

# General Equilibrium Analysis with Stochastic Benchmarking

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**Abstract:** This paper applies a continuous-time model to study the equilibrium of an economy consisting one normal agent and one constrained agent who has to maintain his intermediate wealth above a stochastic benchmark whose value is determined endogenously. After characterizing the optimization problems, I use the martingale approach to derive the equilibrium market dynamics in closed-form to compare with the normal economy. In the benchmarking economy before the constraint date, asset price, volatility and risk premium are higher than those in the normal economy for risky benchmarks. When the benchmark is relatively safe, asset price is higher but volatility, risk premium and the optimal fraction of wealth invested in the risky asset are decreased.

This paper studies the effect of a portfolio benchmarking constraint on the equilibrium market dynamics. The economy has a normal agent with log utility over continuous consumption and a constrained agent (also with log utility) whose intermediate wealth must be no less than a stochastic benchmarking constraint. Using the martingale approach, I solve this continuous-time consumption-based general equilibrium model explicitly to get the equilibrium assets prices, risk premium, volatility, optimal strategy and compare them with those in a normal economy. To my knowledge, this is the first paper to investigate the equilibrium effects of this benchmarking constraint.

The constraint studied here is a requirement that an agent's wealth is no less than a stochastic benchmark index, whose parameters are exogenously given but the value of this benchmark is determined endogenously. The problem of beating a constant floor has been studied by the portfolio insurance literature, e.g. the equilibrium analysis of portfolio insurance by Basak (1995) and Grossman and Zhou (1996). However, the portfolio insurance constraint only ensures the agent doesn't lose more than a certain level without asking for a high return when the state is good. The inclusion of a stochastic benchmark index in the stochastic benchmarking constraint means the agent's performance has to beat the index when market does well and therefore, imposing such a constraint on agent ensures good performance. Since there are only two assets in the economy, a risk-free asset and a risky stock, and markets are complete, the stochastic benchmark is a replication portfolio using the bond and stock. Then, the riskiness of the benchmark is measured by the positions in the risk-free asset. Since the benchmark index is achievable through passive management, it is considered as the minimum return that is required for active portfolio management.

After describing the economy, which apart from the existence of the constraint is the standard set-up following Lucas (1978), I first characterize the portfolio choice problems for the

normal agent and the constrained agent. The approach is the martingale representation approach as in Cox and Huang (1989) and Basak (1995). Then I get explicit solutions for the equilibrium market dynamics using the martingale approach for option pricing and Ito's lemma. Similar procedure has been used by Basak (1995) but the alternative consideration of the stochastic benchmark complicates the discussion of equilibrium effects.

I find that in the stochastic benchmarking economy before the constraint date, the risky asset price is higher than that in the normal economy because the constrained agent consumes less than if he's not constrained. Since the risk-free asset is in zero net supply, the extra investment from the constrained agent goes into the risky asset market and drives up the risky asset's price. After the constraint date, there's no effect. The increase in stock price also reflects the agent's preference for consumption and dividend after the constraint date. When the benchmark is risky, which means the replication portfolio has a short position on the risk-free asset, then the risk premium, volatility, and the optimal fraction of wealth invested in the risky asset by the constrained agent are increased by the presence of the constraint. While if the benchmark is relatively safe, i.e. the replication portfolio contains positive position in the risk-free asset, then the risk premium, volatility, and the optimal fraction of wealth invested in the risky asset by the constrained agent are always decreased by the presence of the constraint. In both cases, the degree of the effects are state-dependent.

The rationale behind these findings is when there are more (less) demand for the risky asset from the constrained agent, the volatility has to increase (decrease) so as to clear the market. It is also because that the stock price in this economy is equal to the normal economy price plus the present value of an option-like payoff that represents the effects from the benchmarking constraint. For the risky benchmark case, the extra payoff is like a call

option payoff and the effect is thus volatility increasing; but for the safe benchmark case extra payoff is like a put option payoff and the effect becomes volatility decreasing. From the modeling aspect, the market price of risk is constant and the SPDs before the constraint date are not directly affected by the existence of the constraint due to the use of the consumption good as numeraire and the existence of intermediate dividend for consumption, which are not true in some other papers that have contrary conclusions for certain parts of the model, for instance Grossman and Zhou (1996). However, the conclusion of this paper is consistent with that for the portfolio insurance model of Basak (1995) which has the similar set-up and approach with this paper.

