\mu} \\ \kappa_{2,t}^0 &= \gamma\sigma_D - \delta_{2,t}^0 \overline{\mu} \end{aligned}$$

where $\delta_{1,t}^0$ and $\delta_{2,t}^0$ are two constants.

[8] The central planner's problem is $\max_{c_1+c_2=c} u_1(c_1) + \lambda(t) u_2(c_2)$.

Portfolio Choices are:

$$\theta_{1,t}^{0} = \frac{\mu_{1,t}}{\sigma_{s,t}^{2}} - \frac{(\gamma-1)\,\sigma_D}{\sigma_{s,t}} - \frac{\beta_{1,t}^{0}}{\sigma_{s,t}}\overline{\mu}$$
$$\theta_{2,t}^{0} = \frac{\mu_{2,t}}{\sigma_{s,t}^{2}} - \frac{(\gamma-1)\,\sigma_D}{\sigma_{s,t}} + \frac{\beta_{2,t}^{0}}{\sigma_{s,t}}\overline{\mu}$$

where $\beta_{1,t}^{0}$ and $\beta_{2,t}^{0}$ are two constants.

This proposition gives us the benchmark case without concerns about relative performance, and all of our results will be compared to this benchmark. Given the state prices, we can easily calculate the stock price, S_t^0, and the volatility, $\sigma_{s,t}^0$, which can be found in the Appendix B (when $k = 0$).

4 The Model with Relative performance

In this section, we solve the model with relative performance ($k > 1$), and compare the equilibrium to the benchmark case.

4.1 Optimization and Equilibrium

Given the indirect utility function, (7) , for each agent i, the optimization problem is:

$$\max E^i\left[\frac{(W_{i,T}f_{i,T})^{1-\gamma}}{1-\gamma}\right]$$
$$\text{s.t.}\quad dW_{i,t} = \theta_{i,t}W_{i,t}\left(\mu_{i,t}dt + \sigma_{s,t}dB_{i,t}\right)$$

By the standard martingale approach (Cox and Huang 1989), manager i's optimization problem is static

$$\max E^i\left[\frac{(W_{i,T}f_{i,T})^{1-\gamma}}{1-\gamma}\right]$$
$$\text{s.t. } E^i\left[\pi_{i,T}W_{i,T}\right] = \frac{S_0}{2} \tag{21}$$

Solving (21), we have the following Lemma.

Lemma 2. *There is an unique equilibrium, where*

$$\widehat{W_{1,T}} = (k+1)^{\frac{1}{\gamma}}(y_1\pi_{1,T})^{-\frac{1}{\gamma}}(W_{2,T})^{\frac{\theta(\overline{\gamma}-1)}{\overline{\gamma}}}$$
$$\widehat{W_{2,T}} = (k+1)^{\frac{1}{\gamma}}(y_2\pi_{2,T})^{-\frac{1}{\gamma}}(W_{1,T})^{\frac{\theta(\overline{\gamma}-1)}{\overline{\gamma}}}$$

where $\widehat{W_{i,T}}$ is the best response of i^{th} investor given $W_{j,T}$ ($i \neq j$), $\overline{\gamma} = \gamma + k(\gamma-1)$ and $\theta = \frac{k}{k+1}$.

Proof. The proof is in Appendix A. □

Lemma 2 shows the optimal responses of both managers. Note that each manager's final wealth is a function of the final wealth[9] of the other manager. By the market clearing condition, $W_{1,T} + W_{2,T} = D_T$, we can solve the final wealth of each agent in Lemma 3.

Lemma 3. *At time T, two agents share the final dividend* D_T

$$W_{1,T} = \frac{D_T}{1 + \lambda(T)^{\frac{1}{\widehat{\gamma}}}} \; ; \qquad W_{2,T} = \frac{\lambda(T)^{\frac{1}{\widehat{\gamma}}} D_T}{1 + \lambda(T)^{\frac{1}{\widehat{\gamma}}}} \tag{22}$$

where $\widehat{\gamma} = \gamma + 2k(\gamma - 1)$.

Proof. The proof is in the Appendix A. □

Comparing to the results in Lemma 1, two managers still share the final dividend D_T. However, the sharing rule now depends on $\lambda(T)^{\frac{1}{\widehat{\gamma}}}$ instead of $\lambda(T)^{\frac{1}{\gamma}}$. $\widehat{\gamma}$ is a function of k so that the relative performance affects the fractions of final dividend shared by two agents. By choosing different values of γ, we have the following lemma.

Lemma 4. *When* $\gamma = 1$, $\widehat{\gamma} = \gamma$, *relative performance has no effect; for* $\gamma > 1$, $\widehat{\gamma} > \gamma$, *we then have:*

When $\lambda(T)$ *is small enough,* $W_{1,T} < W^0_{1,T}$ *and* $W_{2,T} > W^0_{2,T}$;

When $\lambda(T)$ *is large enough,* $W_{1,T} > W^0_{1,T}$ *and* $W_{2,T} < W^0_{2,T}$.

The case of $\gamma = 1$ refers to the log utility, and relative performance does not matter in this case. When $\gamma > 1$, we have two scenarios conditional on the realizations of $\lambda(T)$. As mentioned before, a small $\lambda(T)$ corresponds to a good economy, and a large $\lambda(T)$ corresponds to a bad economy. The results show that in a very good economy, the wealth of the optimistic agent is lower than that in the benchmark case, and in the very bad economy, the opposite is true. Note that the optimistic agent is dominant in the very good economy, and the pessimistic one is dominant in the very bad economy. We then can draw the conclusion that relative performance decreases the impact of the dominant agent in the extreme economy.

Lemma 5. *The state price densities at time T are*

$$\pi_{1,T} = \frac{(k+1)}{y_1} \frac{(1 + \lambda(T)^{\frac{1}{\widehat{\gamma}}})^{\gamma}}{D_T^{\gamma}} \lambda(T)^{\frac{\theta(\overline{\gamma}-1)}{\widehat{\gamma}}} \tag{23}$$

$$\pi_{2,T} = \frac{(k+1)}{y_2} \frac{(1 + \lambda(T)^{\frac{1}{\widehat{\gamma}}})^{\gamma}}{D_T^{\gamma}} \lambda(T)^{-\frac{\theta(\overline{\gamma}-1)+\gamma}{\widehat{\gamma}}} \tag{24}$$

Proof. The proof is in the Appendix A. □

[9] Wealth is a function of portfolio choices which is defined by definition 1.

Similar to the benchmark case in the last section, the term $(1+\lambda(T)^{\frac{1}{\gamma}})^{\gamma}$ can be expressed as $\sum_{i=0}^{\gamma}\left(\begin{array}{c}\gamma\\ i\end{array}\right)\lambda(T)^{\frac{i}{\gamma}}$ when γ is an integer. We then give the equilibrium in the following proposition.

Proposition 2. *When γ is an integer, the state prices are:*

$$\begin{aligned}
\pi_{1,t} &= \frac{k+1}{y_1 D_t^{\gamma}}\sum_{i=0}^{\gamma}\left(\begin{array}{c}\gamma\\ i\end{array}\right)\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}-\gamma\left(\mu_1-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\right](T-t)}\\
\pi_{2,t} &= \frac{k+1}{y_2 D_t^{\gamma}}\sum_{i=0}^{\gamma}\left(\begin{array}{c}\gamma\\ i\end{array}\right)\lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}}e^{\left[\frac{\overline{\mu}^2}{2}\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}-\gamma\left(\mu_2-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\right](T-t)}
\end{aligned}$$

market prices of risk are

$$\begin{aligned}
\kappa_{1,t} &= \gamma\sigma_D+\delta_{1,t}\overline{\mu}\\
\kappa_{2,t} &= \gamma\sigma_D-\delta_{2,t}\overline{\mu}.
\end{aligned}$$

The portfolio choices are:

$$\begin{aligned}
\theta_{1,t} &= \frac{\mu_{1,t}}{\sigma_{s,t}^2}-\frac{(\gamma-1)\,\sigma_D}{\sigma_{s,t}}-\frac{\beta_{1,t}}{\sigma_{s,t}}\overline{\mu}\\
\theta_{2,t} &= \frac{\mu_{2,t}}{\sigma_{s,t}^2}-\frac{(\gamma-1)\,\sigma_D}{\sigma_{s,t}}+\frac{\beta_{2,t}}{\sigma_{s,t}}\overline{\mu}.
\end{aligned}$$

The stock price is

$$S_t=\frac{\sum_{i=0}^{\gamma}\left(\begin{array}{c}\gamma\\ i\end{array}\right)\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}+\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right]^2\right](T-t)}D_t e^{\left(\mu_1-\sigma_D^2\right)(T-t)}}{\sum_{i=0}^{\gamma}\left(\begin{array}{c}\gamma\\ i\end{array}\right)\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left\{-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\right\}(T-t)}}.$$

and the volatility is

$$\sigma_{s,t}=\sigma_D+K\overline{\mu}$$

Note that $\delta_{1,t}$, $\delta_{2,t}$, $\beta_{1,t}$, $\beta_{2,t}$ and K are shown in the Appendix B.

Proof. The proof is in the Appendix B. □

We can see that, comparing to the results in proposition 1, all the equilibrium quantities are affected by the relative performance k. Relative performance affects $\beta_{1,t}$, $\beta_{2,t}$, $\delta_{1,t}$, $\delta_{2,t}$ and K in portfolio choices, Sharpe ratio and the volatility. These parameters are all at play through the disagreement parameter, $\overline{\mu}$. Thus, the relative performance affects those quantities generated by the difference of opinions.

