Guarding the Guardians*

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Abstract

Good government requires some restraints on the powerful, but how can those be imposed if there is no-one above them? This paper studies the equilibrium allocation of power and resources established by self-interested incumbents under the threat of rebellions from inside and outside the group in power. Commitment to uphold individuals' rights can only be achieved if power is not as concentrated as incumbents would like it to be, ex post. Power sharing endogenously enables incumbents to commit to otherwise time-inconsistent laws by ensuring more people receive rents under the status quo, and thus want to defend it.

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> Princes who want to make themselves despotic have always begun by uniting all magistracies in their person.
>
> Montesquieu (1748), *De l'esprit des lois*

1 Introduction

Economic development requires restrictions on the untrammelled exercise of power by incumbents. For example, a system of private property entails a commitment to respect individuals' rights at times when expropriation is physically feasible and in the interests of those in power. But how can the rule of law be upheld when there is no force above the group in the power that can control its acts? This is the classic question of 'who will guard the guardians?'

The methodological contribution of this paper is a framework suitable for studying this question. Incumbents can establish an allocation of power and resources. Power can be shared, so incumbency is endogenous. There is a mechanism through which any collection of individuals (including some or all of the incumbents) can contest the existing allocation. Individuals with power may use it to defend the status quo against such attacks. The model thus determines the allocation of power and resources laid down by self-interested incumbents subject to the struggle for power — the threat that an allocation can always be contested by any group. The framework can be used to understand whether and how there can be protection of individual rights.[1]

In the model, there is no aspect of the allocation of power and resources that will stand outside the struggle for power: the mechanism for contesting an existing allocation is the same no matter what that allocation prescribes. Likewise, there are no individuals who inherently have either a special desire or ability to defend particular features of an allocation: everyone is ex ante identical. There are also no exogenous restrictions on what allocations of resources are feasible through the use of taxes or transfers.

The main result of the paper is that it is possible to guard the guardians by sharing power. Credible commitment to protect individual rights — the 'rule of law' — can be achieved if and only if power is not as concentrated as incumbents would like it to be, ex post. Sharing power (and the rents that go with it) works as a commitment device not because power is shared with individuals who have some specific characteristics or roles, but because it increases the number of individuals willing to defend the status quo against any threats coming from inside or outside the group in power. Although incumbents are free to devise any system of transfers among individuals, power sharing is a necessary and sufficient condition for commitment to otherwise time-inconsistent laws.

One implication of the model is that an increase in investment returns due to new technological discoveries spurs incumbents to share power, which gives rise to the rule of law — in the sense of protecting investors' property from expropriation. The model is thus consistent with the emergence

[1] It would be possible to introduce an exogenous 'higher-level' institutional technology that directly allows for commitment to laws (perhaps at some cost) into a standard model of political economy. However, this would be as a *deus ex machina* that simply overrides the ability of incumbents to exploit their own power, failing to do justice to the question if the aim is to understand how commitment to laws is actually achieved.

of parliaments and independent courts as a consequence of technological progress that increases the gains from long-term investments.[2] Greater returns from investment provide an incentive for incumbents to share power more broadly in order to increase the amount of output available for them to tax, even though this will dilute the rents they receive. Another implication of the model is that the reach of the rule of law in equilibrium will fall short of what is efficient because sharing power entails sharing rents.

The 'rule of law' referred to above is a situation where laws are followed even when incumbents would like to set them aside, that is, where there is some mechanism for guarding the guardians. This ought not to be confused with democracy, which can be seen as a set of laws determining the allocation of resources in society in accordance with the results of elections. The rule of law is thus a more fundamental feature of a state and a crucial building block of any well functioning economy. In fact, an unrestrained median voter would generally not respect property rights because expropriation and redistribution would benefit the majority ex post.

The remainder of the introduction describes the workings of the model and compares it to related literature. Section 2 then presents the model of the power struggle in a simple setting where commitment issues do not arise. The fully fledged model with investment is analysed in section 3. Finally, section 4 draws some conclusions.

1.1 The workings of the model

The model begins with an allocation of power and resources established in the interests of incumbents (who take account of what will happen next). There are then opportunities to contest the allocation, referred to as rebellions. Rebellions can come from any subset of the population, including some or all of those in power. A successful rebellion leads to another round of this process, and so on. When no rebellions occur, the prevailing allocation of resources is implemented. In equilibrium, rebellions will not occur, but the threat of them is crucial in determining the nature of the allocation that will prevail.

At the heart of the model is a theory of the distribution of power and resources, which is first illustrated in a simple endowment economy where commitment issues have no bearing on output. Incentives to rebel depend on the payoffs the rebels expect to receive under a new allocation as the new incumbents. Since all individuals are ex ante identical and since the economic environment does not change, the equilibrium allocation is the fixed point of the constrained maximization problem of the incumbents subject to the threat of rebellion, where subsequent incumbents would be similarly constrained by equivalent threats of rebellion. As in George Orwell's *Animal Farm*, there is no essential difference between the 'pigs' and the 'men' they replace, but in equilibrium, some individuals will be 'more equal' than others.

In the endowment economy, the equilibrium distribution of resources is tied to the distribution

[2] The implication that technological progress leads to greater power sharing is testable in principle. One challenge is that stronger protection of property rights might itself boost technological progress by raising the return to research and development. Testing the model would require disentangling these two channels, for example, through the use of an instrument for technological progress.

of power. Those with equal power receive the same payoff, and those with more power receive a higher payoff. The intuition is that in comparing two individuals of equal power, one with a lower payoff would have more to gain from rebellion and would therefore be willing to exert greater effort in contesting the allocation; while comparing two individuals with the same payoff, one with more power would pose a greater danger. Since any rebellion will be launched by a subset of the population, the incumbents are concerned with the group of individuals having the greatest incentive to rebel. This means rewarding the powerful to keep them on side, while otherwise equalizing payoffs to avoid concentrating disenchantment. Sharing power thus always entails sharing rents.

There is a basic trade-off in this environment that characterizes the equilibrium allocation of power. On the one hand, the greater the number of individuals sharing power as the incumbent group, the greater their ability to defend the status quo against rebellions, which allows them extract more rents from those outside the group. On the other hand, the rents must then be divided more thinly among more individuals. The equilibrium size of the incumbent group maximizes the payoff of each member by striking a balance between these two effects.

The fully fledged model adds an investment technology to the environment described above, which will give rise to a time-inconsistency problem. Individuals who invest incur an immediate effort cost, while the fruits of their investment are realized only after a lag. During this time, there is the ever-present opportunity for any group of individuals to launch a rebellion. Were a rebellion to occur after investments have been made, the group in power following the rebellion would have an incentive to expropriate fully investors' capital because the effort cost of investing would then be sunk.

To provide appropriate incentives for individuals to invest, the allocation of resources laid down prior to investment decisions must offer investors a higher payoff, and importantly, the allocation must survive rebellions so that what it prescribes is actually put into practice. In an endowment economy, the incumbents' principal concern is in avoiding a 'popular uprising', a rebellion of outsiders. When offering incentives to investors, the danger of this type of rebellion increases, but it also becomes essential to avoid a 'coup d'état', a rebellion launched by insiders. The higher payoff enjoyed by investors is only in the interests of the incumbents ex ante, so ex post there is a time-inconsistency problem. In other words, the incumbents themselves want to rebel so they can choose a new allocation of resources that does not protect investors' property rights.

It is therefore necessary to reduce the incentive to rebel simultaneously for those inside and outside the incumbent group. This can only be done by expanding the size of the group in power — the problem cannot be solved by any alternative allocation established ex ante that specifies how resources will be distributed ex post. If higher payoffs were offered to some to reduce their willingness to rebel, resources would have to be taken away from others, increasing their incentive to rebel. Fundamentally, transfers of resources can only redistribute disgruntlement with an allocation. In contrast, enlarging the group in power reduces the attractiveness of all types of rebellions by increasing the number of individuals who will lose power if a rebellion succeeds. This increases the size of the group willing to defend the current allocation.

Simply adding the possibility of investment to the model therefore gives rise to an equilibrium with power sharing among a larger group. Sharing power is an *endogenous* commitment mechanism that allows the incumbents to act as a government bound to a set of laws that would otherwise be time inconsistent. The analysis thus highlights the importance of sharing power as a way of guaranteeing commitment to individuals' rights, allowing in particular for incentives to invest. This resonates with Montesquieu's doctrine of the separation of powers, now accepted and followed in well-functioning systems of government. Note that power is not shared here with those individuals who are actually investing. The additional individuals in power in no sense represent nor care about those who invest — but they do care about their own rents under the status quo. By this means, a group of self-interested individuals is able to act as a government constrained by the rule of law.

Although investors generally receive some protection against expropriation in equilibrium, power sharing and hence the reach of the rule of law fall short of what is socially optimal. Equilibrium investment is inefficiently low in the sense that total output available for consumption could be increased by having a larger group in power to reduce the proportion of investors' returns that is expropriated. The intuition comes from the inseparability of power and rents, which follows from the threat posed by powerful individuals were conflict to occur. It is not possible in equilibrium for incumbents to share power with more individuals yet not grant them the same payoff as their equally strong peers. This places an endogenous and binding limit on the set of possible transfers, so some Pareto-improving deals remain unfulfilled.

1.2 Comparison with the existing literature

The idea of sharing power as a commitment device might be reminiscent of the role played by the extension of the franchise in Acemoglu and Robinson (2000) and Jack and Lagunoff (2006). As in that literature, there is too little protection of property rights in the model here. However, there are substantial differences in terms of both assumptions and results. First, in those papers, extending the franchise means choosing an intrinsically different median voter who will, in future periods, choose policies to which the current elite would like to commit.[3] That type of model assumes the existence of democratic institutions. Here, in contrast, sharing power means simply enlarging the incumbent group (all individuals are ex ante identical). This does not rely on exogenous differences in preferences and a world where electoral outcomes are always respected, but on changes to the costs of rebellions that in turn affect the allocation of power and resources prevailing in equilibrium.

Second, protecting property rights here means protecting the interests of a minority, which is most likely true in practice: expropriation and redistribution will always benefit the majority (ex post) as long as the mean of the distribution of wealth is larger than the median. This suggests democracy alone will be no guarantor of property rights. In contrast, this paper shows how the rule of law emerges through power sharing. By providing a mechanism for guarding the guardians, power sharing allows a minority not in power to invest even though those in power and a majority of the population would like to expropriate their capital ex post.