The closest literatures are the equilibrium analysis of portfolio insurance by Basak (1995) and Grossman and Zhou (1996). Basak (1995) builds a similar consumption-based general equilibrium model and compares the explicit expressions for equilibrium market dynamics in the portfolio insurance economy with those in the normal economy. The portfolio insurers' strategies are similar to the synthetic put approach and the presence of the intermediate portfolio insurance constraints decreases the risk premium, volatility and optimal fraction of wealth invested in the risky asset. Although there are multiple normal and constrained agents, the selection of log utility ensures the SPDs are not affected by the constraints since they are derived by market clearing of intermediate consumption. In contrast, Grossman and Zhou (1996) adopts a different set-up in which the portfolio insurance constraint is on the final date and there's no intermediate consumption so agents only care about consumption at the final date which is financed by a lump-sum of dividend. Therefore, the pricing kernels before the final date are directly affected by the constraint and that makes the overall effect of portfolio insurance to be increasing risk premium and volatility. However, the use of bond price as the numeraire results in different predictions with Basak (1995) and makes the model impossible to be solved explicitly. As mentioned above, these two papers

only consider the case of portfolio insurance which is benchmarking on a constant floor while Tepla (2001) studies the optimal portfolio choice of an agent who performs against a stochastic benchmark similar to the one considered here but doesn't derive the equilibrium results.

The rest of the paper is organized as follows. Section 1 presents the model and characterizes the optimization problems of agents. Section 2 solves for the equilibrium and provides the main results on the effect of the stochastic benchmarking constraint. Then Section 3 presents more discussion on the equilibrium effects. Finally, Section 5 concludes.

# 1 The Model

## 1.1 The Economy

In a finite horizon  $[0, T']$  pure-exchange economy that has a single consumption good, let  $B$  denote a Brownian Motion on a complete probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ . Let  $\{\mathcal{F}_t; t \in [0, T']\}$  be the augmentation by null sets of the filtration generated by  $B$  which represents all uncertainties in the economy.

### 1.1.1 Securities

There are two securities.  $S$  is a risky stock in constant net supply of 1 and pays dividend at rate  $\delta_t$  in  $[0, T']$ . The dividend process follows a Geometric Brownian Motion.

$$d\delta_t = \delta_t(\mu_\delta dt + \sigma_\delta dB_t), t \in [0, T'] \quad (1)$$

$$dS_t + \delta_t dt = S_t(\mu_t dt + \sigma_t dB_t + qA_t), t \in [0, T'] \quad (2)$$

The stock price is a diffusion process with an  $\mathcal{F}_T$ -measurable jump at time  $T$ , which is the constraint date before the final date  $T'$ .

$S^0$  is a risk-less asset (money market account) in zero net supply and also has a jump, which together with the jump in  $S_t$  account for the anticipated discontinuity in the equilibrium.

$$dS_t^0 = S_t^0(r_t dt + q^0 A_t), t \in [0, T'] \quad (3)$$

Here,  $A_t \equiv 1_{t \geq T}$ . The  $\mathcal{F}_T$ -measurable random variable  $q$  is the jump size parameter. To rule out arbitrage,  $q = q^0 = \ln(S_T/S_{T-})$  where  $S_{T-}$  denotes the left limit of  $S_T$ .

The asset price is continuous semi-martingale because the demand for consumption will increase discontinuously immediately after the constraint date  $T$ . Thus the demand for the risky asset will jump down, resulting in a downward jump in asset price. The jump is necessary for the equilibrium conditions to hold, which will be discussed in the section for equilibrium.