In order to analyze the effects of the relative performance, we consider a special case with $\gamma = 2$ as an example, where the equilibrium can be analyzed in more details.[10] For a robustness check, in Section 6, we also analyze those cases when $\gamma = 3$, 4.

5 Special Case: $\gamma = 2$

In this section, we solve the equilibrium by choosing $\gamma = 2$. The purpose of this section is to compare the equilibrium with relative performance to that without relative performance. Since we can solve everything in closed form, the comparative statics are also analyzed in this section.

5.1 Portfolio Choices

The following proposition shows the portfolio choices for managers.

Proposition 3. *When $\gamma = 2$, the portfolio choices are*

$$\theta_{1,t} = \underbrace{\frac{\mu_{1,t}}{\sigma_{s,t}^2}}_{Myopic\ Demand} - \underbrace{\frac{\sigma_D}{\sigma_{s,t}}}_{Hedging\ Demand} - \underbrace{\frac{\beta_{1,t}}{\sigma_{s,t}}\overline{\mu}}_{Heterogeneity\ Demand} \tag{25}$$

$$\theta_{2,t} = \underbrace{\frac{\mu_{2,t}}{\sigma_{s,t}^2}}_{Myopic\ Demand} - \underbrace{\frac{\sigma_D}{\sigma_{s,t}}}_{Hedging\ Demand} + \underbrace{\frac{\beta_{2,t}}{\sigma_{s,t}}\overline{\mu}}_{Heterogeneity\ Demand} \tag{26}$$

$\beta_{1,t}$ and $\beta_{2,t}$ are functions of k, which can be found in Appendix C.

Proof. The proof is in the Appendix C. □

The optimal demands consist of three terms. The first two terms are the myopic demand and the hedging demand for the changing of investment opportunity sets, respectively.[11] The third one is the demand generated by the difference of opinions, and hence is called heterogeneity demand in this paper. From (22), we can see that two agents share the final dividend D_T, and the fraction depends on $\lambda(T)^{\frac{1}{\gamma}}$. Given the realization of different states, agents have state dependent shares of wealth. For example, optimistic agent have larger fraction of wealth than pessimistic one when the economy is good. For this reason, the additional uncertainty originated from different opinions generates heterogeneity demand. Moreover, $\beta_{i,t}$ is a function of k, which means the relative performance only affects heterogeneity demand rather than myopic and hedging demand.

[10] The approach of using an integer for the risk aversion coefficient is the same as Yan (2008) who uses numerical simulation to analyze the equilibrium. Rather than doing the numerical study, we choose a special case with $\gamma = 2$.

[11] This is the same as Merton (1971).

5.1.1 Comparison to Benchmark Case

To analyze the effect of relative performance we need to compare $\beta_{i,t}$ to the benchmark case.

Proposition 4. *The following relationships hold:*

$$\theta_{1,t} - \theta_{2,t} = \overline{\mu}\left[1 - (\beta_{1,t} + \beta_{2,t})\right] \tag{27}$$

$$\beta_{1,t} + \beta_{2,t} - (\beta^0_{1,t} + \beta^0_{2,t}) > 0 \tag{28}$$

moreover,

$$\frac{d\left(\beta_{1,t} + \beta_{2,t}\right)}{dk} > 0 \tag{29}$$

Proof. The proof is in the Appendix D. □

(27) shows that the difference of two agents' demands decreases in $\beta_{1,t} + \beta_{2,t}$, and (28) show $\beta_{1,t} + \beta_{2,t}$ is greater than that in the benchmark case. Thus, with relative performance, two agents trade more *similarly* than they do without relative performance. (29) shows that the more important the relative performance is, the more similarly the managers trade. The following corollary shows the case when the relative performance is infinitely strong ($k \to \infty$).

Corollary 1. *The difference between two demands goes to zero when $k \to \infty$.*

Proof. One can show that both $\beta_{1,t}$ and $\beta_{2,t}$ are smaller than $\frac{1}{2}$. Thus, given (29), we have the above corollary.

Intuitively, when the concerns of the relative performance is infinitely strong, the difference of opinions goes to zero, hence two agents trade like one person. We also show how the heterogeneity demand of each manager changes with respect to the relative performance in the following proposition.

Proposition 5. *For both agents, there exist cutoffs, $g_{c1} < g_{c2}$*

Case 1: when $\lambda(t) < g_{c1}$; $\beta_{1,t} > \beta^0_{1,t}$, $\beta_{2,t} < \beta^0_{2,t}$

Case 2: when $\lambda(t) > g_{c2}$; $\beta_{1,t} < \beta^0_{1,t}$, $\beta_{2,t} > \beta^0_{2,t}$

Case 3: when $g_{c1} < \lambda(t) < g_{c2}$; $\beta_{1,t} > \beta^0_{1,t}$, $\beta_{2,t} > \beta^0_{2,t}$

Proof. The proof is in Appendix E. □

We compare the heterogeneity demand with and without relative performance conditional on $\lambda(t)$. This is intuitive because the realization of $\lambda(T)$ determines the fraction of wealth allocated to each agent, which is shown by (22). We use the following figure to illustrate the three cases in the proposition.

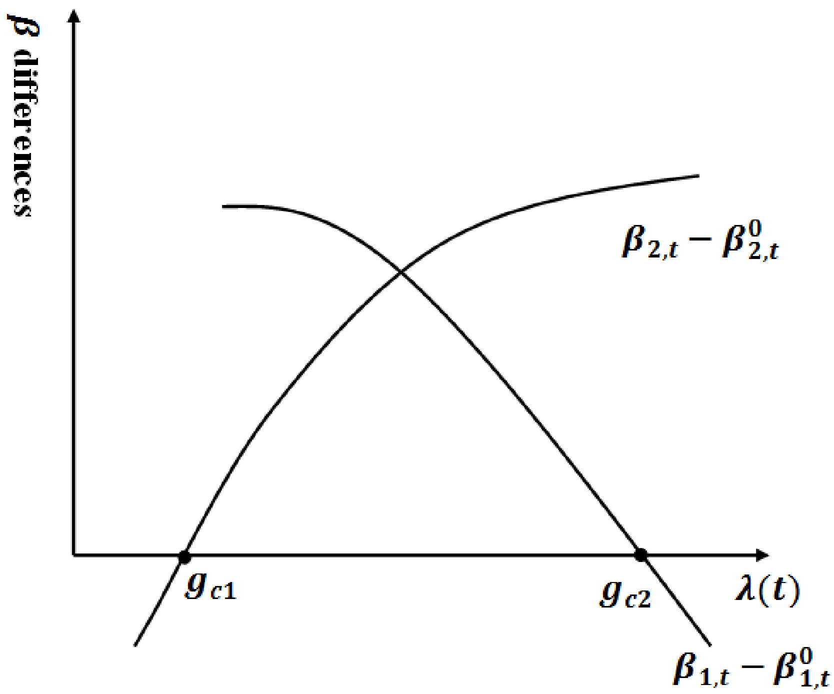

Figure 1: *The difference of β with relative performance to that without relative performance.*

Figure 1 gives the graphical illustration of the proposition. It shows how the difference of heterogeneity demands with and without relative performance changes with respect to $\lambda(t)$. We discuss each case separately.

Case 1 indicates the situation in which *both* agents believe the economy is good. The reason is shown in the following. Comparing to the case without relative performance, the heterogeneity demand of the optimistic (pessimistic) agent is higher (lower). From the results of lemma 5, when $\lambda(T)$ is small, $W_{1,T} < W^0_{1,T}$ and $W_{2,T} > W^0_{2,T}$. Given that $\lambda(t)$ is small, the possibility that $W_{1,T} < W^0_{1,T}$ and $W_{2,T} > W^0_{2,T}$ is high. On expectation, pessimistic agent will have larger share of wealth than the case without relative performance. Consequently, the optimistic (pessimistic) agent will have less (more) fraction of wealth so that she needs to have a larger heterogeneity demand.

Case 2 indicates the situation in which both agents believe the economy is bad. Following the same logic of case 1, given that $\lambda(t)$ is large, the possibility that $W_{1,T} > W^0_{1,T}$ and $W_{2,T} < W^0_{2,T}$ is high. The optimistic agent will end up with higher fraction of wealth, hence have a less heterogeneity demand.

Note that the definition of "good economy" and "bad economy" is subjective to two types of investors' beliefs. In case 3, the optimistic agent believes the economy is "good", and the pessimistic one believes "bad".[12] Thus, on the expectations over the subjective belief, the

[12] There is no other possibility (e.g. the pessimistic agent believes the economy is good, and optimistic one be-

possibility that $W_{1,T} < W_{1,T}^0$ and $W_{2,T} < W_{2,T}^0$ is high for the optimistic and pessimistic agent respectively. In this range, both agents have a higher heterogeneity demand.

5.2 Market Price of Risk (Sharpe Ratio)

The following proposition shows the Sharpe ratios with and without relative performance.