[3]Choosing a different median voter plays a similar role to delegation of monetary policy to a 'conservative' central banker in Rogoff (1985). For a related dynamic analysis of policymaking and political power, see Bai and Lagunoff (2011).

Consistent with this observation, as Acemoglu and Robinson (2000) point out, the extension of the franchise in England in the second half of the nineteenth century led to a large increase in taxes on the rich and a reduction in inequality. By then, the industrial revolution was well under way. The Glorious Revolution, which arguably led to the most decisive break with past government expropriations of property and greatly strengthened the rule of law in England, occurred at the end of the seventeenth century and did not coincide with any significant enfranchisement of the poor. Thus, both theoretically and empirically, it is at best unclear that enfranchisement of the poor has a positive effect on protection of property rights.

As discussed in Acemoglu (2003), if a 'political Coase theorem' were to hold, this would mean that political issues have distributional consequences but no implications for efficiency. However, Acemoglu argues that commitment problems are the major reason why a 'political Coase theorem' fails to apply in practice. This points to understanding better how commitment to rules can actually be maintained, which requires a model in which the ability to commit is not assumed but explained. Here, the breakdown of a 'political Coase theorem' is an implication of the means through which commitment occurs. The model shows that power and rents are inseparable, which implies an endogenous and binding constraint on the set of transfers among individuals. Commitment requires sharing power, which in turn requires sharing rents, and this is a cost from the point of view of those in power.[4]

The literature has proposed a number of mechanisms through which commitment to rules can be sustained, including protection of property rights. Work building on Kydland and Prescott (1977) has studied how commitment can arise in infinitely repeated games in a variety of macroeconomic settings. Moreover, if individual preferences were aggregated through a political process in which wealthy groups have more influence, as in Benabou (2000) and Bai and Lagunoff (2013), property owners would be in a stronger position to uphold their legal rights.

This paper is also related to a burgeoning literature on institutions.[5] The models in Acemoglu and Robinson (2006, 2008) also consider incumbent elites that are constrained by the threat of insurrection. In spite of this similarity, the questions addressed in those papers are very different. In particular, they do not study whether and how commitment can arise, and since they take the size of the group in power as exogenously given, there is no place for power sharing in their analysis. Furthermore, there are important differences in how those papers model conflict. The effects of institutional frictions and political turnover on investments in fiscal capacity and state capacity are studied in Besley and Persson (2009a,b, 2010), but questions about how commitments can be made and how power is shared are beyond the scope of those papers. Greif (2006) also develops game-theoretic models to shed light on political relations, with a focus on medieval history in particular.

The modelling strategy in this paper shares some features of the literature on coalition formation

[4]Other aspects of power sharing have been explored in the literature, for example, Persson, Roland and Tabellini (1997) and Francois, Rainer and Trebbi (2015).

[5]Many researchers have posited institutions as a significant determinant of economic development and the large differences found in the cross-country distribution of income. For example, see North (1990), North and Weingast (1989), Engerman and Sokoloff (1997), Hall and Jones (1999), and Acemoglu, Johnson and Robinson (2005).

(see Ray, 2007).[6] As in that literature, the process of establishing an allocation is non-cooperative, but it is assumed that in the absence of rebellion the allocation is actually implemented.[7] Moreover, the modelling of rebellions here is related to blocking in coalitions (Ray, 2007, part III) in the sense that there is no explicit game-form. One important distinction is the actual modelling of power and conflict in this paper. An incumbent coalition proposes an allocation, but blocking requires (among other things) that an alternative coalition is willing to put in enough fighting effort to displace the current incumbents. Furthermore, while that literature focuses on characterizing stable coalitions, in the sense of allocations for which there is no profitable deviation, this paper studies the allocation of power and resources in the interests of an incumbent group that takes account of the threat of rebellion.

The paper is also related to the literature on social conflict and predation, surveyed by Garfinkel and Skaperdas (2007).[8] It is easy to envisage how conflict could be important in a state of nature: individuals could devote their time to fighting and stealing from others. However, when there are fights, there are deadweight losses. Thus, it would be efficient if individuals could avoid conflict by agreeing and adhering to an allocation of resources. This paper supposes such deals are possible. Hence, differently from the literature on conflict, individuals may fight to be part of the group that determines the allocation, but not directly over what has been produced. Moreover, they fight in groups, not as isolated individuals.

Last, this paper is related to work focusing on political issues that lead to inefficiencies in protecting property rights, such as Glaeser, Scheinkman and Shleifer (2003), Acemoglu (2008), Guriev and Sonin (2009), and Myerson (2010). Those papers make assumptions on institutions to study how they affect the incentives of judges, oligarchs, dictators, and entrepreneurs, and how that translates into economic outcomes.[9] The objective and approach of this paper are different: here, assumptions are made on the power struggle to understand whether and how commitment to protect property rights can emerge in equilibrium.

[6] Baron and Ferejohn (1989) analyse bargaining in legislatures using that approach, while Levy (2004) studies political parties as coalitions. Other contributions include Chwe (1994), Koray (2000), Barbera and Jackson (2004), Acemoglu, Egorov and Sonin (2008), Piccione and Razin (2009), and Acemoglu, Egorov and Sonin (2012).

[7] What ensures the established allocation of resources is actually implemented ex post if no rebellion occurs? As pointed out by Basu (2000) and Mailath, Morris and Postlewaite (2001), laws and institutions do not change the physical nature of the game, all they can do is affect how individuals coordinate on some pattern of behaviour. One possible interpretation of this approach is similar to the application put forward by Myerson (2009) of Schelling (1960)'s notion of focal points in the organization of society. The 'rules of the game' are self enforcing as long as society coordinates on punishing whomever deviates from the established allocation — and whomever deviates from punishing the deviator. Following this, theorizing about institutions is theorizing about (i) how allocations (or focal points) are chosen, and (ii) how allocations can change. For example, Myerson (2004) explores the idea of justice as a focal point influencing the allocation of resources in society. This paper takes a more cynical view of our fellow human beings: those in power choose an allocation to maximize their own payoffs subject to the threat of rebellions.

[8] See also Grossman and Kim (1995) and Hirshleifer (1995).

[9] Glaeser, Scheinkman and Shleifer (2003) focus on democratic societies and thus assume the existence of a legal system. Acemoglu (2008) compares democratic and oligarchic societies, assuming different institutions in each case. Guriev and Sonin (2009) study the interplay between a dictator and oligarchs, assuming oligarchs can choose between having a weak or a strong dictator (which can be seen as an institution that determines the set of possible actions of the dictator). Myerson (2010) assumes a dictator can choose political liberalization, which is modelled as a probability that the dictator loses power if he expropriates capital.

2 A model of the power struggle

This section presents an analysis of the equilibrium allocation of power and resources in a simple endowment economy where commitment issues are absent.

2.1 Preferences, technologies, allocations, and rebellions

There is an area containing a measure-one population of ex-ante identical individuals indexed by $\imath \in \Omega$. Individuals receive utility $\mathscr{U}(C, F)$ from their own consumption C of a homogeneous good and disutility if they exert *fighting effort* F:

$$\mathscr{U}(C, F) = u(C) - F, \tag{2.1}$$

where $u(\cdot)$ is a strictly increasing and weakly concave function.

In this simple economy, individuals who become *workers* receive an exogenous endowment of q units of goods.

There will be an *allocation* of power and resources that is determined endogenously through a process referred to as the power struggle. An allocation specifies the set $\mathcal{W}$ of workers, and the set $\mathcal{P}$ of individuals currently in *power*, referred to as the *incumbents*. Each position of power confers an equal advantage on its holder in the event of any conflict, as described below. The term *power sharing* refers to a variable p, defined as the measure of the group $\mathcal{P}$. The incumbent group $\mathcal{P}$ can have any size between 0% and 50% of the population.[10] Those individuals in power cannot simultaneously be workers.[11]

An allocation also prescribes how goods are distributed across all individuals. Let $C_{\mathrm{p}}(\imath)$ and $C_{\mathrm{w}}(\imath)$ denote the individual-specific consumption levels of incumbent $\imath \in \mathcal{P}$ and worker $\imath \in \mathcal{W}$ respectively specified by an allocation. The allocation must satisfy the resource constraint

$$\int_{\mathcal{P}} C_{\mathrm{p}}(\imath)\mathrm{d}\imath + \int_{\mathcal{W}} C_{\mathrm{w}}(\imath)\mathrm{d}\imath = \int_{\mathcal{W}} \mathrm{q}\mathrm{d}\imath, \tag{2.2}$$

and each individual's consumption level must be non-negative.

Formally, an allocation is a collection $\mathscr{A} = \{\mathcal{P}, \mathcal{W}, C_{\mathrm{p}}(\imath), C_{\mathrm{w}}(\imath)\}$, where the sets $\mathcal{P}$ and $\mathcal{W}$ satisfy $\mathcal{P} \cup \mathcal{W} = \Omega$, $\mathcal{P} \cap \mathcal{W} = \emptyset$, and $p < 1/2$, and where the consumption levels $C_{\mathrm{p}}(\imath)$ and $C_{\mathrm{w}}(\imath)$ are consistent with the resource constraint (2.2) and non-negativity constraints.

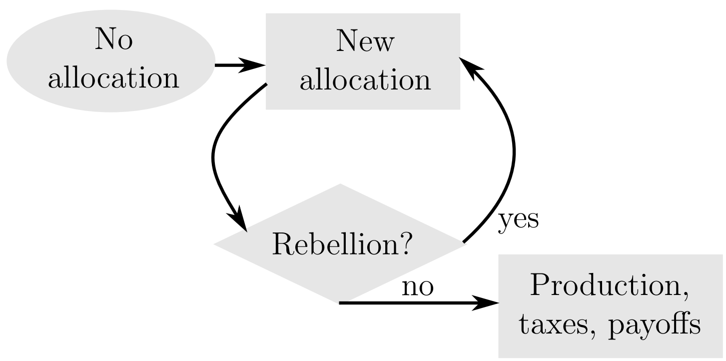
The *power struggle* is depicted in Figure 1. An allocation is first established. There are then opportunities for *rebellions*, which are described in detail below. If a rebellion succeeds, a new allocation is established, followed by more opportunities for rebellions. When no rebellions occur,

[10] This is a simple way of modelling decreasing returns in strength with respect to the size of the group in power. The size constraint is generally not binding, but without it, it could be possible that there would be too few individuals outside the incumbent group to launch a rebellion, implying there would be increasing returns to the size of the group in power.