### 1.1.2 State Price Density

To apply the martingale approach, consider the state price density process (SPD),

$$\pi_t = \frac{1}{S_0^0} \exp\left(-\int_0^t r_s ds - \int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds - q A_t\right) \quad (4)$$

where  $\theta_t = (\mu_t - r_t)/\sigma_t$  is the market price of risk. The SPD process also contains a jump.

Apply Ito's lemma to  $\pi_t$ ,

$$d\pi_t = -\pi_t(r_t dt + \theta_t dB_t + q A_t), t \in [0, T'] \quad (5)$$

and use the SPD process, the relationship between the stock price and future dividends is

$$S_t = \frac{1}{\pi_t} E\left[\int_t^{T'} \pi_s \delta_s ds \mid \mathcal{F}_t\right], t \in [0, T'] \quad (6)$$

### 1.1.3 Agents

There are two agents in the economy, i.e. agent  $n$  and agent  $m$ , each endowed with  $x_{n0}$  and  $x_{m0}$  at time zero. Let  $X_{it}$  denote the optimal invested wealth for agent  $i$  at time  $t$ ,  $i = n, m$ .

$$dX_{it} = (1 - \Phi_{it})X_{it}(r_t dt + qA_t) + \Phi_{it}X_{it}(\mu_t dt + \sigma_t dB_t + qA_t) - c_{it} dt, i = n, m, t \in [0, T'] \quad (7)$$

where  $\Phi_{it}$  denotes the optimal fraction of wealth invested in the risky asset and  $c_{it}$  is the consumption process for agent  $i$ . Both agents have a time-additive state-independent utility function for consumption. The function  $u(c_{it})$  is the same for both agents and is continuous with continuous first derivatives, strictly increasing, strictly concave. Hereafter, all optimal quantities are denoted with a caret ( $\hat{\phantom{x}}$ ).

$$\hat{X}_{it} = \frac{1}{\pi_t} E\left[\int_t^{T'} \pi_s \hat{c}_{is} ds \mid \mathcal{F}_t\right], t \in [0, T'] \quad (8)$$

Agent  $n$  is the normal agent while agent  $m$  is the constrained benchmarker who faces a stochastic benchmarking constraint at time  $T$  so that he has to maintain his time  $T$  wealth above a stochastic benchmark  $aS_T + b$  where  $a$  and  $b$  are exogenously given constants. So the constraint is set to be  $X_{mT-} \geq aS_T + b$ . Refine  $aS_0 + b = x_{m0}$  then the value of  $b$  characterize the riskiness of the benchmark and the choice of parameter values has important impact on the equilibrium properties.

## 1.2 The Optimization Problems

The normal agent solves the following problem:

$$\begin{aligned} & \max_{c_n} E\left[\int_0^{T'} u(c_{ns}) ds\right] \\ & \text{subject to } E\left[\int_0^{T'} \pi_s c_{ns} ds\right] \leq \pi_0 x_{n0} \end{aligned}$$

Assuming a solution exists, the solution is:

$$\hat{c}_{nt} = I(\lambda_n \pi_t) \quad t \in [0, T'] \quad (9)$$

where  $I(\cdot)$  is the inverse of  $u'(\cdot)$  and  $\lambda_n$  solves:

$$E\left[\int_0^{T'} \pi_s I(\lambda_n \pi_s) ds\right] = \pi_0 x_{n0} \quad (10)$$

The constrained agent  $m$  solves:

$$\begin{aligned} & \max_{c_m, X_{mT-}} E\left[\int_0^{T'} u(c_{ms}) ds\right] \\ & \text{subject to } E\left[\int_0^T \pi_s c_{ms} ds + \pi_{T-} X_{mT-}\right] \leq \pi_0 x_{n0} \\ & E\left[\int_T^{T'} \pi_s c_{ms} ds \mid \mathcal{F}_T\right] \leq \pi_{T-} x_{mT-} \text{ almost surely,} \\ & X_{mT-} \geq aS_T + b \text{ almost surely} \end{aligned}$$