Proposition 6. *When $\gamma = 2$, the market prices of risk are:*

$$\kappa_{1,t} = 2\sigma_D + a_k\overline{\mu}; \;\; \kappa_{2,t} = 2\sigma_D - (1 - a_k)\overline{\mu} \tag{30}$$

where a_k is a function of k and is shown in the Appendix F. Moreover, the market prices of risk in the benchmark case are:

$$\kappa_{1,t}^0 = 2\sigma_D + a_0\overline{\mu}; \quad \kappa_{2,t}^0 = 2\sigma_D - (1 - a_0)\overline{\mu} \tag{31}$$

then we have

$$\begin{array}{ll} \textit{Case 1: when} & \lambda(t) < \exp[-2\overline{\mu}\sigma_D(T-t)],\ a_k > a_0,\ \textit{and}\ \dfrac{\partial a_k}{\partial k} > 0 \\ \textit{Case 2: when} & \lambda(t) > \exp[-2\overline{\mu}\sigma_D(T-t)],\ a_k < a_0,\ \textit{and}\ \dfrac{\partial a_k}{\partial k} < 0 \end{array}$$

Proof. The proof is in the Appendix F. □

(30) shows that some risk is actually transferred from pessimistic agent to the optimistic agent since $\kappa_{1,t} - \kappa_{2,t} = \overline{\mu}$. This is the standard result for asset pricing with heterogeneous beliefs. Given a_k is a function of k, we know that the transferred risk is affected by the relative performance. (31) gives Sharpe ratios without relative performance ($k = 0$), so the analysis depends on the comparison between a_k and a_0, which is shown in the two cases of the proposition.

Similar to the analysis of the portfolio choices, case 1 corresponds to a good economy. We show that the market price of risk with relative performance is higher than that without relative performance, and the more important the relative performance is, the higher the market price of risk is. In case 2 (bad economy), the market price of risk with relative performance is smaller than that without relative performance, and the more important the relative performance is, the smaller the market price of risk is. This is because: When the economy is good, optimistic investors possess less wealth with relative performance than she does without relative performance. Although the optimistic agent still dominates the market, the stock is less overvalued

lieves bad.) given the priors of two agents because the optimistic agent, by definition, is always more "optimistic" than the pessimistic one.

with relative performance. Hence, the market price of risk is higher with relative performance than that without relative performance. When the economy is bad, by the similar logic, the market price of risk is smaller with relative performance than that without relative performance.

Proposition 7. *The first order condition of κ_i on λ is positive for both investors, i.e.*

$$\frac{d\kappa_{i,t}}{d\lambda(t)} > 0, \qquad i = 1, 2 \tag{32}$$

Proof. The proof is in the Appendix G. □

(32) shows that Sharpe ratios of both investors are *counter-cyclical*. Intuitively, when the market is good, the optimistic agent dominates the market so the stock is overvalued. The excess return is lower, hence the Sharpe ratio is lower.

5.3 Stock Price and Volatility

In this section, we solve the equilibrium price and the volatility, and compare those to the benchmark case.

Proposition 8. *The stock price is*

$$S_t = \frac{\left\{1 + \lambda(t)^{\frac{1}{2+2k}} 2e^{\left[-\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2+2k}\overline{\mu}\sigma_D\right](T-t)} + \lambda(t)^{\frac{2}{2+2k}} e^{\frac{2}{2+2k}\overline{\mu}\sigma_D(T-t)}\right\} D_t e^{\left(\mu_1 - 2\sigma_D^2\right)(T-t)}}{\left\{1 + 2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)} + \lambda(t)^{\frac{1}{1+k}} e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\} e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\frac{k}{1+k}\right)(T-t)}} \tag{33}$$

Denote S_t^0 as the stock price when $k = 0$, then we have

$$\begin{aligned} &\textit{Case 1} \quad : \quad \textit{when } \lambda(t) \to 0,\ S_t < S_t^0 \\ &\textit{Case 2} \quad : \quad \textit{when } \lambda(t) \to \infty,\ S_t > S_t^0 \end{aligned}$$

Proof. The proof is in the Appendix H. □

(33) shows the expression of the stock price, and we compare it to the benchmark case price, S_t^0, in two extreme cases. Case 1 and Case 2 depend on the process $\lambda(t)$, so we have similar interpretation to that of portfolio choices. However, in the proposition, we only consider the cases to the extreme situation. In case 1, $\lambda(t) \to 0$ so we interpret it as the extremely good economy. We show that the stock price is lower with the relative performance than that without relative performance. Case 2 is interpreted as the extremely bad economy, and the stock price is higher with the relative performance than that without.

When $\lambda(t)$ is small, the aggregate demands are lower than the benchmark case. When $\lambda(t)$ is large, the aggregate demands are higher. Given that the stock has fixed supply, the price is lower in case 1 and higher in case 2 relative to the benchmark case. Figure 2 explains this proposition.

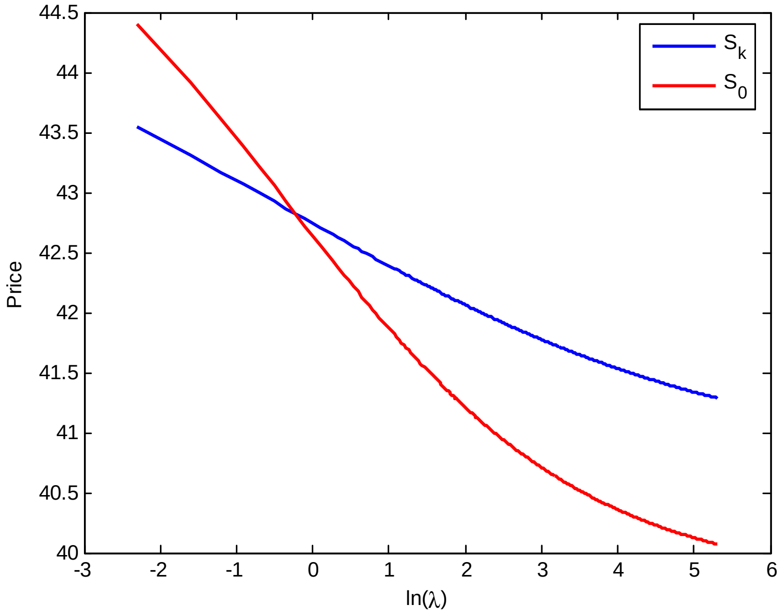

Figure 2: *This figure shows the comparison of stock prices with and without relative performance for different λ. The model parameters are $\sigma_D = 0.3$, $\overline{\mu} = 0.5$, $k = 0.5$.*

From the graph and the proposition, we can see that when $\lambda(t)$ is very large (very bad economy), the stock price is higher than the benchmark case. When $\lambda(t)$ is very small (very good economy), the stock price is lower. For the middle range of $\lambda(t)$, the stock price can be either higher or lower than that in the benchmark case. Note that the middle range corresponds to the case 3 in proposition 5 in which both agents disagree with the "good" or "bad" economies.

Corollary 2. *When $k \to \infty$, $S_t = D_t e^{\left(\frac{\mu_1+\mu_2}{2} - 2\sigma_D^2\right)(T-t)}$.*

This corollary is the extension of corollary 1. When $k \to \infty$, we know that, from corollary 1, both agents submit the same demand so that the economy is the same as that with one representative investor. Moreover, this investor has the average belief, $\frac{\mu_1+\mu_2}{2}$, on the dividend process. As a result, we have the stock price in the corollary.

Proposition 9. *The volatility is*

$$\sigma_{s,t} = \sigma_D + \frac{1}{1+k}\left\{ \begin{array}{l} \left[\frac{e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}+e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\lambda(t)^{\frac{1}{1+k}}}{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\lambda(t)^{\frac{1}{1+k}}}\right] \\ -\left[\frac{e^{\left[-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2+2k}\overline{\mu}\sigma_D\right](T-t)}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}+e^{\frac{2}{2+2k}\overline{\mu}\sigma_D(T-t)}\lambda(t)^{\frac{1}{1+k}}}{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left[-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2+2k}\overline{\mu}\sigma_D\right](T-t)}+e^{\frac{2}{2+2k}\overline{\mu}\sigma_D(T-t)}\lambda(t)^{\frac{1}{1+k}}}\right] \end{array} \right\}\overline{\mu} \quad (34)$$

Comparing the volatility with relative performance, $\sigma_{s,t}$, to the benchmark case $\sigma_{s,t}^0$, there exist *two cutoffs, d_{c1} and d_{c2}, where $d_{c1} < d_{c2}$.*

$$\textit{Case 1: When} \quad d_{c1} < \lambda(t) < d_{c2},\ \sigma_{s,t} < \sigma^0_{s,t}$$

$$\textit{Case 2: When} \quad \lambda(t) < d_{c1} \text{ or } \lambda(t) > d_{c2},\ \sigma_{s,t} > \sigma^0_{s,t}$$

Moreover, the larger the k is, the larger the $d_{c2}-$ d_{c1} is.

Proof. The proof is in the Appendix B. □

From (34), the volatility with relative performance is greater than σ_D. That is, the difference of opinions generates excess volatility. In case 1, $\lambda(t)$ has upper and lower bounds so that we interpret it as normal days (the economy is not very good or very bad). We show that the volatility is smaller with relative performance than that without relative performance. Case 2 indicates extreme economy (very good or very bad), we show that the volatility is larger with relative performance than that without relative performance. Moreover, the stronger the relative performance is (large k), the wider range of the normal days is in. Figure 3 depict numerical simulations.