[11] The assumption that those in power do not receive the endowments of workers is not essential for the main results. It does mean there is an opportunity cost of individuals being in power, so strictly speaking, this is not a pure endowment economy. However, the resulting 'guns versus butter' inefficiency is *not* the focus of this paper.

workers receive their endowments, resources are distributed according to the prevailing allocation, and payoffs are received.

Figure 1: *The power struggle*



In calculating payoffs using the utility function (2.1), note that consumption is received only at the end of the process in Figure 1, and that any disutility from fighting effort is additively separable between different stages of the power struggle. There is no discounting of utility based on the number of iterations of the power struggle.

Rebellions are the only mechanism for changing an established allocation. A rebellion is described by a rebel army $\mathcal{R}$, an amount of (non-negative) fighting effort $F(\imath)$ for each individual $\imath \in \mathcal{R}$ who belongs to the rebel army, and an incumbent army $\mathcal{D}$ that defends the current allocation. In a given rebellion, the rebel army can comprise those outside or inside the group currently in power, or a mixture of both. The incumbent army is drawn from those currently in power who do not join the rebel army. Formally, a rebellion is a collection $\{\mathcal{D}, \mathcal{R}, F(\imath)\}$, where the sets $\mathcal{D}$ and $\mathcal{R}$ satisfy $\mathcal{D} \subseteq \mathcal{P}$ and $\mathcal{D} \cap \mathcal{R} = \emptyset$.

A rebellion succeeds if

$$\int_{\mathcal{R}} F(\imath)\mathrm{d}\imath > \int_{\mathcal{D}} \delta \mathrm{d}\imath, \qquad [2.3]$$

which requires that the fighting strength of the rebel army exceeds the fighting strength of the incumbent army. Each army's fighting strength is the integral of the fighting strengths of its members. The fighting strength of individual $\imath \in \mathcal{R}$ in the rebel army is the amount of fighting effort $F(\imath)$ he exerts. Each individual $\imath \in \mathcal{D}$ in the incumbent army has fighting strength measured by a parameter δ (the power parameter), which is obtained at no utility cost to these individuals.[12]

Although the term rebellion is used to describe the process of changing an allocation, the formal definition encompasses 'popular uprisings' (the rebel army comprises only workers, $\mathcal{R} \subset \mathcal{W}$), 'coups d'état' (the rebel army is a subset of the incumbent group, $\mathcal{R} \subset \mathcal{P}$), 'suspensions of the constitution' (the rebel army includes all incumbents, $\mathcal{R} = \mathcal{P}$), and 'revolutions' that receive the backing of some

[12] Rebellions in this model play a similar role to blocking in coalitions (see Ray, 2007). One particularity of the model is that the effort rebels make in fighting is endogenous, and this effort is costly. Section 2.4 further discusses the assumptions on the power struggle underlying the rebellion mechanism.

insiders from the current regime (the rebel army includes a mixture of workers and incumbents).

2.2 Equilibrium

The requirements for an allocation to be an equilibrium of the power struggle described above are now stated. In what follows, let $U(\imath) = u(C(\imath))$ denote individual $\imath$'s continuation utility under a particular allocation, and let U_{p} and U_{w} denote the average payoffs of incumbents and workers:

$$U_{\mathrm{p}} \equiv \frac{1}{p}\int_{\mathcal{P}} u(C_{\mathrm{p}}(\imath))\mathrm{d}\imath, \quad \text{and} \;\; U_{\mathrm{w}} \equiv \frac{1}{1-p}\int_{\mathcal{W}} u(C_{\mathrm{w}}(\imath))\mathrm{d}\imath. \tag{2.4}$$

These payoffs assume that a given allocation prevails (with no further fighting effort incurred if there are no rebellions). The payoffs also exclude any past fighting effort.

An equilibrium allocation must be optimal in the sense of maximizing the average payoff of the incumbents, taking into account the threat of rebellions. Any rebellions must be rational in the sense defined below. The notation $'$ is used to signify any aspect of the allocation that would be established following a rebellion.

Definition 1 *A rebellion $(\mathcal{D}, \mathcal{R}, F(\imath))$ against the current allocation $\mathscr{A}$ is rational given the subsequent allocation $\mathscr{A}' = \{\mathcal{P}', \mathcal{W}', C'_{\mathrm{p}}(\imath), C'_{\mathrm{w}}(\imath)\}$ if:*

(i) All individuals in the rebel army $\mathcal{R}$ expect a position of power under the subsequent allocation yielding a payoff no lower than what the individual would receive under the current allocation, and each individual's fighting effort $F(\imath)$ does not exceed his expected gain:

$$\mathcal{R} = \{\imath \in \mathcal{P}' \mid U'_{\mathrm{p}} \geq U(\imath)\}, \quad \text{and} \;\; F(\imath) \leq U'_{\mathrm{p}} - U(\imath). \tag{2.5a}$$

(ii) The incumbent army $\mathcal{D}$ (drawn from those currently holding positions of power who do not rebel) comprises those who would be worse off under the subsequent allocation:

$$\mathcal{D} = \{\imath \in \mathcal{P} \backslash \mathcal{R} \mid U(\imath) > U'_{\mathrm{w}}\}. \tag{2.5b}$$

(iii) Condition (2.3) for a successful rebellion holds. □

In a rational rebellion, the rebel army $\mathcal{R}$ includes only individuals who will be in power under the subsequent allocation, which is an assumption designed to capture the incentive problems in inducing individuals to fight. As discussed in section 2.4, this would allow for credible punishment of individuals who do not exert fighting effort, namely exclusion from the group subsequently in power. The maximum amount of fighting effort exerted by an individual in the rebel army corresponds to his utility gain from changing the allocation.[13] Analogously, an individual in power will join the

[13]The payoff from rebellion is the expected payoff net of fighting effort from being in the incumbent group under the subsequent allocation, but with the assumption that if there is a non-degenerate distribution of consumption among those in power, individuals do not know in advance where they will be in this distribution. In equilibrium, there is a degenerate consumption distribution within the incumbent group, but this assumption simplifies the analysis.

incumbent army $\mathcal{D}$ to defend the current allocation if this is in his own interest. Note that restricting the rebels to those who would be in power under a subsequent allocation will actually restrict only the maximum number of rebels, not the identities or current status of those who can rebel.

For an allocation established at a stage of the power struggle in Figure 1 to be an equilibrium, the allocation must be optimal from the point of view of incumbents after taking account of the threat of any rational rebellions, where any allocation established following a rebellion must itself be an equilibrium (and so on for subsequent stages of the power struggle). Of those allocations satisfying these conditions, any that depend (apart from individual identities) on payoff-irrelevant histories are then deleted to leave a set of equilibrium allocations.

Definition 2 *An allocation* $\mathscr{A} = \{\mathcal{P}, \mathcal{W}, C_{\mathrm{p}}(\imath), C_{\mathrm{w}}(\imath)\}$ *is an equilibrium of a stage of the power struggle if the following conditions are satisfied:*

(i) ***Optimality for incumbents****: The allocation* $\mathscr{A}$ *maximizes the average utility* U_{p} *of incumbents when it is established, subject to:*

(ii) ***Rationality of rebels****: A rational rebellion occurs if according to Definition 1 there exists any rational rebellion against the current allocation* $\mathscr{A}$ *for some subsequent allocation* $\mathscr{A}'$*, subject to:*

(iii) ***Threats of rebellion are credible****: Any allocation* $\mathscr{A}'$ *established following a rebellion is itself an equilibrium of that stage of the power struggle.*

(iv) ***Independence of irrelevant history****: Allocations* $\mathscr{A}$ *and* $\mathscr{A}'$ *established at any two stages of the power struggle where fundamental (payoff-relevant) state variables are the same are identical up to a permutation of identities.* □

The first equilibrium condition is that an allocation must be established in the interests of incumbents after taking account of the threat of rebellions. The second condition states that there will be a rebellion if and only if there is some rebellion in the interests of those who take part in it.[14]

It is never in the interests of incumbents to have an allocation that triggers a rational rebellion, and since there is no uncertainty in the model, no rebellions occur in equilibrium. Nevertheless, the problem of finding an allocation that maximizes the average payoff of incumbents is effectively constrained by the threat of rebellions. With a slight abuse of language, the term ‘no-rebellion constraint’ is used below to refer to the restrictions on an allocation such that there is no rational rebellion for a particular rebel army (associated with a particular incumbent group under a subsequent allocation). The set of ‘no-rebellion constraints’ is the collection of these constraints for all possible compositions of the rebel army (associated with different subsequent incumbent groups).

[14] In political-economy models (for example, Acemoglu and Robinson, 2006 and Besley and Persson, 2009a), it is conventional to treat groups (rather than individuals) as agents. Here, an allocation maximizes the average payoff of individuals in an incumbent group, but participation in a rebellion must be individually rational, and the sizes and composition of groups are not exogenously fixed.

The nature of the threat posed by rebel armies depends on what the post-rebellion allocation would be. The third equilibrium condition is that only subsequent allocations that are themselves equilibria of the power struggle can be considered when determining whether a rebellion is rational. Essentially, subsequent allocations must be in the interests of subsequent incumbents after taking account of further threats of rebellion. This excludes the possibility that rebels make a binding commitment to an allocation that is not in their interests ex post — for example, an allocation that would give rebels an incentive to exert more fighting effort now, but which would not be optimal once the fighting is over. Note that because there will be many subsequent equilibrium allocations with different compositions of the group in power, and as any one of these could follow a rational rebellion, the third requirement of equilibrium does not in itself restrict who can rebel.

The fourth condition is that equilibrium allocations depend only on fundamental state variables (fundamental meaning payoff relevant, for example, variables that appear in resource constraints), with the exception of individual identities. This Markovian restriction disciplines the equilibrium concept.[15] In the basic model so far, there are no fundamental state variables, therefore equilibrium allocations at any stage of the power struggle must be the same apart from changes in the identities of those in power. In the model of section 3 below, the capital stock will be a fundamental state variable.

In summary, Definition 2 states that an equilibrium allocation must maximize incumbents' payoff (first condition) subject to avoiding any opportunity for rational rebellion (second condition), where the range of possible rational rebellions is itself limited by the set of equilibrium allocations that could be established following a rebellion (third condition). Note that the fourth (Markovian) condition does not operate as an additional constraint on the allocations that can be chosen in the interests of incumbents. Its role is in restricting which rebellions are rational (in conjunction with the third equilibrium condition) because the rationality of a rebellion depends on what allocation rebels anticipate being established following the rebellion. When any subsequent equilibrium allocation must depend only on fundamental state variables, an optimal allocation taking account of threats of rebellion need only depend on fundamental state variables.