**Lemma 1.**

Assuming a solution exists, the solution is:

$$\hat{c}_{mt} = I(\lambda_{m1} \pi_t) \quad t \in [0, T) \quad (11)$$

$$\hat{c}_{mt} = I(\lambda_{m2} \pi_t) \quad t \in [T, T'] \quad (12)$$

where  $\lambda_{m1}$  and  $\lambda_{m2}$  solve:

$$E\left[\int_0^T \pi_s I(\lambda_{m1} \pi_s) ds + \pi_{T-} \max\left\{aS_T + b, \frac{1}{\pi_{T-}} E\left[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds \mid \mathcal{F}_T\right]\right\}\right] = \pi_0 x_{m0} \quad (13)$$

$$E\left[\int_T^{T'} \pi_s I(\lambda_{m2} \pi_s) ds \mid \mathcal{F}_T\right] = \pi_{T-} \max\left\{aS_T + b, \frac{1}{\pi_{T-}} E\left[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds \mid \mathcal{F}_T\right]\right\} \quad (14)$$

N.B.(1)  $\lambda_n, \lambda_{m1}$  are constant,  $\lambda_{m2}$  is  $\mathcal{F}_T$ -measurable random variable. (2) If  $x_{n0} = x_{m0}$ , then  $\lambda_{m1} \geq \lambda_n$ . (3)  $\lambda_{m1} = \lambda_{m2}$  if the benchmarking constraint is not binding and  $\lambda_{m1} > \lambda_{m2}$  if

it is binding.

For the optimal invested wealth at time  $T$ ,

$$\hat{X}_{nT-} = \frac{1}{\pi_{T-}} E\left[\int_T^{T'} \pi_s I(\lambda_n \pi_s) ds \mid \mathcal{F}_T\right] \quad (15)$$

$$\hat{X}_{mT-} = \max\left\{aS_T + b, \frac{1}{\pi_{T-}} E\left[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds \mid \mathcal{F}_T\right]\right\} \quad (16)$$

## 2 The Equilibrium

In the rest of this paper, I first solve for the equilibrium in this benchmarking economy and then compare it with the normal economy in which only normal agents exist. For simplicity, we only consider log-utility while similar approach can be applied to power and negative exponential utilities.

The equilibrium condition is given by market clearing:

$$\delta_t = \hat{c}_{nt} + \hat{c}_{mt} \quad t \in [0, T'] \quad (17)$$

Thus in this case, the derivation of SPD follows Basak (1995):

$$\delta_t = I(\lambda_n \pi_t) + I(\lambda_{m1} \pi_t) \quad t \in [0, T] \quad (18)$$

$$\delta_t = I(\lambda_n \pi_t) + I(\lambda_{m2} \pi_t) \quad t \in [T, T'] \quad (19)$$

For  $u(c) = \log(c)$ , the solution for  $\pi_t$  is:

$$\pi_t = \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}}\right) \frac{1}{\delta_t} \quad t \in [0, T] \quad (20)$$

$$\pi_t = \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}}\right) \frac{1}{\delta_t} \quad t \in [T, T'] \quad (21)$$

Apply Ito's lemma on  $\pi_t$  which follows the dynamics in equation (5),

$$r_t = \mu_\delta - \sigma_\delta^2 = r \quad (22)$$

$$\theta_t = \sigma_\delta = \theta \quad (23)$$

$$q = \ln\left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}}\right) - \ln\left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}}\right) \leq 0 \quad (24)$$

The short-rate and market price of risk are constant, while the SPD has a jump at time  $T$ . If the SPD is continuous, the normal agent's demand for consumption will also be continuous. But when agent  $m$ 's constraint is binding, since  $\lambda_{m1} > \lambda_{m2}$  and  $\hat{c}_{mT-} = 1/\lambda_{m1}\pi_{T-}$ ,  $\hat{c}_{mT} = 1/\lambda_{m2}\pi_T$ , his demand for consumption will jump upwards immediately after time  $T$ , leading the aggregate demand for consumption jump upwards, which is impossible given a continuous dividend process. Hence, there must be a jump in the SPD to smooth agent  $m$ 's demand for consumption. And the jump in SPD is upward because agent  $m$  values consumption after the constraint date more than before the date.