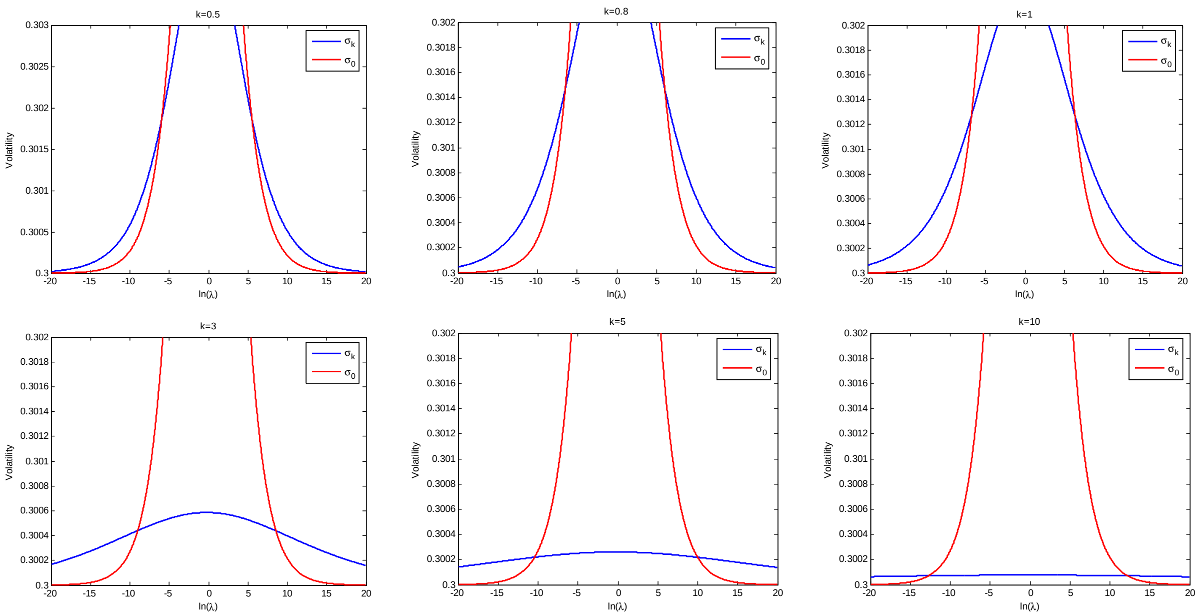

Figure 3: *Comparison of volatilities with and without relative performance. In this figure, we plot the volatility vs. the value of $\lambda(t)$.* σ_k and σ_0 *denote the volatility with and without relative performance, respectively. We choose different k for different graphs. Other model parameters are $\sigma_D = 0.3$, $\overline{\mu} = 0.5$.*

The relative performance has two effects on the volatility. First, it leads agents to trade similarly which effectively decreases the difference of opinions. Thus, it has the effect of decreasing volatility. This is what we observe in the middle range of the economy. Second, it changes the fraction of wealth that two agents will share at time T. The existence of relative performance decreases the fraction held by the optimistic (dominant) agent, hence decreases her impact to the economy. On the other hand, the impact of pessimistic agent is increased. Thus, the second effect of relative performance is to increase the "difference of opinions" which in turn increases the volatility when the economy is extreme. Intuitively, when the economy is extremely good, optimistic agent dominates the market. We can imagine that the pessimistic agent is driven out of the market when she loses a lot of money. Thus, only optimistic agent survives in the market. As a result, there is no difference of opinions, hence no excess volatility. However, with concerns of relative performance, the pessimistic agent trades more like an optimistic one so she can stay in the market even the economy is extremely good. This exaggerate the difference of opinions and hence results in an excess volatility.

Overall, in normal days, the first effect dominates the second one so that the volatility with relative performance is smaller. In extreme cases, the second effect dominates the first so that the volatility is larger. The stronger the concerns of relative performance, the more similarly agents trade; as a result, the wider the range of normal days is. This is shown by the changes of the middle range (increasing) from the first to the sixth graph in Figure 3. However, when k goes to infinity, we have the one agent economy again which is shown in the following corollary.

Corollary 3. *When $k \to \infty$, $\sigma_{s,t} \to \sigma_D$.*

It is easy to see that, in (34), the expression in the bracket is between (0,1) so that when $k \to \infty$, $\sigma_{s,t} \to \sigma_D$. The intuition is similar to those in corollary 1 and 2. When the relative performance is infinitely strong, we have the representative agent economy.

6 More Special Cases

In the last section, we use one special case with $\gamma = 2$ to illustrate our general model. However, in order to show that our general model works for more cases, we do some numerical studies for different risk aversion parameters. We use the results in proposition 2 (the general case), by choosing $\gamma = 3$ and 4, and simulate numerically volatilities in different cases.

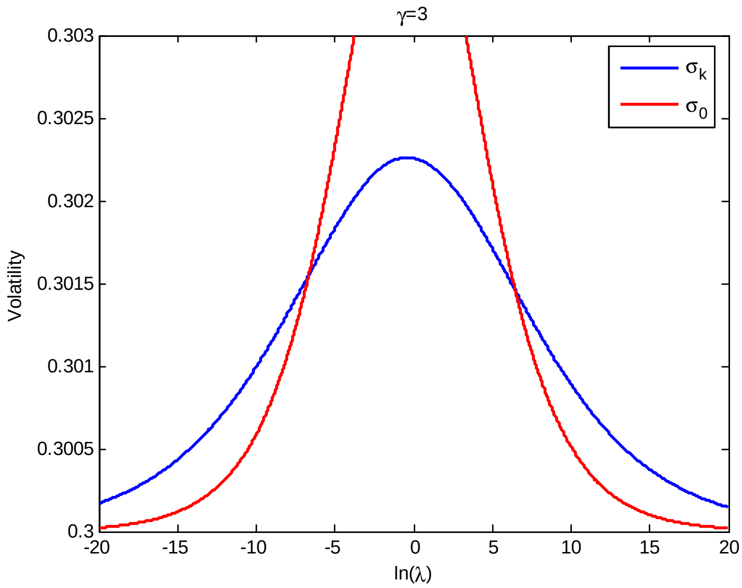

Figure 4.1: *Comparison of volatilities with and without relative performance for* $\gamma = 3$*.* σ_k and σ_0 denote *the volatility with and without relative performance, respectively. Other model parameters are* $k = 0.5$, $\sigma_D = 0.3$, $\overline{\mu} = 0.5$.

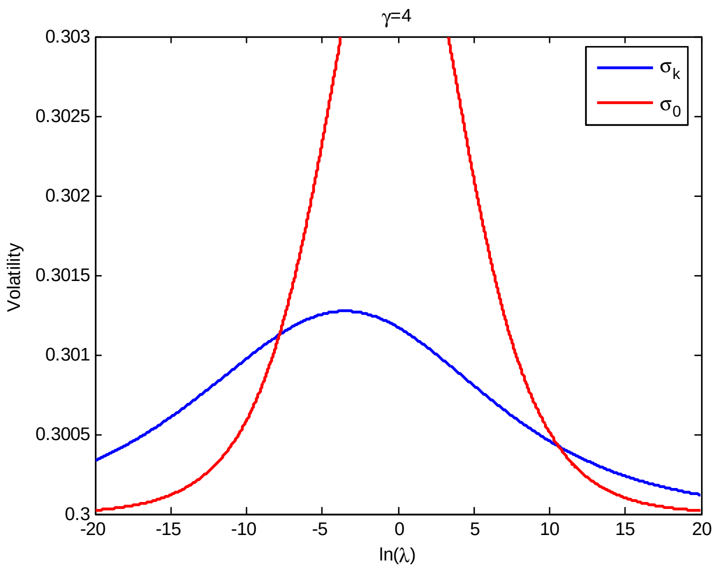

Figure 4.2: *Comparison of volatilities with and without relative performance for* $\gamma = 4$*.* σ_k and σ_0 denote *the volatility with and without relative performance, respectively. Other model parameters are* $k = 0.5$, $\sigma_D = 0.3$, $\overline{\mu} = 0.5$.

We use the volatility because it can best illustrate the theory in normal and extreme economy. Similar to the special case when $\gamma = 2$, the volatility with relative performance is smaller than that without relative performance in normal days. This reflects that the difference of opinions is smaller in normal days. However, in the extreme economy, the opposite is true.

7 Irrational Traders' Survivalship

In this section, we discuss one important implication for our model, i.e. the survival of irrational traders in the long run. Without loss of generality, we suppose the first trader has rational belief and is always right about the economy, the second trader has irrational belief, but both of the traders care about their relative performance to each other. We relax our assumption that $\overline{\mu} > 0$ so that the irrational trader can be either optimistic or pessimistic. Furthermore, we define "survival" as

Definition 1. *Agent 2 is defined as relative extinction in the long run if*

$$\lim_{T\to\infty} \frac{W_{2,T}}{W_{1,T}} = 0.^{13}$$

From the competitive equilibrium derived above, we have the following result:

Proposition 10. *Define* $\overline{\mu}^* := -2\sigma_D(\widehat{\gamma} - 1)$. *For* $\gamma > 1$ *and* $\overline{\mu} \neq \overline{\mu}^*$, *only one of the traders survives in the long run. In particular, we have:*

$\overline{\mu} > 0$, *pessimistic irrational trader* ⇒*Rational trader survives*

$\overline{\mu}^* < \overline{\mu} < 0$, *moderately optimistic irrational trader* ⇒*Irrational trader survives*

$\overline{\mu} < \overline{\mu}^*$, *strongly optimistic irrational trader* ⇒*Rational trader survives*

Proof. The proof is in the Appendix I. □

Note that for $\overline{\mu} = \overline{\mu}^*$, both rational and irrational traders survive. This proposition identifies three distinct regions on which type of trader will survive in the long run[14]. The range in which the irrational trader survives depends on $\overline{\mu}^*$. Without relative performance ($k = 0$), $\overline{\mu}^* = -\sigma_D 2(\gamma - 1)$. Hence, comparing to the benchmark case without relative performance, we have the following corollary.

Corollary 4. *The range,* $(\overline{\mu}^*, 0)$, *where the irrational trader survives is larger in the case of relative performance than that without relative performance.*

Proof. By the expression of $\widehat{\gamma}$, we can easily get the result. □

Our analysis reveals that the region where the irrational trader survives in the long run is larger when traders care about relative performance. This is consistent with our above results. Because both type of traders care about relative performance and hence they trade more similarly, thus irrational trader has higher probability of survival.