Since individuals are ex ante identical, there is an essential indeterminacy in the identities of the incumbents and in the assignment of particular consumption levels to incumbents and workers in the case of non-degenerate distributions. If a certain allocation $\mathscr{A}$ is an equilibrium, otherwise identical allocations with any permutation of identities will also be equilibria. Therefore, the characterization of the set of equilibrium allocations will determine power sharing p and the distribution of resources $C_{\mathrm{p}}(\cdot)$ and $C_{\mathrm{w}}(\cdot)$, but not specific identities. As a shorthand, these features of the set of equilibrium allocations are referred to as 'the' equilibrium allocation in what follows.

[15] Without this restriction, it might in principle be possible to find a sequence of different allocations associated with different stages of the power struggle that would justify different equilibrium allocations at an earlier stage of the power struggle.

2.3 Characterizing the equilibrium allocation

In the simple endowment-economy model, there are no fundamental state variables, so the equilibrium allocation can be determined as a fixed point of a constrained maximization problem where the subsequent incumbents would face an identical problem following any rebellion. The equilibrium values of p^*, $C^*_{\mathrm{p}}(\cdot)$ and $C^*_{\mathrm{w}}(\cdot)$ are thus found in two steps. First, the value of p and the distributions $C_{\mathrm{p}}(\cdot)$ and $C_{\mathrm{w}}(\cdot)$ are chosen to maximize the average payoff of those in power subject to no rational rebellion, where the post-rebellion allocation p', $C'_{\mathrm{p}}(\cdot)$ and $C'_{\mathrm{w}}(\cdot)$ is taken as given. Conditional on the values of p' and U'_{p}, incumbents want to ensure that all 'no-rebellion constraints' hold, that is, there is no rational rebellion for each permutation of the identities of those in $\mathcal{P}'$. Second, the equilibrium conditions of identical power sharing $p' = p^*$ before and after rebellions and identical distributions of consumption for both groups $C'_{\mathrm{p}}(\cdot) = C^*_{\mathrm{p}}(\cdot)$ and $C'_{\mathrm{w}}(\cdot) = C^*_{\mathrm{w}}(\cdot)$ are imposed.

The following proposition derives some necessary features of any equilibrium allocation in this environment.

Proposition 1 *Any equilibrium allocation must have the following properties:*

(i) Equalization of workers' payoffs: $U^*_{\mathrm{w}}(\imath) = U^*_{\mathrm{w}}$ for all $\imath \in \mathcal{W}$ (with measure one).

(ii) Sharing power implies sharing rents: $U^*_{\mathrm{p}}(\imath) = U^*_{\mathrm{p}}$ for all $\imath \in \mathcal{P}$ (with measure one).

(iii) Power determines rents: $U^*_{\mathrm{p}} - U^*_{\mathrm{w}} = \delta$.

(iv) The allocation maximizes the payoff of incumbents subject to a single 'no-rebellion constraint' for a rebel army comprising only workers:

$$U'_{\mathrm{p}} - U_{\mathrm{w}} \leq \delta \frac{p}{p'}, \qquad [2.6]$$

with all other 'no-rebellion constraints' being redundant.

(v) An equilibrium always exists and is unique up to permutations of identities. Equilibrium power sharing p^* satisfies $0 < p^* \leq 2 - \varphi$, where $\varphi \equiv (1 + \sqrt{5})/2 \approx 1.618$.[16]

Proof See appendix A.1. ■

The first two parts of the proposition demonstrate that incumbents have a strong incentive to avoid inequality except where it is matched by differences in power. These results hold even when the utility function is linear in consumption, and so do not depend on strict concavity.

The intuition for the payoff-equalization results is that the most dangerous composition of a rebel army is the one including those individuals with the greatest incentive to fight. A rebel army will always be a subset of the whole population. As a consequence, if there were payoff inequality among workers, the most dangerous rebel army would not include those workers who receive a relatively high payoff. The incumbents could then reduce the effectiveness of this rebel army by redistributing

[16]The constraint $p < 1/2$ is always slack in equilibrium.

from relatively well-off workers to the worse off.[17] This slackens the set of 'no-rebellion constraints', allowing the incumbents to achieve a higher payoff. These gains are exploited to the maximum possible extent when all workers' payoffs are equalized.[18] Similarly, inequality in incumbent payoffs is undesirable because incumbents receiving a relatively low payoff can defect and join a rebel army. Equalizing incumbent payoffs by redistributing consumption does not directly lower their average utility when the utility function is weakly concave, while it has the advantage of weakening the most dangerous rebel army. Since defections from the group in power weaken the incumbent army, there is no version of this argument that calls for equalization of payoffs *between* workers and incumbents.

Making use of the payoff-equalization results in Proposition 1 and the resource constraint (2.2), the utilities of workers and incumbents are:

$$U_{\mathrm{w}} = u(C_{\mathrm{w}}), \quad \text{and } \; U_{\mathrm{p}} = u(C_{\mathrm{p}}) = u\left(\frac{(1-p)(\mathrm{q}-C_{\mathrm{w}})}{p}\right). \tag{2.7}$$

As a consequence of the payoff-equalization results, all that matters for the composition of a rebel army are the fractions σ_{w} and σ_{p} of its total numbers drawn from workers and from the group in power. The allocation must then be a solution of the problem

$$\max_{p, C_{\mathrm{w}}} U_{\mathrm{p}} \;\text{ s.t. }\; \sigma_{\mathrm{w}} \max\{U'_{\mathrm{p}} - U_{\mathrm{w}}, 0\} + \sigma_{\mathrm{p}}(U'_{\mathrm{p}} - U_{\mathrm{p}} + \delta)\mathbb{1}[U'_{\mathrm{p}} \geq U_{\mathrm{p}}] \leq \delta \frac{p}{p'}, \tag{2.8}$$

for all non-negative proportions σ_{w} and σ_{p} that are feasible.[19] The first term in the constraint is the fighting effort that would be exerted by workers in a rebellion. The second term is the fighting effort exerted by any turncoat incumbents in a rebel army plus the loss of fighting strength of the incumbent army by not having these individuals in it (the term $\sigma_{\mathrm{p}}\delta$). The general form of the no-rebellion constraints stated in (2.8) is derived from the successful rebellion condition (2.3) and the participation constraints (2.5a) and (2.5b) on membership of the rebel and incumbent armies in the requirements for a rational rebellion,[20] with $\mathbb{1}[\cdot]$ denoting the indicator function.

The fourth claim in Proposition 1 states that the equilibrium allocation can be determined subject to a single no-rebellion constraint (2.6), which is equivalent to setting $\sigma_{\mathrm{w}} = 1$ and $\sigma_{\mathrm{p}} = 0$ in the general constraints of (2.8) (and noting that U'_{p} will exceed U_{w} in equilibrium). Satisfaction of (2.6) is clearly necessary, but the proposition shows that this single constraint is also *sufficient* to characterize the equilibrium allocation.[21] The constrained maximization problem (2.8) thus reduces

[17] Payoff equalization implies equalization of the maximum fighting effort workers are willing to put in.

[18] This result is different from those found in some models of electoral competition such as Myerson (1993). In the equilibrium of that model, politicians offer different payoffs to different agents. But there is a similarity with the model here because in neither case will agents' payoffs depend on their initial endowments.

[19] Given the size of the rebel army p' and the sizes of the groups of workers and incumbents under the current allocation, feasibility requires $\sigma_{\mathrm{w}} \leq (1-p)/p'$, $\sigma_{\mathrm{p}} \leq p/p'$, and $\sigma_{\mathrm{w}} + \sigma_{\mathrm{p}} = 1$.

[20] The condition $U_{\mathrm{p}} > U'_{\mathrm{w}}$ for those in power who do not join a rebel army being willing to belong to the incumbent army defending the current allocation is always satisfied in equilibrium.

[21] This finding is specific to the simple endowment-economy model of this section. In the fully fledged model with investment, other no-rebellion constraints become binding. Which compositions of the rebel army are associated with binding no-rebellion constraints in equilibrium is endogenous and will depend on the situation being analysed.

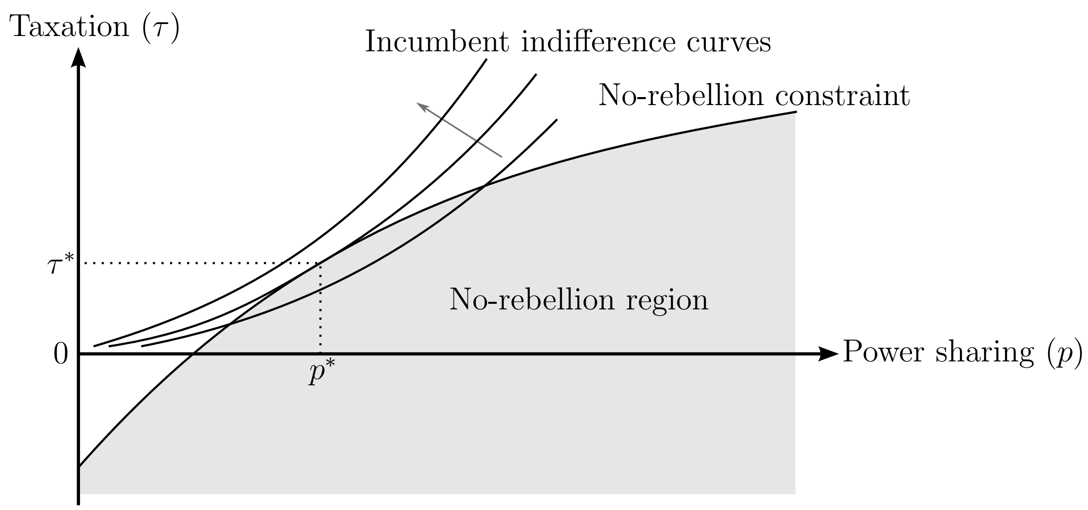
to

$$\max_{p, C_w} U_p \text{ s.t. } U'_p - U_w \leq \delta \frac{p}{p'}, \quad [2.9]$$

where U_w and U_p are as in equation (2.7), with p' and U'_p taken as given. In equilibrium, these are equal to the corresponding values p^* and U^*_p that solve the constrained maximization problem.