Then, define the equity market as the total optimal wealth invested in the stock and denote it by  $\hat{X}_{emt}$ . Since the money market account is in zero net supply, the equity market is equal to the total optimal wealth invested by the two agents.

$$\hat{X}_{emt} = \hat{X}_{nt} + \hat{X}_{mt} \quad t \in [0, T'] \quad (25)$$

Also, since the risky asset is in net supply of 1,

$$S_t = \hat{X}_{emt} = \hat{X}_{nt} + \hat{X}_{mt} \quad t \in [0, T'] \quad (26)$$

## 2.1 Asset Prices

Now, we can derive explicit solutions for  $X_{emt}$ ,  $S_t$ ,  $\mu_t$ ,  $\sigma_t$ ,  $\phi_t$ . To compare with the normal economy, denote the normal economy terms by  $\bar{X}_{emt}$ ,  $\bar{S}_t$ ,  $\bar{\mu}_t$ ,  $\bar{\sigma}_t$ ,  $\bar{\phi}_t$ .

In the normal economy:

$$\begin{aligned}\bar{S}_t = \bar{X}_{emt} = \bar{X}_{nt} &= \frac{1}{\bar{\pi}_t} E\left[\int_t^{T'} \bar{\pi}_s \frac{1}{\lambda_n \bar{\pi}_s} ds \mid \mathcal{F}_t\right] \\ &= (T' - t)\delta_t \quad t \in [0, T']\end{aligned}\tag{27}$$

In the benchmarking economy, for  $t \in [T, T']$ :

$$\begin{aligned}S_t = X_{emt} &= \hat{X}_{nt} + \hat{X}_{mt} \\ &= \frac{1}{\pi_t} E\left[\int_t^{T'} \pi_s \frac{1}{\lambda_n \pi_s} ds \mid \mathcal{F}_t\right] + \frac{1}{\pi_t} E\left[\int_t^{T'} \pi_s \frac{1}{\lambda_{m2} \pi_s} ds \mid \mathcal{F}_t\right] \\ &= \frac{1}{\left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}}\right) \frac{1}{\delta_t}} \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}}\right) (T' - t) \\ &= (T' - t)\delta_t\end{aligned}\tag{28}$$

So  $S_t = \bar{S}_t = (T' - t)\delta_t$ , for  $t \in [T, T']$ . This is true because the benchmarking economy becomes the normal economy after  $T$ . Then we study the economy before  $T$ .

**Lemma 2.** For  $t \in [0, T)$ :

$$\begin{aligned}S_t = X_{emt} &= \hat{X}_{nt} + \hat{X}_{mt} \\ &= \frac{1}{\pi_t} E\left[\int_t^{T'} \pi_s \frac{1}{\lambda_n \pi_s} ds \mid \mathcal{F}_t\right] + \frac{1}{\pi_t} E\left[\int_t^T \pi_s \frac{1}{\lambda_{m1} \pi_s} ds + \pi_{T-} \hat{X}_{mT-} \mid \mathcal{F}_t\right] \\ &= \frac{\lambda_{m1}}{\lambda_n + \lambda_{m1}} (T' - t)\delta_t + \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T - t)\delta_t \\ &\quad + \frac{1}{\pi_t} E\left[\pi_{T-} \max\{aS_T + b, \frac{1}{\pi_{T-}} E\left[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds \mid \mathcal{F}_T\right]\} \mid \mathcal{F}_t\right] \\ &= \frac{\lambda_{m1}}{\lambda_n + \lambda_{m1}} (T' - t)\delta_t + \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T - t)\delta_t + \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T)\delta_t \\ &\quad + \frac{1}{\pi_t} E\left[\pi_{T-} \max\left\{aS_T + b - \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T)\delta_{T-}, 0\right\} \mid \mathcal{F}_t\right] \\ &= \bar{S}_t + \frac{1}{\pi_t} E\left[\pi_{T-} \max\left\{aS_T + b - \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T)\delta_{T-}, 0\right\} \mid \mathcal{F}_t\right]\end{aligned}\tag{29}$$

N.B.  $S_t \geq \bar{S}_t$ , for  $t \in [0, T)$ . This is true because before time  $T$  agent  $m$  consumes less than if he's unconstrained. So he invests more than he would if unconstrained. Since the risk-free asset is in zero net supply, the extra investment from agent  $m$  goes into the stock market and drives the price up before time  $T$ .