[13] For further discussion of this definition, please refer to Kogan, Ross, Wang and Westerfield (2006).

[14] This analysis is similar to Proposition 4 in Kogan, Ross, Wang and Westerfield (2006).

8 Conclusion

This paper studies an equilibrium asset pricing model in which institutional investors with heterogeneous beliefs care about relative performance. We focus on the investor's portfolio choices, asset prices, volatility and market prices of risks. Relative performance has two effects: On the one hand, relative performance leads agents to trade similarly, which effectively decreases the difference of opinions among investors. As a result, the volatility is lower with relative performance than that without relative performance. On the other hand, relative performance decreases the fraction of the dominant agent in extreme economy which effectively increases the difference of opinions. As a result, the volatility is smaller with relative performance than that without relative performance in normal economy; and larger in the extreme economy. The asset price is lower with relative performance than without relative performance when the economy is extremely good; it is higher when the economy is extremely bad. The model shows that the portfolio choices for both investors can be decomposed into three parts: myopic demand, hedging demand and heterogeneity demand. Only the heterogeneity demand is influenced by the relative performance. Regarding the market price of risk, when the economy is good, the risk premium is higher relative to the case without relative performance; and it is lower when the economy is bad.

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Appendixes:

A Proof of Lemma 1, 2, 3, 5

The Lagrangian for (21) is:

$$E^i\left[\frac{\left[W_{i,T}\left(\frac{W_{i,T}}{W_{j,T}}\right)^k\right]^{1-\gamma}}{1-\gamma}\right]+y_i\left[\frac{S_0}{2}-E^i\left(\pi_{i,t}W_{i,T}\right)\right]$$

By FOC, we have

$$\widehat{W_{i,T}}=(k+1)^{\frac{1}{(k+1)(\gamma-1)+1}}(y_i\pi_{i,T})^{-\frac{1}{(k+1)(\gamma-1)+1}}(W_{j,T})^{\frac{k(\gamma-1)}{(k+1)(\gamma-1)+1}}=(k+1)^{\frac{1}{\overline{\gamma}}}(y_i\pi_{i,T})^{-\frac{1}{\overline{\gamma}}}(W_{j,T})^{\frac{\theta(\overline{\gamma}-1)}{\overline{\gamma}}}$$

where $\overline{\gamma}=(k+1)(\gamma-1)+1=\gamma+k(\gamma-1)$ and $\theta=\frac{k}{k+1}$. This concludes the proof of Lemma 3. From the expressions in Lemma 3, we calculate $\frac{\widehat{W_{1,T}}}{\widehat{W_{2,T}}}$, and have:

$$\frac{\widehat{W_{1,T}}}{\widehat{W_{2,T}}}=\left(\frac{y_1\pi_{1,t}}{y_2\pi_{2,t}}\right)^{-\frac{1}{\theta(\overline{\gamma}-1)+\overline{\gamma}}}=\left(\frac{y_1\pi_{1,t}}{y_2\pi_{2,t}}\right)^{-\frac{1}{\gamma+2k(\gamma-1)}}=\lambda(T)^{-\frac{1}{\widehat{\gamma}}}$$

where $\lambda(t)=\frac{y_1\pi_{1,t}}{y_2\pi_{2,t}}$ and $\widehat{\gamma}=\gamma+2k(\gamma-1)$. Together with the market clearing clearing conditions, $W_{1,T}+W_{2,T}=D_T$, we get the (22) in Lemma 4. (15) in Lemma 2 is just a special case of (22). Together with the FOC (which shows the relationship between $\widehat{W_{i,T}}$ and $\pi_{i,T}$), we can solve $\pi_{i,T}$ in lemma 6.

B Proof of Proposition 2 and 9:

B.1 State Prices

When γ is an integer, (23) and (24) become:

$$\pi_{1,T}=\frac{(k+1)}{y_1}\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(T)^{\frac{i}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}}\lambda(T)^{\frac{k(\gamma-1)}{\gamma+2k(\gamma-1)}}=\frac{(k+1)}{y_1}\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(T)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}}$$

$$\pi_{2,T}=\frac{(k+1)}{y_2}\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(T)^{\frac{i}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}}\lambda(T)^{-\frac{k(\gamma-1)+\gamma}{\gamma+2k(\gamma-1)}}=\frac{(k+1)}{y_2}\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(T)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}}$$

Given the dynamics of $\lambda(t)$ and D_t w.r.t. $B_{1,t}$, we have

$$\lambda(T)=\lambda(t)\exp\left[-\frac{\overline{\mu}^2}{2}(T-t)-\overline{\mu}\left(B_{1,T}-B_{1,t}\right)\right]$$

$$D_T = D_t \exp\left[\left(\mu_1 - \frac{\sigma_D^2}{2}\right)(T-t) + \sigma_D\left(B_{1,T} - B_{1,t}\right)\right]$$

Then we can rewrite $\pi_{1,T}$ as:

$$\pi_{1,T} = \frac{k+1}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\begin{array}{c} \left[-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right)\right](T-t) \\ -\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]\left(B_{1,T} - B_{1,t}\right) \end{array}}$$

by $\pi_{1,t} = E_t^1\left(\pi_{1,T}\right)$, we have

$$\pi_{1,t} = \frac{k+1}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[\begin{array}{c} -\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) \\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array}\right](T-t)}$$

Similarly, for the dynamics w.r.t $B_{2,t}$, we have

$$\lambda(T) = \lambda(t) \exp\left[\frac{\overline{\mu}^2}{2}(T-t) - \overline{\mu}\left(B_{2,T} - B_{2,t}\right)\right]$$

$$D_T = D_t \exp\left[\left(\mu_2 - \frac{\sigma_D^2}{2}\right)(T-t) + \sigma_D\left(B_{2,T} - B_{2,t}\right)\right]$$

Following the similar procedure, we have

$$\pi_{2,t} = \frac{k+1}{y_2 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}} e^{\left[\begin{array}{c} \frac{\overline{\mu}^2}{2}\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_2 - \frac{\sigma_D^2}{2}\right) \\ +\frac{1}{2}\left[\frac{i-k(\gamma-1)+\gamma}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array}\right](T-t)}$$

B.2 Market Price of Risk

By Ito's lemma on $\pi_{1,t}$ and matching the diffusion terms, we can get Market price of risk

$$\begin{aligned} -\pi_{1,t}\kappa_{1,t} &= \frac{(k+1)}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \exp\left(\left[\begin{array}{c} -\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) \\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array}\right](T-t)\right) \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \\ &\quad \cdot \left(-\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} - \gamma\sigma_D\right) \end{aligned}$$

Then, we can get $\kappa_{1,t}$ in the proposition with

$$\delta_{1,t} = \frac{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \exp\left(\left[\begin{array}{c} -\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \\ \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) + \\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array}\right](T-t)\right) \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \exp\left(\left[\begin{array}{c} -\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \\ \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) + \\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array}\right](T-t)\right)}.$$

Similarly, we can get $\kappa_{2,t}$ with

$$\delta_{2,t}=\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}}\exp\left(\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}-\\ \gamma\left(\mu_2-\frac{\sigma_D^2}{2}\right)+\\ \frac{1}{2}\left[\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)\right)\frac{k(\gamma-1)+\gamma-i}{\gamma+2k(\gamma-1)}\overline{\mu}}{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}}\exp\left(\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}-\\ \gamma\left(\mu_2-\frac{\sigma_D^2}{2}\right)+\\ \frac{1}{2}\left[\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)\right)}.$$

B.3 Portfolio Choices

For agent 1, we have $\pi_{1,t}W_{1,t}=E_t^1\left(W_{1,T}\pi_{1,T}\right)$. By some manipulation, it can be written as

$$\pi_{1,t}W_{1,t}=\frac{k+1}{y_1D_t^{\gamma-1}}\sum_{i=0}^{\gamma-1}\begin{pmatrix}\gamma-1\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\exp\left(\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}-(\gamma-1)\left(\mu_1-\frac{\sigma_D^2}{2}\right)\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)\right)$$

by Ito's lemma and matching the diffusion terms

$$\pi_{1,t}W_{1,t}\left(\theta_{1,t}\sigma_{s,t}-\kappa_{1,t}\right)dB_{1,t}=\frac{(k+1)}{y_1D_t^{\gamma-1}}\left\{\begin{array}{c}\sum_{i=0}^{\gamma-1}\begin{pmatrix}\gamma-1\\ i\end{pmatrix}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}-\\ (\gamma-1)\left(\mu_1-\frac{\sigma_D^2}{2}\right)+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)}\\ \times\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\left(-(\gamma-1)\,\sigma_D-\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}\right)\end{array}\right\}dB_{1,t}$$

then $\theta_{1,t}=\frac{\mu_{1,t}}{\sigma_{s,t}^2}-\frac{(\gamma-1)\sigma_D}{\sigma_{s,t}}-\beta_{1,t}\overline{\mu}$ with

$$\beta_{1,t}=\frac{\sum_{i=0}^{\gamma-1}\begin{pmatrix}\gamma-1\\ i\end{pmatrix}\exp\left(\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)\right)\times\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\overline{\mu}}{\sum_{i=0}^{\gamma-1}\begin{pmatrix}\gamma-1\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)}}$$