The equilibrium allocation of resources can be implemented by having each worker $\imath \in \mathcal{W}$ face a tax (or transfer if negative) $\tau = \mathrm{q} - C_w$. The payoff of each incumbent is increasing in the tax τ, and decreasing in power sharing p because the total tax revenue must be distributed more widely (and because there are fewer workers to tax). The indifference curves of the incumbents over τ and p and the single binding no-rebellion constraint are plotted in Figure 2.

Figure 2: *Trade-off between power sharing and taxation*



An increase in τ reduces the payoffs of workers, making them more willing to fight in a rebellion, while an increase in the size of the group in power boosts the fighting strength of the incumbent army, making rebellion less attractive to workers. The incumbents thus have two margins to avoid rebellions. They can reduce taxes τ (the 'carrot'), or increase their size p (the 'stick'). This corresponds to the upward-sloping no-rebellion constraint depicted in Figure 2.[22]

After taking into account the binding no-rebellion constraint, the key decision is how widely to share power. Incumbents face a fundamental trade-off in determining their optimal size: a larger size will make rebellions more costly and allow higher taxes to be extracted from workers, but will also spread the revenue from these taxes more thinly among a larger number of individuals (and reduce the tax base). Proposition 1 shows that it is not possible in equilibrium to include extra individuals in the group in power without offering these individuals the same high payoff received by other members. Thus, sharing power entails sharing rents. The incumbents then share power with an extra individual if and only if this allows them to increase their average payoff. The allocation

[22]When utility is linear in consumption, the no-rebellion constraint is a straight line. The diagram shows the general case where utility is strictly concave in consumption, resulting in the constraint being a concave function.

of power therefore reflects the interests of the incumbents, rather than the interests of society. In equilibrium, the utility value of the rents received by those in power depends only on the exogenous parameter δ.

2.4 Discussion

The model described above is designed to capture in a simple manner the 'power struggle' for control of the allocation of resources. An allocation of power and resources can be overthrown by rebellions and replaced by a new one. The success of a rebellion is settled by a basic 'conflict technology' that avoids going into the punches and sword thrusts of battle. Everyone has access on the same terms to opportunities for rebellion, irrespective of their current status. Several assumptions are made for simplicity. The combatants' strengths are linear in the strengths of the individuals that make up the rebel and incumbent armies. Members of the incumbent army have a predetermined fighting strength (the power parameter δ), so this is inelastic with respect to current fighting effort. Given fighting strengths, there is no uncertainty about which side will emerge victorious.[23]

In modelling conflict, it is necessary to introduce some asymmetry between the rebel and incumbent armies for the notion of being 'in power' to be meaningful. The parameter δ represents the advantage incumbents derive from their entrenched position. It is fighting strength that is obtained at no *current* effort cost (though past fighting effort may have been expended in becoming an incumbent), while the rebels can only obtain fighting strength from current effort. The inelasticity of incumbent-army members' fighting strength with respect to current effort can be seen as an inessential simplification.[24] When the current allocation is established, there are several margins that can be adjusted to avoid rebellions, such as varying the number of individuals in power, or increasing the consumption allocated to those who would otherwise join a rebel army. The ability to adjust each incumbent's power δ at some cost in addition to these does not fundamentally change the problem. The rebel army, however, lacks these alternative margins, so it is essential that its fighting strength is responsive to the effort put in by its members.

One interpretation of the parameter δ is that the individuals currently in power possess some defensive fortifications, such as a castle, which place them at an immediate fighting advantage over any rebels who must breach these from outside. A broader interpretation is that δ represents the more severe coordination problems faced by rebels from outside the incumbent group. Authority depends on a chain of command, where individuals follow instructions in expectation of punishment

[23]An extension of the model where there is uncertainty about the outcome a rebellion is available upon request. This extension does not change any of the key results of the paper. In many models in the political-economy literature, the threat of conflict plays an important role, but conflict does not actually occur in equilibrium. For example, see Acemoglu and Robinson (2006) and Francois, Rainer and Trebbi (2015). A discussion of models where conflict does occur in equilibrium is found in the survey by Blattman and Miguel (2010).

[24]An extension of the model with effort-elastic incumbent fighting strength is available upon request. This extension does not change any of the key results of the paper. In some applications it might be important to consider other ways of increasing the cost of rebellions. Campante, Do and Guimaraes (2013) extend the framework developed here to build a model where the cost of rebellion depends on how far a citizen is from the capital city. This means that isolating the capital city raises the cost of rebellions. Increasing military spending is another way the cost of rebellion might be raised. This is also analysed in Campante, Do and Guimaraes (2013), and the finding is that military spending and isolating the capital city work as substitutes.

from others if they disobey. The rebels confront the challenge of persuading enough individuals that they should fear punishment for disobeying them rather than the incumbents people are accustomed to.[25]

If a rebellion occurs and the existing allocation of power and resources is replaced by a new one, the old allocation has no effect on payoffs because consumption is assumed to occur only when an equilibrium is reached.[26] In effect, this assumption on timing disregards any short-lived effects of the old allocation after it is overthrown by a rebellion. Besides its simplicity, one advantage of this formulation for the purposes of the paper is that it rules out any mechanism through which commitment to time-inconsistent laws can arise owing to the transitional dynamics that would follow a rebellion. If an allocation were to influence payoffs even though it has been overthrown then this could potentially deter (or stimulate) rebellions. It is also assumed that the occurrence of a rebellion does not destroy any of the goods available for consumption.[27]

The assumptions of the model allow for coordination among individuals in launching rebellions, but subject to some restrictions. These restrictions are intended as a simple representation of the plausible constraints that ought to be placed on the set of possible deals or 'contracts' among the rebels. The fundamental contracting problem is the issue of enforceability in a world with no exogenous commitment technologies. The rebellion mechanism is intended to be as flexible as possible in allowing for 'deals' subject to the limits of enforceability.

The rebellion 'contract' implicit in the model requires a prescribed amount of fighting effort from each rebel in return for a place in the group in power under the new allocation. There is a restriction that rebel armies are open only to those who expect a place in the group subsequently in power. This rebellion contract raises two questions. Why are other forms of contract ruled out? And what suggests a contract of this form is not susceptible to enforceability concerns?

Consider a group of individuals with incentives to come together and agree to fight, enabling a new allocation to be established that offers each one of them a better payoff than what they would currently receive. The maximum fighting effort each would be willing to agree to is equal to each's expected utility gain. There are two facets of the enforceability problem for this 'deal'. First, after the success of the rebellion, will there be incentives to establish the particular allocation that was agreed beforehand? Second, will individual rebels honour their agreed levels of fighting effort?

A rebellion contract prescribing exactly what allocation is to be established faces formidable enforceability problems. Once the status quo is overthrown, there is no higher authority that can compel the group now in power to act against its interests ex post. This type of contract is therefore ruled out. The new allocation of power and resources must maximize the payoffs of those who are

[25] Under this interpretation, the successful rebellion condition (2.3) can be seen as the rebels' effort requirement to demonstrate that they have the strength and the organization to overcome these problems. Once the incumbents see this tipping point is reached, they surrender and no actual fighting takes place.

[26] This is different from the literature on dynamically stable coalitions, for example, Becker and Chakrabarti (1995) and Kalandrakis (2004), and closer to the literature on rules as self-selected fixed points, for example, Baron and Ferejohn (1989), Chwe (1994), and Koray (2000).

[27] An extension of the model where rebellions cause economic damage is available upon request. This extension does not change any of the key results of the paper.

now the incumbents[28] starting afresh from the world as they find it, unconstrained by history except for fundamental state variables.[29] Any bygones will be bygones. In particular, this precludes the trigger strategies of repeated games as commitment devices.[30]

As Proposition 1 shows, incumbents have a strict preference to avoid payoff inequality except where it is matched by differences in power. It follows that the group now in power would have incentives not to honour contracts that specified either transfers to those who contributed fighting effort during the rebellion or fines for those who did not, hence such contracts cannot be written ex ante.

Now consider the second facet of the enforceability problem. Taking as given the payoff improvement an individual expects if a rebellion succeeds (subject to the restrictions on what can be agreed in advance regarding the new allocation), will it be possible to enforce the agreed amount of fighting effort from an individual who is party to a rebellion contract? The basic problem is that each atomistic individual (correctly) does not perceive himself as pivotal in determining whether the rebellion succeeds. Thus, left to his own devices, he would have an incentive to shirk and free-ride on others' fighting effort. To a large extent, rebel armies may be able to control individual members through internal discipline, but a non-negligible enforceability problem remains when some necessary fighting is done at an individual's discretion.

To ensure that all agreed fighting effort is actually exerted, there needs to be a credible punishment that can be imposed on shirkers after the fact.[31] However, according to Proposition 1, only differences in payoffs that reflect differences in power are in the interests of incumbents ex post. This suggests that the offer of a position of power conditional on the requisite amount of fighting effort can provide a credible incentive not to shirk.[32] While the rebellion contract cannot determine the total number of positions of power under the new allocation, it can control the identities of those who will receive these positions.[33] But for those who anticipate becoming workers, even if they gain

[28]In the model, an allocation being in the interests of the incumbent group is interpreted to mean maximizing the average payoff of its members. Moving away from this simplifying assumption would require modelling the hierarchy and power relations within the incumbent group. See Myerson (2008) for a model which addresses that question.

[29]For simplicity, the occurrence of conflict does not itself affect any fundamental state variables. This implicitly assumes members of the rebel army can be demobilized costlessly once the fighting is over if, off the equilibrium path, there were more rebels than places in the new incumbent group. Adding a cost of demobilization would make the size of the previous rebel army a state variable at the stage a new allocation is established, which would add a significant complication to the model without obviously delivering any new insights.

[30]Models that allow trigger strategies face the problem of multiple equilibria because there is always a range of possible punishments consistent with equilibrium.

[31]The model abstracts from imperfections of information. Taking account of less than perfect information might place additional restrictions on the range of punishments that are feasible.