Now focus on the explicit solution of  $S_t$  for  $t \in [0, T)$ . From the equation (28),  $S_T = (T' - T)\delta_T$ , and by the continuity of  $\delta_t$ ,  $\delta_{T-} = \delta_T$ . So, replace  $S_T$  by  $(T' - T)\delta_{T-}$ , rewrite equation (29) as:

$$S_t = (T' - t)\delta_t + \frac{1}{\pi_t} E[\pi_{T-} \max\{(a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})(T' - T)\delta_{T-} + b, 0\} | \mathcal{F}_t] \quad (30)$$

Since the process  $\delta_t$  is a Geometric Brownian Motion, we can apply the martingale approach to take the expectation in equation (30). Hereafter, we assume  $(a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})b < 0$  which means there's uncertainty over the relative performance of the benchmark and the unconstrained strategy. Now, solve equation (30), we have:

**Proposition 1.** In the benchmarking economy,

$$S_t = (T' - t)\delta_t + (a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})(T' - T)N(z_1)\delta_t + be^{-r(T-t)}N(z_2) \quad (31)$$

where

$$\begin{aligned} z_1 &= z_2 + \sigma_\delta \sqrt{T-t} \\ z_2 &= \frac{\ln[(a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})(T' - T)\delta_t / -b] + (r - \frac{1}{2}\sigma_\delta^2)(T-t)}{\sigma_\delta \sqrt{T-t}} \end{aligned}$$

$N(\cdot)$  is the distribution function of normal random variable. And we also have:

$$\hat{X}_{mt} = S_t - \hat{X}_{nt} = S_t - \frac{\lambda_{m1}}{\lambda_n + \lambda_{m1}}(T' - t)\delta_t \quad (32)$$

## 2.2 Volatility and Risk Premium

Apply Ito's lemma on  $S_t$ , we get explicit solutions for the volatility and risk premium in both the normal and benchmarking economies.

### *Proposition 2.*

In the normal economy, for  $t \in [0, T']$ ,

$$\bar{\sigma}_t = \sigma_\delta \quad (33)$$

$$\bar{\mu}_t - r = \sigma_\delta^2 \quad (34)$$

In the benchmarking economy, for  $t \in [T, T']$ ,

$$\sigma_t = \sigma_\delta \quad (35)$$

$$\mu_t - r = \sigma_\delta^2 \quad (36)$$

While for  $t \in [0, T)$ ,

$$\sigma_t = \left[1 - \frac{be^{-r(T-t)}N(z_2)}{S_t}\right]\sigma_\delta \quad (37)$$

$$\mu_t - r = \left[1 - \frac{be^{-r(T-t)}N(z_2)}{S_t}\right]\sigma_\delta^2 \quad (38)$$

Whether the volatility and risk premium in the benchmarking economy are greater or

less than those in the normal economy only depends on the sign of the fraction term inside the bracket. Therefore, for  $t \in [0, T)$ :

***Corollary 1.***

For  $b < 0$  ( $b > 0$ ), the volatility and risk premium in the benchmarking economy are higher (lower) than those in the normal economy.

The volatility and risk premium in the normal economy are constant. However, they're time-varying in the presence of the stochastic benchmarking constraint. In general, the volatility and risk premium will increase when the benchmark is risky ( $b < 0$  means the benchmark is equivalent to borrowing money to invest in the risky asset). While for relatively safe benchmarks (with  $b > 0$ ), volatility and risk premium will decrease. In both cases, the degree of the increase or decrease is state-dependent. To better understand the results for volatility, I further investigate agents' optimal strategies.