Similarly,

$\theta_{2,t}=\frac{\mu_{2,t}}{\sigma_{s,t}^2}-\frac{(\gamma-1)\sigma_D}{\sigma_{s,t}}+\beta_{2,t}\overline{\mu}$ with

$$\beta_{2,t}=\frac{\begin{array}{c}\sum_{i=0}^{\gamma-1}\begin{pmatrix}\gamma-1\\ i\end{pmatrix}e^{\left[\begin{array}{c}\frac{\overline{\mu}^2}{2}\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left(\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right)^2\end{array}\right](T-t)}\times\\ \frac{k(\gamma-1)1+\gamma-1-i}{\gamma+2k(\gamma-1)}\lambda(t)^{\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}}\overline{\mu}\end{array}}{\sum_{i=0}^{\gamma-1}\begin{pmatrix}\gamma-1\\ i\end{pmatrix}\lambda(t)^{\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}\frac{\overline{\mu}^2}{2}\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left(\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right)^2\end{array}\right](T-t)}}$$

B.4 Stock Price

By the martingale property,

$$S_t = \frac{E_t^1(\pi_{1,T} D_T)}{\pi_{1,t}} S_t = E_t^1 \left[\frac{\frac{(k+1)}{y_1} \frac{\sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(T)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma-1}}}{\pi_{1,t}} \right]$$

$$= \frac{\sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[-\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} + \frac{\sigma_D^2}{2} + \frac{1}{2} \left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} \overline{\mu} + (\gamma-1)\sigma_D \right]^2 \right](T-t)} D_t e^{\left(\mu_1 - \sigma_D^2\right)(T-t)}}{\sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[-\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} + \frac{1}{2} \left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} \overline{\mu} + \gamma\sigma_D \right]^2 \right](T-t)}}$$

B.5 Volatility

Denote the stock price as

$$S_t = \frac{X_t}{Y_t}$$

set $X_t = \sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \exp\left\{ \left[\left[-\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} + \frac{\sigma_D^2}{2} + \frac{1}{2} \left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} \overline{\mu} + (\gamma-1)\sigma_D \right]^2 \right] (T-t) \right] \right\} D_t e^{\left(\mu_1 - \sigma_D^2\right)(T-t)}$ and $Y_t = \sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \exp\left\{ \left[\left[-\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} + \frac{1}{2} \left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} \overline{\mu} + \gamma\sigma_D \right]^2 \right] (T-t) \right] \right\}$, then for diffusion term, we only need to consider $d\frac{X_t}{Y_t} = \frac{Y_t dX_t - X_t dY_t}{Y_t^2}$. Apply Ito's Lemma to calculate diffusion terms of dX_t and dY_t.

The diffusion term for dX_t is

$$e^{\left(\mu_1 - \sigma_D^2\right)(T-t)} \left\{ \begin{array}{c} \sum_{i=0}^{\gamma} \binom{\gamma}{i} e^{\left[-\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} + \frac{\sigma_D^2}{2} + \frac{1}{2} \left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} \overline{\mu} + (\gamma-1)\sigma_D \right]^2 \right](T-t)} \times \\ \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \left(D_t \sigma_D - \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} D_t \overline{\mu} \right) \end{array} \right\} dB_t$$

The diffusion term for dY_t is

$$\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)}dB_t$$

by matching the diffusion terms, we have $\frac{X_t}{Y_t}\sigma_{s,t}=\frac{Y_t\sigma_X-X_t\sigma_Y}{Y_t^2}$ and $\sigma_{s,t}=\frac{Y_t\sigma_X-X_t\sigma_Y}{Y_tX_t}$.

$$\begin{aligned}\sigma_{s,t} \quad = \quad & \sigma_D-\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}}{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)}D_te^{\left(\mu_1-\sigma_D^2\right)(T-t)}}\overline{\mu}\\ & +\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)}}{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)}}\overline{\mu}\end{aligned}$$

then, we can get the expression in the proposition.

Similarly, by setting $\gamma=2$, we can get (34). Thus, Proposition 9 is proved.

C Proof of Proposition 3

Similar to the proof of proposition 2 by setting $\gamma=2$, we can get (25) and (26) where:

$$\beta_{1,t}=\frac{\frac{k}{2+2k}+\frac{1}{2}\lambda(t)^{\frac{1}{2(k+1)}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)}}{1+\lambda(t)^{\frac{1}{2(k+1)}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)}};\quad \beta_{2,t}=\frac{\frac{k}{2+2k}\lambda(t)^{\frac{1}{2(1+k)}}e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)}+\frac{1}{2}}{\lambda(t)^{\frac{1}{2(1+k)}}e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)}+1}$$

D Proof of Proposition 4

Define $\exp(a)=\lambda(t)^{\frac{1}{2}}\exp\left[\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)\right]$ and $b=\frac{\overline{\mu}^2}{8}$, then we have:

$$\beta_{1,t}=\frac{\frac{k}{2+2k}+\frac{1}{2}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}}{1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}}\ ;\quad \beta_{2,t}=\frac{\frac{1}{2}+\frac{k}{2+2k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}}{1+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}}$$

when $k=0$, we have $\beta_{1,t}^0 = \frac{\frac{1}{2}e^{-b+a}}{1+e^{-b+a}}$; $\beta_{2,t}^0 = \frac{\frac{1}{2}}{1+e^{b+a}}$. By some manipulation, we can write the difference between two investors' portfolio choices as

$$\begin{aligned}
\theta_{1,t} - \theta_{2,t} &= \frac{\overline{\mu}}{\sigma_{s,t}}\left[1-(\beta_{1,t}^k+\beta_{2,t}^k)\right] \\
\text{where} &: \beta_{1,t}^k < \frac{1}{2},\ \beta_{2,t}^k < \frac{1}{2}
\end{aligned}$$

For this reason, relative performance's effect on portfolio choice depends on $\beta_{1,t}^k + \beta_{2,t}^k - (\beta_{1,t}^0 + \beta_{2,t}^0)$ which can be calculated as

$$\frac{\frac{1+2k}{2(1+k)} + e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}} + \frac{k}{1+k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}} + \frac{1+2k}{2(1+k)}e^{\frac{2a}{(k+1)}}}{1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}} + e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}} + e^{\frac{2a}{(k+1)}}} - \frac{\frac{1}{2}+e^{-b+a}+e^{2a}}{1+e^{-b+a}+e^{b+a}+e^{2a}}$$

If we use notation $\frac{A}{B}-\frac{C}{D}$ for above expression, its sign depends on $AD-CB$, which can be calculated as:

$$\begin{aligned}
&\frac{k}{2(1+k)} - \frac{1}{2(1+k)}e^{-b+a} + \frac{1+2k}{2(1+k)}e^{b+a} + \frac{k}{2(1+k)}e^{2a} + e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}\left(\frac{1}{2}+e^{-b+a}+e^{2a}\right) \\
&+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}\left[\frac{k}{1+k}-\frac{1}{2}-\frac{1}{(1+k)}e^{-b+a}+\frac{k}{1+k}e^{b+a}+\left(\frac{k}{1+k}-\frac{1}{2}\right)e^{2a}\right] \\
&+e^{\frac{2a}{(k+1)}}\left[\frac{k}{2(1+k)}-\frac{1}{2(1+k)}e^{-b+a}+\frac{1+2k}{2(1+k)}e^{b+a}+\frac{k}{2(1+k)}e^{2a}\right]
\end{aligned}$$

Then we can conclude

$$\beta_{1,t}^k + \beta_{2,t}^k - (\beta_{1,t}^0 + \beta_{2,t}^0) = \begin{cases} >0 & \text{when } k>1 \\ <0 & \text{when } k<1 \text{ and } a \text{ is large enough} \end{cases}$$

Moreover, we have:

$$\frac{d\left(\beta_{1,t}^k+\beta_{2,t}^k\right)}{dk} = \frac{d\left[\frac{\frac{1+2k}{2(1+k)}+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{k}{1+k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1+2k}{2(1+k)}e^{\frac{2a}{(k+1)}}}{1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{2a}{(k+1)}}}\right]}{dk}$$

After some manipulation, $\frac{d\left(\beta_{1,t}^k+\beta_{2,t}^k\right)}{dk}$ can be expressed as:

$$\begin{aligned}
&\left[\frac{1}{2(1+k)^2}+\frac{1}{(1+k)^2}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1}{2(1+k)^2}e^{\frac{2a}{(k+1)}}\right]\left[1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{2a}{(k+1)}}\right] \\
&+\frac{2b}{(1+k)^3}\left[\frac{1}{2(1+k)}+\frac{1}{1+k}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1}{2(1+k)}e^{\frac{2a}{(k+1)}}\right]e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}} \\
&+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}\frac{2b}{(1+k)^3}\left[\frac{1}{2(1+k)}+\frac{1}{1+k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1}{2(1+k)}e^{\frac{2a}{(k+1)}}\right] \\
&-\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}e^{\frac{2a}{(k+1)}} \\
&-\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}e^{\frac{2a}{(k+1)}}
\end{aligned}$$

It is easy to see that the above expression is greater than 0.