[32]Suppose that a fraction ξ of an individual's agreed fighting effort (associated with some necessary tasks) cannot be directly enforced at the time through the rebel army's own discipline mechanisms, but that the full amount of fighting effort F must be exerted otherwise the individual does not obtain fighting strength F. Suppose also that each individual's total fighting effort is verifiable after the rebellion. After agreeing to the rebellion contract, the individual is directly compelled to exert effort $(1-\xi)F$. If he exerts the remaining effort ξF then he subsequently receives his position of power with continuation payoff U'_{p}. If he shirks and exerts no further effort, he is demoted to worker status and receives payoff U'_{w}. Therefore, for individual $\imath$ to join the rebel army and contribute to the fighting, incentive compatibility requires $U'_{\mathrm{p}} - U'_{\mathrm{w}} \geq \xi F(\imath)$, while the maximum-effort participation constraint is $U'_{\mathrm{p}} - U(\imath) \geq F(\imath)$. If ξ is positive but sufficiently small then the incentive compatibility constraint is satisfied but not binding, while the participation constraint binds, as was implicitly assumed in the description of the rebellion mechanism.

[33]As the identities of the incumbent group have no effect on the maximum attainable payoffs of those in power,

from the success of the rebellion, there is no worse position they can be credibly assigned if they fail to put in the agreed level of fighting effort. There is nothing to prevent such individuals from shirking,[34] which leads to the restriction that all rebels must expect a position of power.

2.5 Example I: utility linear in consumption

There are three exogenous parameters in the model: the power parameter δ of an incumbent, the endowment q of a worker, and the utility function $u(\cdot)$ in consumption. This example illustrates the workings of the model with a linear utility function $u(C) = C$. The constrained maximization problem (2.9) becomes

$$\max_{p, C_{\mathrm{w}}} \frac{(1-p)(\mathrm{q} - C_{\mathrm{w}})}{p} \quad \text{s.t.} \quad C'_{\mathrm{p}} - C_{\mathrm{w}} \leq \delta \frac{p}{p'}, \tag{2.10}$$

after substituting the expressions for U_{p} and U_{w} from (2.7). The single no-rebellion constraint is binding, and can be used to solve explicitly for the consumption of a worker $C_{\mathrm{w}} = C'_{\mathrm{p}} - \delta p/p'$. Substituting that into the objective function yields the consumption of an incumbent:

$$C_{\mathrm{p}} = \frac{1-p}{p}\left(\mathrm{q} - C'_{\mathrm{p}} + \delta \frac{p}{p'}\right). \tag{2.11a}$$

The problem is now an unconstrained choice of power sharing p to maximize each incumbent's consumption, with p' and C'_{p} taken as given. The first-order condition is

$$\frac{C^*_{\mathrm{p}}}{1-p^*} = (1-p^*)\frac{\delta}{p'}. \tag{2.11b}$$

The equilibrium conditions $p^* = p'$ and $C^*_{\mathrm{p}} = C'_{\mathrm{p}}$ are now imposed in (2.11a), which leads to $C^*_{\mathrm{p}} = (\mathrm{q} + \delta)(1-p^*)$. Combining this with equation (2.11b) (and using $p^* = p'$ again) yields the equilibrium allocation:[35]

$$p^* = \frac{\delta}{\mathrm{q} + 2\delta}, \quad C^*_{\mathrm{p}} = \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta}, \quad \text{and} \quad C^*_{\mathrm{w}} = \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} - \delta. \tag{2.12}$$

Notice in this case that power sharing is a function of the ratio δ/q.

The power parameter δ affects the equilibrium in three ways. First, an increase in δ makes the

carrying out the punishment would not affect others in the incumbent group. Intuitively, since all individuals are ex ante identical, individuals in power do not care about the identities of those with whom they share power, only the total number of such people.

[34] The incentive compatibility constraint discussed in footnote 32 would be violated for these individuals with any positive discretionary effort fraction ξ, no matter how small. One alternative approach that could incentivize more individuals not to shirk offers a lottery in return for an agreed amount of fighting effort, where the prize is a position of power. While this mechanism could induce fighting effort from more individuals, the amount of effort each individual would agree to is lower because the lottery is less valuable than a position of power with certainty.

[35] The parameter restriction $\delta/\mathrm{q} \leq \varphi$ is assumed, where φ is as defined in Proposition 1. When the utility function is linear, this restriction is necessary and sufficient for an equilibrium in which the non-negativity constraint on workers' consumption is not binding.

incumbents stronger because the rebels have to exert greater fighting effort to defeat the incumbent army. This 'income effect' leads to a reduction in C_w and a decrease in p. Second, the payoff that the rebels would receive once in power increases because their position will also be stronger once they have supplanted the current incumbents, making rebellion more attractive. This offsetting 'income effect' increases C_w and increases p. Third, an increase in δ raises the effectiveness of the marginal fighter in the incumbent army, leading to a 'substitution effect' whereby the incumbents increase their size in order to extract higher rents. As long as the non-negativity constraint on workers' consumption is not binding, the third effect dominates and power sharing is increasing in δ.

The allocation (2.12) can be implemented by a lump-sum tax τ on each worker given by

$$\tau = \frac{\delta(\mathrm{q}+\delta)}{\mathrm{q}+2\delta}, \qquad [2.13]$$

where the proceeds of the tax are shared equally among incumbents.

2.6 Example II: public goods

In the model so far, there is no scope for incumbents to do what governments are customarily supposed to do, such as provide public goods. This example extends the model by introducing a technology that allows for production of public goods. The allocation of power and resources now also specifies spending on public goods. It is then natural to ask whether resources will be efficiently allocated to public-good production.

The new technology converts units of output into public goods. If g units of goods per person are converted then everyone receives an extra $\Gamma(g)$ units of the consumption good. The function $\Gamma(\cdot)$ is strictly increasing, strictly concave, and satisfies the usual Inada conditions. The definition of an allocation $\mathscr{A}$ from section 2.2 is augmented to specify public-good provision g, hence $\mathscr{A} = \{\mathcal{P}, \mathcal{W}, C_\mathrm{p}(\imath), C_\mathrm{w}(\imath), g\}$. All individuals observe the choice of g and take it into account — along with all other aspects of an allocation — when considering the participation and maximum-fighting-effort conditions for a rational rebellion. There is no other change to the environment.

The consumption level C in the utility function (2.1) is now

$$C = c + \Gamma(g), \qquad [2.14]$$

which represents an individual's overall consumption, comprising private consumption c and the consumption $\Gamma(g)$ each person obtains from the public good. The resource constraint is now

$$pC_\mathrm{p} + (1-p)C_\mathrm{w} = (1-p)\mathrm{q} - g + \Gamma(g). \qquad [2.15]$$

A benevolent social planner would choose the first-best level of public-good provision $g = \hat{g}$, determined by $\partial\Gamma(\hat{g})/\partial g = 1$, which maximizes the total amount of goods available for consumption.

In determining the equilibrium allocation, the payoff-equalization insights of Proposition 1 continue to apply to this new environment, hence it is possible without loss of generality to focus on allocations specifying power sharing p, the consumption C_{w} of all workers, and the necessarily common public-good provision g. The resource constraint (2.15) can be used to find the consumption level of each incumbent under a particular allocation:

$$C_{\mathrm{p}} = \frac{(1-p)(\mathrm{q}-C_{\mathrm{w}})}{p} + \frac{\Gamma(g)-g}{p}. \qquad [2.16]$$

The argument of Proposition 1 that the equilibrium allocation can be characterized by maximizing the incumbent payoff subject only to the no-rebellion constraint for a rebel army composed entirely of workers also carries over to this new environment. Thus, the equilibrium allocation is the solution of

$$\max_{p,C_{\mathrm{w}},g} u\left(\frac{(1-p)(\mathrm{q}-C_{\mathrm{w}})}{p} + \frac{\Gamma(g)-g}{p}\right) \text{ s.t. } U'_{\mathrm{p}} - u\left(C_{\mathrm{w}}\right) \leq \delta\frac{p}{p'}, \qquad [2.17]$$

where p' and U'_{p} are taken as given, but with $p' = p^*$, $C'_{\mathrm{w}} = C^*_{\mathrm{w}}$ and $g' = g^*$ in equilibrium. The first-order condition for public-good provision g is:

$$\frac{\partial\Gamma(g^*)}{\partial g} = 1. \qquad [2.18]$$

This is identical to the condition for the provision $\hat{g}$ chosen by a benevolent social planner, so $g^* = \hat{g}$. The equilibrium allocation is therefore economically efficient in respect of public-good production.[36]

To understand this result, observe that the no-rebellion constraint implies incumbents cannot disregard the interests of workers, even though they do not care about them directly. Provision of the public good slackens the no-rebellion constraint, while the resources appropriated to finance it tighten the constraint. By optimally trading off the benefits of the public good against the cost of production, the incumbents effectively maximize the size of the pie, making use of transfers to ensure everyone is indifferent between rebelling or not. This efficiency result can be seen as a 'political' analogue of the Coase theorem, where the contestability of allocations through rebellion plays the role of legal property rights. The assumption that allocations are actually implemented in the absence of rebellions is important to this finding, but more importantly, the constraints imposed here by the power struggle do not interfere with the transfers that are needed for an efficient allocation of resources.

The equilibrium allocation can be implemented by a lump-sum tax on all individuals of size g^* implicitly defined by (2.18) to finance the public good, and a lump-sum tax τ on workers with the revenue equally distributed among incumbents. With utility linear in consumption, the tax τ on each worker would be the same as in (2.13).

The analysis shows that although incumbents are extracting rents from workers, this does not preclude them from acting as if they were benevolent in other contexts. Hence, the overall welfare of workers might be larger or smaller compared to a world in which no-one can compel others to act

[36] The distribution of total output between workers and incumbents depends on the other parameters of the model, including the utility function $u(\cdot)$. In equilibrium, all individuals will receive a higher overall payoff as a result of the public-good technology being available, though in general, the benefits will not be shared equally.

against their will. This reflects the ambivalent effects on ordinary people of having a ruling elite.[37]

The result found with this example can be obtained in several other settings, as discussed by Persson and Tabellini (2000) in the context of voting and elections. Here the result provides a benchmark case where the equilibrium allocation is economically efficient.

3 Investment

This section analyses the implications of the possibility of investment for the equilibrium allocation of power and resources. Individuals can now exert effort to obtain a greater quantity of goods, but there is a time lag between the effort being made and the fruits of the investment being received. During this span of time, there are opportunities for rebellion against the prevailing allocation. The model is otherwise identical to that of section 2. In particular, there are no changes to the mechanism through which an allocation of power and resources is established and changed. However, the occurrence of investment changes incentives for rebellion, and thus affects the 'no-rebellion constraints' the equilibrium allocation must satisfy. The following analysis considers how the equilibrium allocation will be able to provide credible incentives for individuals to invest, and to what extent it will be done — in particular, whether the allocation will achieve economic efficiency.