## 2.3 The Optimal Strategy

Denote the optimal fraction of wealth invested in the risky asset by  $\hat{\Phi}_{nt}$  and  $\hat{\Phi}_{mt}$  for agent  $n$  and agent  $m$ . Apply Ito's lemma on  $\pi_t \hat{X}_{nt}$  and  $\pi_t \hat{X}_{mt}$ , we get explicit solutions for  $\hat{\Phi}_{nt}$  and  $\hat{\Phi}_{mt}$  in the benchmarking economy.

***Proposition 3.***

For  $t \in [0, T']$ ,

$$\hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2 \quad (39)$$

While for  $\hat{\Phi}_{mt}$ , when  $t \in [T, T']$ ,

$$\hat{\Phi}_{mt} = \hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2 \quad (40)$$

When  $t \in [0, T)$ ,

$$\hat{\Phi}_{mt} = [1 - \frac{be^{-r(T-t)}N(z_2)}{\hat{X}_{mt}}] \frac{\mu_t - r}{\sigma_t^2} \quad (41)$$

Similar to the analysis on volatility and risk premium, we have:

***Corollary 2.***

For  $b < 0$  ( $b > 0$ ), the constrained agent invests more (less) fraction of wealth in the risky asset than the normal agent.

### 3 Discussion of the Equilibrium

We have seen that when the benchmark is equivalent to borrowing an amount of money and investing in the risky asset, the existence of this risky benchmark increases risk premium, volatility conditionally. While when the benchmark is relatively safe, those terms are smaller in the benchmarking economy than in the normal economy. In both cases, the risky asset price is always higher in the presence of the benchmarking constraint than in the normal economy. The conclusion here is consistent with that of Basak (1995), in which the portfolio insurance constraint can be viewed as a special case of what we study here.

Such findings may be contrary to conventional wisdom, e.g. Grossman and Zhou (1996), which says the inclusion of the portfolio insurance constrained agent should increase volatility and risk premium. However, the results here are understandable in the following ways:

Firstly, seen from proposition 3 and corollary 2, the effect of volatility can be explained by the optimal fraction of wealth invested in the stock by both agent. In the normal economy, both agents optimally invest all their wealth into the stock (as implied by the log-utility). In the safe benchmark case, the constrained agent has to invest less in the stock so as to hold some risk-free asset. To clear the market, it means the unconstrained agent has to take more than he would in the normal economy. Since we have,

$$\hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2 = \theta/\sigma_t \quad (42)$$

so the only way to make the unconstrained agent be willing to hold more stock is to decrease  $\sigma_t$ . While if the benchmark is risky, the constrained agent may have to hold more stock. So the volatility has to increase to induce the unconstrained agent to sell. This suggests that in this economy which has a constant market price of risk ( $\theta$ ), the volatility of the stock has to decrease (increase) so as to make it more (less) attractive to the unconstrained agent. In both case, since the constrained agent has to adjust his demand for the stock conditionally, the degree to which the volatility is increased or decreased is therefore state-dependent.

In another attempt to explain the effect on volatility, recall the expression for  $S_t$  before  $T$  in equation (30). The stock price in the benchmarking economy is the normal economy price plus an expectation term which is the present value of an option-like payoff, which is a call option payoff when the benchmark is risky, and a put option payoff when the benchmark is safe. For the safe benchmark case, when the market goes down, the put option value goes up, which helps stabilize stock price. So the volatility is decreased in the presence of the constraint. However, in the risky benchmark case, when the market goes down the call option value goes down as well so the stock price falls even further and the constraint increases volatility.

The results for the safe benchmark case is consistent with Basak (1995) which has the similar set-up in using consumption as numeraire and having intermediate consumption, dividend and constraint date. These features result in a constant market price of risk as compared to the price of consumption. But in Grossman and Zhou (1996), the market price of risk is time-varying as they used bond price as numeraire and there's no intermediate consumption so the pre-constraint pricing kernels are conditional expectations of the final one and therefore directly affected by the constraint. Therefore, the volatility is affected by both the change in market price of risk and the change in agent's risk aversion, and they show that the overall effect is a higher volatility.