E Proof of Proposition 5

Given $\beta_{i,t}$ in the proof of proposition 3, we can easily get $\beta^0_{i,t}$ by setting k=0. Then for optimistic agent, the sign of $\beta_{1,t} - \beta^0_{1,t}$ depends on

$$\frac{\frac{k}{1+k} + \lambda(t)^{\frac{1}{2(k+1)}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)}}{1 + \lambda(t)^{\frac{1}{2(k+1)}} e} - \frac{\lambda(t)^{\frac{1}{2(}} e^{\left(-\frac{\overline{\mu}^2}{8} + \frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}}{1 + \lambda(t)^{\frac{1}{2}} e^{\left(-\frac{\overline{\mu}^2}{8} + \frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}}$$

by some manipulation, the sign depends on

$$\frac{k}{1+k} + \lambda(t)^{\frac{1}{2(k+1)}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)} - \frac{1}{1+k}\lambda(t)^{\frac{1}{2(}} e^{\left(-\frac{\overline{\mu}^2}{8} + \frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}$$

Let $x \equiv \lambda(t))^{\frac{1}{2}} e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}$,define $F(x)$ as

$$F(x) = \frac{k}{1+k} + e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)} x^{\frac{1}{1+k}} - \frac{1}{1+k}x$$

When

$$F^{'}(x) = \frac{1}{1+k} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)} x^{\frac{1}{1+k}-1} - \frac{1}{1+k} = 0$$

We have$x = e^{\frac{-\overline{\mu}^2}{8(1+k)k}(T-t)}$.When $x > e^{\frac{-\overline{\mu}^2}{8(1+k)k}(T-t)}$, $F^{'}(x) < 0$, $F(x)$ is a monotonically decreasing function of x; when $x < e^{\frac{-\overline{\mu}^2}{8(1+k)k}(T-t)}$, $F^{'}(x) > 0$, $F(x)$ is a monotonically increasing function of x. For this reason, there exist one cutoff x_{c1}, when $x > x_{c1}$, $F(x) < 0$, when $x < x_{c1}$, $F(x) > 0$

For pessimistic agent, the sign of $\beta_{2,t} - \beta^0_{2,t}$ depends on

$$\frac{\frac{k}{1+k}\lambda(t)^{\frac{1}{2(1+k)}} e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)} + 1}{\lambda(t)^{\frac{1}{2(1+k)}} e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)} + 1} - \frac{1}{\lambda(t)^{\frac{1}{2}} e^{\left(\frac{\overline{\mu}^2}{8} + \frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)} + 1}$$

by some manipulation, the sign depends on

$$\frac{k}{1+k} + e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)} \left[\lambda(t)^{\frac{1}{2}} e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}\right]^{-\frac{1}{1+k}} - \frac{1}{1+k} e^{-\frac{\overline{\mu}^2}{8}(T-t)} \left[(\lambda(t))^{\frac{1}{2}} e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}\right]^{-1}$$

Let $x := \lambda(t)^{\frac{1}{2}} e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}$, define $G(x)$ as

$$G(x) = \frac{k}{1+k} + e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)} x^{-\frac{1}{1+k}} - \frac{1}{1+k}x^{-1}$$

When

$$G^{'}(x) = -\frac{1}{1+k} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)} x^{-\frac{1}{1+k}-1} + \frac{1}{1+k}x^{-2} = 0$$

We have $x = e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}$. When : $x > e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}$, $G^{'}(x) < 0$, $G(x)$ is a monotonically decreasing function of x; when $x < e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}$, $G^{'}(x) > 0$, $G(x)$ is a monotonically increasing function of x. Consequently, there exist one cutoff x_{c2}, when $x > x_{c2}$, $G(x) > 0$, when $x < x_{c2}$, $G(x) > 0$.

Now we study the x_{c1} and x_{c2}. We have $F(1)=G(1)=\frac{k}{1+k}+e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}-\frac{1}{1+k}>0$. Because $0=G(x_{c2})<G(1)<G\left[e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}\right]$, we have: $x_{c2}<1$. In addition, because $0=F(x_{c1})<F(1)<F\left[e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}\right]$, we have: $x_{c1}>1$. To sum up, $x_{c1}>x_{c2}$. Let $g_{c1}:=x_{c1}^2e^{(-\overline{\mu}\sigma_D)(T-t)}$ and $g_{c2}:=x_{c2}^2e^{(-\overline{\mu}\sigma_D)(T-t)}$, Proposition 5 is proved.

F Proof of Proposition 6

Similar to the proof of proposition 2 by setting $\gamma=2$, we can get (30) and (31) where

$$a_k=\frac{\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}.$$

$$a_0=\frac{\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\overline{\mu}\sigma_D\right)(T-t)}+\lambda(t)e^{(2\overline{\mu}\sigma_D)(T-t)}}{1+2\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\overline{\mu}\sigma_D\right)(T-t)}+\lambda(t)e^{(2\overline{\mu}\sigma_D)(T-t)}}$$

If $a_k\geq(\leq)a_0$,then $\frac{a_k}{1-a_k}\geq(\leq)\frac{a_0}{1-a_0}$. The sign of $\frac{a_k}{1-a_k}-\frac{a_0}{1-a_0}$depends on

$$\frac{\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}{\frac{2+k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}$$
$$-\frac{\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\overline{\mu}\sigma_D\right)(T-t)}+\lambda(t)e^{(2\overline{\mu}\sigma_D)(T-t)}}{1+\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\overline{\mu}\sigma_D\right)(T-t)}}$$

After some manipulation, we can show that the sign depends on

$$\frac{k}{2+2k}\left(1-\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+2}\right)+e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left(\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}}-\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}+2}\right)$$
$$+\frac{2+k}{2+2k}\left(\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}}-\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{2}\right)$$
$$+\frac{1}{1+k}e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left(\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1}-\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right)$$

If $\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}<1$, we have $\frac{k}{2+2k}\left(1-\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+2}\right)>0$ and $\frac{2+k}{2+2k}\left(\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}}-\left\{\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right\}^{2}\right)>0$. Moreover,

$$
\begin{aligned}
& e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left(\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}}-\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}+2}\right) \\
> \; & e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left(\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}-\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1}\right) \\
> \; & e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left(\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}-\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1}\right) \\
> \; & \frac{1}{1+k} e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left(\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}-\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1}\right)
\end{aligned}
$$

We have

$$
e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}} \\ -\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\}+\frac{1}{1+k} e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1} \\ -\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\end{array}\right\}>0
$$

Hence, in this case, $\frac{a_k}{1-a_k}>\frac{a_0}{1-a_0}$, and it is easy to show that $a_k>a_0$.

If $\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}>1$, we have $\frac{k}{2+2k}\left\{1-\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+2}\right\}<0$ and $\frac{2+k}{2+2k}\left\{\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}}-\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]\right\}^2<0$. Then,

$$
\begin{aligned}
& e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}}- \\ \left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\}+\frac{1}{1+k} e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1}- \\ \left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]\end{array}\right\} \\
< \; & e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}}- \\ \left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\}+e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1}- \\ \left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]\end{array}\right\} \\
= \; & e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}}-\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}+\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1} \\ -\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\} \\
< \; & 0
\end{aligned}
$$

In this case $\frac{a_k}{1-a_k}-\frac{a_0}{1-a_0}<0$, it is easy to show $a_k<a_0$.

For comparative statics, we consider how the ratio $\frac{a_k}{1-a_k}$ changes w.r.t. k.

$$
\frac{a_k}{1-a_k}=\frac{\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}} e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}{\frac{2+k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{k}{2+2k}\lambda(t)^{\frac{1}{1+k}} e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}
$$

Let $a=\overline{\mu}\sigma_D(T-t)$, $b=\frac{\overline{\mu}^2}{8}(T-t)$, we calculate $\frac{\partial\frac{a_k}{1-a_k}}{\partial k}$. After some calculation, we find that the sign depends on

$$
\begin{aligned}
&-\left[\frac{1}{(1+k)^2}+\frac{2a}{(1+k)^3}\right]\left[e^{\frac{2a}{1+k}}-1\right]-\frac{2a}{(1+k)^3}\\
&-e^{-\frac{b}{(1+k)^2}+\frac{a}{1+k}}\left(e^{\frac{2a}{1+k}}-1\right)\left[\frac{1}{(1+k)^2}+\frac{a}{(1+k)^3}+\frac{2b}{(1+k)^4}\right]\\
&-\frac{2a}{(1+k)^3}e^{-\frac{b}{(1+k)^2}+\frac{a}{1+k}}-\frac{1}{2(1+k)^3}\left[e^{\frac{2a}{1+k}}-1\right]^2
\end{aligned}
$$

If $e^{\frac{a}{1+k}}-1>0$, then $a>0$, and the above expression is smaller than zero. If $e^{\frac{a}{1+k}}-1<0$, then $a<0$, and we can show it is greater than zero. Since $\frac{\partial\frac{a_k}{1-a_k}}{\partial k}$ has the same sign with $\frac{\partial a_k}{\partial k}$, the proposition is thus proved.

G Proof of Proposition 7

$$
\frac{d\kappa_{1,t}}{d\lambda(t)}=\overline{\mu}\frac{\left\{\frac{1}{2}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{2+k}{2+2k}\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\}}{\left\{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\}}-
$$

$$
\overline{\mu}\frac{\left\{\begin{array}{c}\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\\ \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{\begin{array}{c}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}\\ +\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}}{\left\{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\}^2}
$$

The numerator is:

$$
\begin{aligned}
=&\left\{\begin{array}{c}\frac{1}{2}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}\\ +\frac{2+k}{2+2k}\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{\begin{array}{c}1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\\ \lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\\
&-\left\{\begin{array}{c}\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\\ \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{\begin{array}{c}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}\\ +\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}
\end{aligned}
$$

then $\frac{d\kappa_{1,t}}{d\lambda(t)}$ is:

$$
\begin{aligned}
=&\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}\left[\frac{1}{2}e^{(-\frac{\overline{\mu}^2}{8(1+k)^2})(T-t)}+\frac{1}{2}\frac{2+k}{1+k}[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}]^{\frac{1}{1+k}}\right]\\
&\times\left\{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\}\\
&-\left\{\begin{array}{c}\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\\ \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}+\left[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}\right]^{\frac{1}{1+k}}\right\}
\end{aligned}
$$