3.1 Environment

The sequence of events is depicted in Figure 3. Before any investment decisions are made, an allocation is first established through a process identical to that described in section 2 (compare Figure 1). The allocation specifies the group in power and the amount of consumption each individual receives, which can now be contingent on individuals' investment decisions. Once an allocation that does not immediately trigger rebellion is established, there are opportunities to invest. After investment decisions are made, there is another round of opportunities for rebellion, with a new allocation established if a rebellion succeeds.

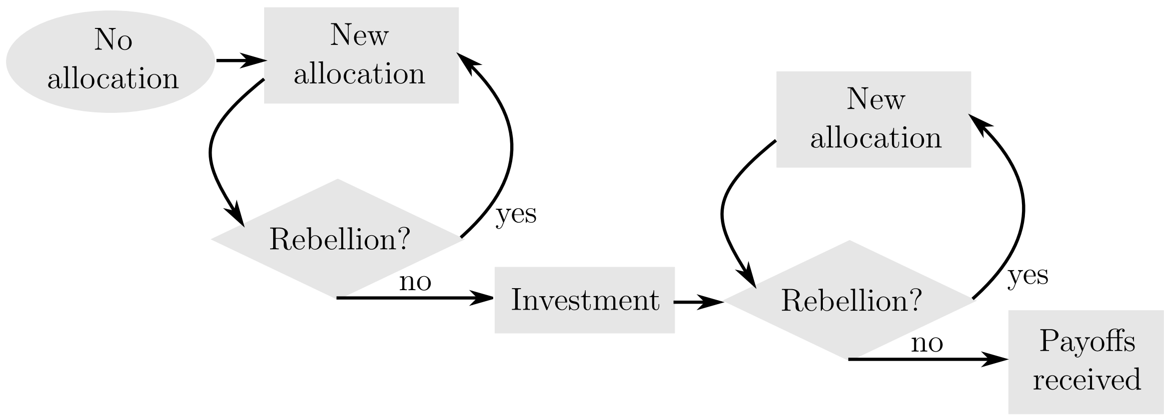
Individuals who are in power (incumbents, denoted by $\imath \in \mathcal{P}$) have fighting strength δ in defence, as in the model of section 2. Individuals not belonging to the incumbent group (denoted by $\imath \in \mathcal{N}$) at the post-investment stage receive an endowment of q units of goods.

There are μ *investment opportunities*, all of which arrive at the same point in the sequence of events in Figure 3. An investment opportunity is the option to produce κ units of capital in the future in return for incurring a present effort cost θ (in utility units), which is sunk by the time the capital is produced. *Capital* here simply means more units of the consumption good. All investment opportunities yield the same amount of capital κ, but each features an effort cost that is an independent draw from the distribution

$$\theta \sim \text{Uniform}\,[\psi, \kappa]\,, \tag{3.1}$$

[37] This trade-off is mentioned in the Bible (1 Samuel 8:10–20). The people want a king to provide public goods, despite being warned by the prophet Samuel that the king would use his power in his own interests. Many centuries later, in far too many cases, the warnings of Samuel remain as relevant as ever.

Figure 3: *Opportunities for investment and the power struggle*



where $0 < \psi < \kappa$, with ψ being the minimum effort cost.[38] An individual's receipt of an investment opportunity, the required effort cost θ, and whether the opportunity is taken, are private information at the time of the investment decision, while the individual's production of capital becomes common knowledge after the investment stage.[39] Investment decisions are made rationally by individuals who are able to understand what equilibrium allocation will prevail given the sequence of events in Figure 3. It is further assumed that investment opportunities are received only by those individuals outside the incumbent group,[40] and that investment opportunities are randomly assigned at the investment stage with no-one having prior knowledge of whether he will receive one, nor the required effort cost θ if so.[41]

An individual's utility function is now

$$\mathscr{U}(C, I, F) = C - \theta I - F, \tag{3.2}$$

where $I \in \{0, 1\}$ is an indicator variable for whether an investment opportunity is received and taken, and F denotes any fighting effort, as in the model of section 2. Note that utility from consumption is received at the final stage of the sequence of events in Figure 3, disutility from any

[38]The uniform distribution is chosen for simplicity. The choice of distribution does not affect the qualitative results.

[39]Since investment opportunities are private information when taken, it is not feasible to have a 'command economy' allocation where individuals perform investments by decree.

[40]Allowing those in power to invest adds extra complications to the model. It might be thought important to have investors inside the incumbent group to provide appropriate incentives. As will be seen, this is not the case, and moreover, the advantage of having investors in power most likely applies to the case where they are brought into the incumbent group *conditional* on taking an investment opportunity. From the incumbents' perspective, the incentive to do this disappears once investments have come to fruition. But given the information restrictions, which represent the not implausible difficulties of identifying talented investors in advance, bringing in individuals conditional on investment at an earlier stage would not be feasible.

[41]This modelling device places individuals behind a 'veil of ignorance' about their talents as investors when the pre-investment stage allocation is determined. Doing this avoids having to track whether talented investors are disproportionately inside or outside the group in power, which would add a (relevant) state variable to the problem of determining the pre-investment stage allocation, significantly complicating the analysis. However, it will turn out that the 'no-rebellion constraints' are slack at the pre-investment stage, so this assumption need not significantly affect the results.

fighting effort is additively separable between different stages of the power struggle, and there is no discounting of payoffs based on the number of rounds of the power struggle that occur. At the pre-investment stage, since those outside the incumbent group do not know their type yet (whether they will be workers or investors), their payoffs are to be interpreted as expected payoffs. At the post-investment stage, continuation payoffs are independent of any earlier effort (this is the sense in which effort costs are sunk). The utility function is assumed to be linear in consumption for analytical tractability.

The following parameter restrictions are imposed:

$$\frac{\delta}{q} \leq \varphi \equiv \frac{1+\sqrt{5}}{2}, \quad \mu < \frac{q}{2(q+2\delta)}, \quad \text{and} \ \ \kappa < \delta. \tag{3.3}$$

The first restriction is the bound from section 2.5 needed to ensure non-negativity constraints are always slack in equilibrium. The second restriction states that the measure μ of individuals who receive an investment opportunity is not too large, which ensures there is never a shortage of workers to fill a rebel army after investments have occurred.[42] The third restriction places a physical limit on the economy's maximum capital stock.

In this environment, an allocation is formally defined as $\mathscr{A} = \{\mathcal{P}, \mathcal{N}, C_{\mathrm{p}}(\imath), C_{\mathrm{w}}(\imath), C_{\mathrm{k}}(\imath)\}$, where $\mathcal{P}$ is the set of incumbents and $\mathcal{N}$ is the set of all other individuals. The consumption allocation of the incumbents is denoted by $C_{\mathrm{p}}(\imath)$. For an individual $\imath \in \mathcal{N}$ outside the incumbent group, the consumption allocation is potentially contingent on whether the individual has produced capital. The term *worker* now refers to an individual outside the incumbent group who did not produce any capital ($\imath \in \mathcal{W}$), in which case the consumption allocation is $C_{\mathrm{w}}(\imath)$. Those who did produce capital ($\imath \in \mathcal{K}$), the *capitalists*, receive consumption allocation $C_{\mathrm{k}}(\imath)$.

Investment decisions are required to be individually rational, taking account of the equilibrium allocation that will emerge from the power struggle. The equilibrium concept is unchanged from that introduced in Definition 2. An equilibrium allocation must be in the interests of incumbents taking into account threats of rebellion, where a rebellion occurs if and only if one is rational. Any allocation established following a rebellion must itself be an equilibrium, and equilibrium allocations are restricted to those that depend only on fundamental state variables (apart from individual identities). Note that since the aggregate capital stock is a fundamental state variable at the post-investment stage, any aspect of an equilibrium allocation of power and resources established following a rebellion after investments have been made is permitted to depend on the capital stock.

Since any equilibrium allocation would remain so after a permutation of identities, the relevant variables characterizing 'the' equilibrium allocation are power sharing p, and the distributions $C_{\mathrm{p}}(\cdot)$, $C_{\mathrm{w}}(\cdot)$, and $C_{\mathrm{k}}(\cdot)$ of consumption for incumbents, workers, and capitalists respectively.

[42] If investors were to become the predominant group then the nature of the binding constraints might change and a general analysis would be significantly more convoluted. This extension would not invalidate the main result of the paper that commitment can be achieved through power sharing.

3.2 Characterizing the equilibrium allocation

Characterizing the equilibrium allocation requires working backwards, determining the new equilibrium allocation if a rebellion were to occur at the post-investment stage, and then analysing what allocation will be established at the pre-investment stage. Incumbents want the established allocation to survive threats of rebellion so that they remain in power. In this environment, an allocation established at the pre-investment stage must now avoid rebellions both before and after the investment stage.

3.2.1 Post-investment stage allocation after a rebellion

Suppose that a rebellion occurs at some point after investment decisions have been made, which is necessarily off the equilibrium path. Let K denote the predetermined stock of capital that has been produced. In what follows, the superscript † is used to distinguish a new consumption allocation established at the post-investment stage from one established earlier.

The effort costs θ of investing are sunk at the post-investment stage, so the continuation utility function $\mathscr{U}(C, F) = C - F$ is the same for both a capitalist and any other individual. The resource constraint at this stage is

$$\int_{\mathcal{P}^\dagger} C_{\mathrm{p}}^\dagger(\imath)\mathrm{d}\imath + \int_{\mathcal{W}^\dagger} C_{\mathrm{w}}^\dagger(\imath)\mathrm{d}\imath + \int_{\mathcal{K}^\dagger} C_{\mathrm{k}}^\dagger(\imath)\mathrm{d}\imath = (1 - p^\dagger)\mathrm{q} + K,$$

where the sets of workers $\mathcal{W}^\dagger$ and capitalists $\mathcal{K}^\dagger$ partition on the basis of past investment decisions the set $\mathcal{N}^\dagger$ of those now outside the incumbent group.