## 4 Concluding Remarks

This paper studies the equilibrium effect of a stochastic benchmarking constraint. The economy has a normal agent with log utility over continuous consumption and a constrained agent whose intermediate wealth must meet a stochastic benchmarking constraint. Using the martingale approach, I solve this consumption-based general equilibrium model explicitly to get the equilibrium assets prices, risk premium, volatility, optimal strategy and compare them with those in a normal economy. The problem of the equilibrium effects of portfolio insurance studied by Basak (1995) and Grossman and Zhou (1996) is a special case of the problem studied here.

In the stochastic benchmarking economy before the constraint date, the risky asset price is higher than that in the normal economy because the constrained agent consumes less than if he's not constrained. Since the risk-free asset is in zero net supply, the extra investment flows into the risky asset market and drives up the risky asset's price. When the benchmark

is chosen to be a risky one, i.e. can be replicated by borrowing an amount of money to invest in the risky asset, then the risk premium, volatility are increased by the presence of the constraint. While if the benchmark is relatively safe, i.e. the replication portfolio of which contains positive position in the risk-free asset, the risk premium, volatility are decreased. In both cases, the degree of the increase or decrease are state-dependent.

The rationale behind these findings is when there are more (less) demand for the risky asset from the constrained agent, the volatility has to increase (decrease) so as to clear the market. This is true because in this model the market price of risk is constant and the SPDs before the constraint date are not directly affected by the existence of the constraint due to the existence of intermediate dividend for consumption. The model has consistent results with the portfolio insurance model of Basak (1995) that has the similar set-up and approach with this paper.

In an earlier version of the paper, I considered a more general case in which the constrained agent has to maintain time  $T$  wealth above the maximum of the stochastic constraint and a constant floor (i.e.  $\max[aS_T + b, K]$ ). Thus we can view the overall effect of this constraint as the mixture of the effect from the benchmark constraint and the effect from the floor constraint. The individual effect of the floor constraint is volatility-decreasing, so does that for a safe benchmark constraint. But the effect of a risky benchmark constraint is volatility-increasing. So the individual effects from the benchmark constraint and the floor constraint are in the same direction if the benchmark is safe; and they're in opposite directions in the benchmark is risky. Therefore, for a risky benchmark, when market is good, a risky benchmark constraint is more likely to bind. Its volatility-increasing effect overshadows the volatility-decreasing effect of the floor constraint so the overall effect is volatility-increasing. But when market is bad, the floor constraint's effect is stronger as

it's more like to bind, so the overall effect is volatility-decreasing. For the safe benchmark case, the two individual effects are both volatility-decreasing, so does the overall effect.

## 5 Appendix (very brief proof of some results)

**Proof of Lemma 1:** see Basak (1995)'s proof of lemma 3, replace  $K$  by  $aS_T + b$ .

**Proof of Proposition 1:** Follow the martingale approach for the Black-Scholes option pricing formula.

**Proof of Proposition 2:** Apply Ito's lemma on the explicit solution for  $\bar{S}_t$  and  $S_t$ , get the expression for the drift and diffusion terms. Also, we have

$$dS_t + \delta_t dt = S_t(\mu_t dt + \sigma_t dB_t), t \in [0, T) \quad (43)$$

by definition, so we can solve for  $\mu_t - r$  and  $\sigma_t$ . After getting the explicit solution for  $\sigma_t$ , we can then use the relationship that  $(\mu_t - r)/\sigma_t = \theta_t = \sigma_\delta$  to derive  $\mu_t - r$ .

**Proof of Proposition 3:** Applying Ito's lemma on the product  $\pi_t \hat{X}_t$  gives

$$d(\pi_t \hat{X}_t) + \pi_t \hat{c}_t dt = \pi_t \hat{X}_t [\hat{\Phi}_t \sigma_t - \theta_t] dB_t \quad (44)$$

again apply Ito's lemma on the explicit solutions of  $\hat{X}_{nt}$  and  $\hat{X}_{mt}$ , equalizing the diffusion term solves  $\hat{\Phi}_{nt}$  and  $\hat{\Phi}_{mt}$ .

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