After some manipulation, it becomes:

$$\frac{1}{2(1+k)}e^{-\frac{\bar{\mu}^2}{8(1+k)^2}(T-t)}+\frac{1}{(1+k)}\left[\lambda(t)^{\frac{1}{2}}e^{(\bar{\mu}\sigma_D)(T-t)}\right]^{\frac{1}{1+k}}+\frac{1}{2(1+k)}e^{-\frac{\bar{\mu}^2}{8(1+k)^2}(T-t)}\left[\lambda(t)^{\frac{1}{2}}e^{(\bar{\mu}\sigma_D)(T-t)}\right]^{\frac{2}{1+k}}$$

$$\begin{aligned}\frac{d\kappa_{1,t}}{d\lambda(t)} &= \lambda(t)\bar{\mu}\frac{1}{1+k}[\lambda(t)^{\frac{1}{2}}e^{(\bar{\mu}\sigma_D)(T-t)}]^{\frac{1}{1+k}}\left\{\begin{array}{c}\frac{1}{2(1+k)}e^{-\frac{\bar{\mu}^2}{8(1+k)^2}(T-t)}+\frac{1}{(1+k)}[\lambda(t)^{\frac{1}{2}}e^{(\bar{\mu}\sigma_D)(T-t)}]^{\frac{1}{1+k}}\\ +\frac{1}{2(1+k)}e^{-\frac{\bar{\mu}^2}{8(1+k)^2}(T-t)}[\lambda(t)^{\frac{1}{2}}e^{(\bar{\mu}\sigma_D)(T-t)}]^{\frac{2}{1+k}}\end{array}\right\}\\ &= \lambda(t)\bar{\mu}\frac{1}{(1+k)^2}\left[\frac{1}{2}e^{-\frac{b}{(1+k)^2}+\frac{a}{1+k}}+e^{\frac{2a}{1+k}}+\frac{1}{2}e^{-\frac{b}{(1+k)^2}+\frac{3a}{1+k}}\right]>0\end{aligned}$$

where $\exp(a)=\lambda(t)^{\frac{1}{2}}\exp\left(\bar{\mu}\sigma_D\right)(T-t)$, $b=\frac{\bar{\mu}^2}{8}$.

Since $\kappa_{1,t}-\kappa_{2,t}$ is a constant, it is easy to show $\frac{d\kappa_{2,t}}{d\lambda(t)}>0$.

H Proof of Proposition 8

Similar to the proof of proposition 2 by setting $\gamma=2$, we can get (33). When $k=0$, we have $S_t^0=\dfrac{\left\{\begin{array}{c}1+\lambda(t)^{\frac{1}{2}}2e^{\left[-\frac{\bar{\mu}^2}{8}+\frac{1}{2}\bar{\mu}\sigma_D\right](T-t)}\\ +\lambda(t)e^{\bar{\mu}\sigma_D(T-t)}\end{array}\right\}D_te^{\left(\mu_1-\sigma_D^2\right)(T-t)}}{\left\{\begin{array}{c}1+2\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\bar{\mu}^2}{8}+\bar{\mu}\sigma_D\right)(T-t)}+\\ \lambda(t)e^{2\bar{\mu}\sigma_D(T-t)}\end{array}\right\}e^{\sigma_D^2(T-t)}}$. Denote $S_t=\frac{X_t^k}{Y_t^k}D_t\exp\left(\mu_1-\sigma_D^2\right)(T-t)$ and $S_t^0=\frac{X_t^0}{Y_t^0}D_t\exp\left(\mu_1-2\sigma_D^2\right)(T-t)$. For $S_t^k\leq(\geq)S_t^0$, we need $\frac{X_t^k}{Y_t^k}\leq(\geq)\frac{X_t^0}{Y_t^0}$, we need to look at the sign of $X_t^kY_t^0-Y_t^kX_t^0$

$$\begin{aligned}X_t^k &= 1+\lambda(t)^{\frac{1}{2+2k}}2e^{\left[-\frac{\bar{\mu}^2}{8(1+k)^2}+\frac{1}{2+2k}\bar{\mu}\sigma_D\right](T-t)}+\lambda(t)^{\frac{2}{2+2k}}e^{\frac{2}{2+2k}\bar{\mu}\sigma_D(T-t)}\\ &= 1+2\left\{\lambda(t)e^{\bar{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{2+2k}}e^{-\frac{\bar{\mu}^2}{8(1+k)^2}(T-t)}+\left\{\lambda(t)e^{\bar{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{2+2k}}=1+2e^{-\frac{\bar{\mu}^2}{8(1+k)^2}(T-t)}K^{\frac{1}{2+2k}}+K^{\frac{2}{2+2k}}\end{aligned}$$

$$\begin{aligned}Y_t^k &= \left\{\begin{array}{c}1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\bar{\mu}^2}{8(1+k)^2}+\bar{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\\ \lambda(t)^{\frac{1}{1+k}}e^{\bar{\mu}\sigma_D\frac{2}{1+k}(T-t)}\end{array}\right\}e^{\frac{k}{2+2k}\bar{\mu}\sigma_D(T-t)}\\ &= e^{\frac{k}{2+2k}\bar{\mu}\sigma_D(T-t)}+2K^{\frac{1}{2}\frac{1}{1+k}}e^{\frac{1}{2}\bar{\mu}\sigma_D(T-t)}e^{-\frac{\bar{\mu}^2}{8(1+k)^2}(T-t)}+K^{\frac{1}{1+k}}e^{\frac{2+k}{2+2k}\bar{\mu}\sigma_D(T-t)}\end{aligned}$$

$$\begin{aligned}X_t^0 &= 1+\lambda(t)^{\frac{1}{2}}2e^{\left[-\frac{\bar{\mu}^2}{8}+\frac{1}{2}\bar{\mu}\sigma_D\right](T-t)}+\lambda(t)e^{\bar{\mu}\sigma_D(T-t)}\\ &= 1+2\left\{\lambda(t)e^{\bar{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{2}}e^{-\frac{\bar{\mu}^2}{8}(T-t)}+\lambda(t)e^{\bar{\mu}\sigma_D(T-t)}=1+2e^{-\frac{\bar{\mu}^2}{8}(T-t)}K^{\frac{1}{2}}+K\end{aligned}$$

$$Y_t^0=1+2\lambda(t)^{\frac{1}{2}}e^{\bar{\mu}\sigma_D(T-t)}e^{-\frac{\bar{\mu}^2}{8}(T-t)}+\lambda(t)e^{2\bar{\mu}\sigma_D(T-t)}=1+2K^{\frac{1}{2}}e^{\frac{1}{2}\bar{\mu}\sigma_D(T-t)}e^{-\frac{\bar{\mu}^2}{8}(T-t)}+Ke^{\bar{\mu}\sigma_D(T-t)}$$

where $K=\lambda(t)e^{\bar{\mu}\sigma_D(T-t)}$. After some calculation, the sign of $X_t^kY_t^0-Y_t^kX_t^0$ depends on

$$1-e^{\frac{k}{2+2k}\overline{\mu}\sigma_D(T-t)}+2\left[e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}-e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}\right]K^{\frac{1}{2+2k}}+K^{\frac{1}{1+k}}\left[1-e^{\frac{2+k}{2+2k}\overline{\mu}\sigma_D(T-t)}\right]$$

$$\begin{aligned}
+ \quad & 2\left[e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}-e^{\left(\frac{k}{2+2k}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}\right]K^{\frac{1}{2}}+\left[e^{\overline{\mu}\sigma_D(T-t)}-e^{\frac{2+k}{2+2k}\overline{\mu}\sigma_D(T-t)}\right]K^{\frac{2+k}{1+k}} \\
& +2\left[e^{\left(\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}-e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}\right]K^{\frac{3+2k}{2+2k}} \\
+ \quad & \left[e^{\overline{\mu}\sigma_D(T-t)}-e^{\frac{k}{2+2k}\overline{\mu}\sigma_D(T-t)}\right]K+2\left[e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}-e^{\left(\frac{2+k}{2+2k}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}\right]K^{\frac{3+k}{2+2k}}
\end{aligned}$$

Then when $K\to\infty$, $S_t>S_t^0$. When $K\to 0$, $S_t<S_t^0$

I Proof of Proposition 10

$W_{1,T}=\frac{D_T}{1+\lambda(T)^{\frac{1}{\widehat{\gamma}}}}$;$W_{2,T}=\frac{\lambda(T)^{\frac{1}{\widehat{\gamma}}}D_T}{1+\lambda(T)^{\frac{1}{\widehat{\gamma}}}}$, where $\widehat{\gamma}=\gamma+2k(\gamma-1)$. We thus obtain

$$\frac{W_{2,T}}{W_{1,T}}=\lambda(T)^{-\frac{1}{\widehat{\gamma}}}=\exp\left[\frac{1}{\widehat{\gamma}}\left(-\frac{1}{2}\overline{\mu}^2-\overline{\mu}\sigma_D\left(\widehat{\gamma}-1\right)\right)T-\frac{1}{\widehat{\gamma}}\overline{\mu}B_T\right]$$

Using the strong Law of Large Numbers for Brownina motion (see Karatzas ande Shreve (1991), for any value of σ, we have:

$$\lim_{T\to\infty}\exp(aT+\sigma B_T)=\left\{\begin{array}{c}0, a<0\\ \infty, a>0\end{array}\right\}$$

so we have:

$$\lim_{T\to\infty}\frac{W_{2,T}}{W_{1,T}}=\left\{\begin{array}{c}0, -\frac{1}{2}\overline{\mu}^2-\overline{\mu}\sigma_D\left(\widehat{\gamma}-1\right)<0\\ \infty, -\frac{1}{2}\overline{\mu}^2-\overline{\mu}\sigma_D\left(\widehat{\gamma}-1\right)>0\end{array}\right\}$$

then we can easily get the result in the proposition