An argument similar to Proposition 1 shows that the new equilibrium allocation would equalize continuation payoffs for all individuals outside the incumbent group ($\imath \in \mathcal{N}^\dagger$). This means that any notional claims to capital will be set aside and individuals' payoffs will be determined according to their power, with capital redistributed according to a new allocation. Capitalists are not compensated for any past efforts because those effort costs are sunk. Let $C_{\mathrm{n}}^\dagger = C_{\mathrm{w}}^\dagger(\imath) = C_{\mathrm{k}}^\dagger(\imath)$ denote the common consumption level of all individuals outside the incumbent group (workers and expropriated capitalists). Also as in Proposition 1, payoffs of incumbents will be equalized, with $C_{\mathrm{p}}^\dagger = C_{\mathrm{p}}^\dagger(\imath)$ denoting this common payoff. The resource constraint implies the incumbents' payoff is

$$U_{\mathrm{p}}^\dagger = C_{\mathrm{p}}^\dagger = \frac{(1 - p^\dagger)(\mathrm{q} - C_{\mathrm{n}}^\dagger) + K}{p^\dagger}. \qquad [3.4]$$

Using the argument of Proposition 1, the equilibrium allocation following a rebellion at the post-investment stage can be characterized by maximizing the payoff of incumbents in (3.4) subject to a single 'no-rebellion constraint'

$$U_{\mathrm{n}}^\dagger \leq U_{\mathrm{p}}'(K) - \delta\frac{p^\dagger}{p'(K)},$$

where $p'(K)$ and $U_{\mathrm{p}}'(K)$ are features of the equilibrium allocation that would follow a further rebellion.[43] In equilibrium, p' and U'_{p} may be functions of the aggregate capital stock K, which is a fundamental state variable at this stage. Given payoff equalization and the resource constraint, the variables to be determined in the constrained maximization problem are power sharing $p^\dagger$ and the consumption $C^\dagger_{\mathrm{n}}$ of those outside the incumbent group.

The equilibrium allocation is found by solving the constrained maximization problem and then imposing the equilibrium conditions $p^\dagger(K) = p'(K)$ and $U^\dagger_{\mathrm{p}}(K) = U'_{\mathrm{p}}(K)$. The unique equilibrium values of each variable are as follows:

$$p^\dagger = \frac{\delta}{\mathrm{q}+2\delta}, \quad U^\dagger_{\mathrm{p}}(K) = \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} + K, \quad \text{and} \;\; U^\dagger_{\mathrm{w}}(K) = \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} - \delta + K. \tag{3.5}$$

The equilibrium allocation is such that everyone outside the incumbent group receives utility $U^\dagger_{\mathrm{w}}(K)$. Equilibrium power sharing $p^\dagger$ is independent of K and is the same as that found in the endowment-economy model with linear utility from section 2.5.[44] The results show that were a rebellion to occur at the post-investment stage, the entire capital stock would be expropriated and equally distributed among the whole population. This is because the presence of capital increases incentives for further rebellions.

The allocation (3.5) can be implemented by taxing all non-incumbents an amount τ given in (2.13) and distributing the proceeds equally among incumbents (as in section 2.5), and by a 100% tax on holdings of capital with the proceeds divided equally among all individuals.

3.2.2 Pre-investment stage allocation

The equilibrium allocation will avoid rebellions at both the pre- and post-investment stages. Potential rebellions at these stages need to be considered separately owing to the change in the environment as a result of accumulation of capital and the revelation of information after the investment stage.

Given that there will be no rebellion against the allocation in equilibrium, an individual with an investment opportunity will decide to take it or not on the basis of the prevailing investment-contingent consumption allocation. An *investor* is an individual who receives and takes an investment opportunity (and who will be referred to as a *capitalist* at the post-investment stage). If individual $\imath$ receives an investment opportunity with effort cost θ then he obtains utility $U_{\mathrm{i}}(\imath;\theta) = C_{\mathrm{k}}(\imath) - \theta$ by taking the opportunity. If he chooses not to take it, he becomes a worker and obtains utility $U_{\mathrm{w}}(\imath) = C_{\mathrm{w}}(\imath)$. This gives rise to an incentive compatibility constraint (conditional on the realization of θ) that must hold if individual $\imath$ is to invest:

$$C_{\mathrm{k}}(\imath) - C_{\mathrm{w}}(\imath) \geq \theta.$$

Let s denote the proportion of those receiving an investment opportunity for whom the incentive

[43] Under the parameter restrictions in (3.3), in equilibrium, all non-negativity constraints on consumption are slack, the bound $p^\dagger < 1/2$ is respected, and the condition $U^\dagger_{\mathrm{p}}(K) > U^\dagger_{\mathrm{w}}(K)$ for those in power to belong to the incumbent army is satisfied.

[44] This analytically convenient finding is owing to the linearity of utility in consumption.

constraint is satisfied. The resource constraint is then

$$\int_{\mathcal{P}} C_{\mathrm{p}}(\imath)\mathrm{d}\imath + \int_{\mathcal{W}} C_{\mathrm{w}}(\imath)\mathrm{d}\imath + \int_{\mathcal{K}} C_{\mathrm{k}}(\imath)\mathrm{d}\imath = (1-p)\mathrm{q} + \mu\kappa s,$$

and the post-investment capital stock K is:

$$K = \mu\kappa s.$$

The expected utility of individual $\imath \in \mathcal{N}$ outside the incumbent group (who does not yet know whether he will become a worker or an investor) is

$$U_{\mathrm{n}}(\imath) = (1-\alpha)C_{\mathrm{w}}(\imath) + \alpha\mathbb{E}_{\theta}\max\{C_{\mathrm{k}}(\imath) - \theta, C_{\mathrm{w}}(\imath)\}, \quad \text{where } \alpha = \frac{\mu}{1-p}. \tag{3.6}$$

In this expression, α is the probability of any individual outside the incumbent group receiving an investment opportunity, which is the total measure of such opportunities divided by the measure of those individuals.[45]

The following proposition presents the key features of the equilibrium allocation.

Proposition 2 *Any equilibrium allocation with $s > 0$ must have the following features:*

(i) Payoff equalization among all workers, and payoff equalization among all incumbents: $U_{\mathrm{w}} = U_{\mathrm{w}}(\imath)$, $U_{\mathrm{p}} = U_{\mathrm{p}}(\imath)$.

(ii) Consumption equality among capitalists ($C_{\mathrm{k}} = C_{\mathrm{k}}(\imath)$) is consistent with equilibrium without loss of generality, and all no-rebellion constraints for rebel armies with a positive measure of capitalists are slack. In equilibrium, there is a binding incentive compatibility constraint for capitalists that can be expressed as a threshold condition $\theta \leq \tilde{\theta}$ for investment. The effort cost threshold $\tilde{\theta}$ and the implied fraction $s = \mathbb{P}_{\theta}[\theta \leq \tilde{\theta}]$ of investment opportunities that are taken are given by:

$$\tilde{\theta} = C_{\mathrm{k}} - C_{\mathrm{w}}, \quad \textit{and} \;\; s = \frac{\tilde{\theta} - \psi}{\kappa - \psi}. \tag{3.7a}$$

(iii) All no-rebellion constraints at the pre-investment stage are slack. The equilibrium allocation can be characterized by only two binding no-rebellion constraints at the post-investment stage, one for rebellions including only workers, and one for rebellions including only incumbents:

$$U_{\mathrm{w}} \geq U_{\mathrm{p}}^{\dagger}(K) - \delta\frac{p}{p^{\dagger}}, \quad \textit{and} \;\; U_{\mathrm{p}} \geq U_{\mathrm{p}}^{\dagger}(K) - \delta\frac{(p-p^{\dagger})}{p^{\dagger}}, \tag{3.7b}$$

or equivalently, any two linearly independent combinations of these no-rebellion constraints.

[45] Note that the parameter restrictions in (3.3) imply $\mu < 1/2$, and since $p < 1/2$, individuals outside the group in power are always more than 50% of the population, and hence there are more of them than investment opportunities.

(iv) The binding incentive-compatibility constraint (3.7a) *and the binding no-rebellion constraints* (3.7b) *imply that power sharing p must satisfy*

$$p = p^{\dagger} + \frac{\mu\tilde{\theta}s}{\delta}. \qquad [3.7c]$$

Credible incentives for investment ($s > 0$) thus require that power is not as concentrated as incumbents would like it to be, ex post ($p > p^{\dagger}$).

(v) Given the power-sharing constraint (3.7c)*, the payoff of an incumbent in terms of s is*

$$U_{\mathrm{p}} = \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} + \mu\left(\kappa - \left(\frac{\mathrm{q}+2\delta}{\delta}\right)(\psi + (\kappa-\psi)s)\right)s. \qquad [3.7d]$$

Proof See appendix A.2. ■

Conditional on the fraction s of investment opportunities that are taken, which is still to be determined, the equilibrium allocation is completely characterized by the results of Proposition 2. It can be implemented by a lump-sum tax τ_{q} on both workers and capitalists with the proceeds distributed equally among incumbents (as in section 2.5), combined with a tax on capitalists τ_{k} with the proceeds divided equally among everyone (including the capitalists themselves). The required values of τ_{q} and τ_{k} are:

$$\tau_{\mathrm{q}} = \frac{\delta(\mathrm{q}+\delta)}{\mathrm{q}+2\delta} + \frac{\mu(\mathrm{q}+\delta)(\psi+(\kappa-\psi)s)s}{\delta}, \quad \text{and} \quad \tau_{\mathrm{k}} = (\kappa-\psi)(1-s).$$

The equilibrium allocation is such that the marginal investor receives no surplus from investing, while the most efficient investor keeps a fraction s of the whole surplus. Although capital is not fully expropriated, it is still taxed.

The intuition for the equalization of payoffs in equilibrium among workers and among incumbents is the same as in Proposition 1. The second part of Proposition 2 shows that no investor belongs to a rebel army associated with a binding no-rebellion constraint. The basic reason is that providing incentives to investors means granting them higher consumption than workers, and thus higher utility ex post once the sunk effort cost of investing has already been incurred (ex ante, the marginal investor has the same utility as a worker). The analysis of section 3.2.1 shows that the new equilibrium allocation established following a rebellion at the post-investment stage would not respect individuals' holdings of capital prior to the rebellion. Thus, what investors stand to receive by participating in a rebellion (net of fighting costs) is no different from that of workers who rebel (their power is identical), while what they lose is superior if they currently hold capital. Accordingly, they are less willing to fight to replace the current allocation. This implies the distribution of income needed to provide incentives to invest is not one that investors themselves could enforce by a credible threat to participate in rebellions now or at some future stage.

The fundamental problem lying behind the results of Proposition 2 is that the distribution of income needed to encourage investment diverges from that consistent with the distribution of power, and so there are incentives for groups to rebel against allocations granting property rights to investors. As usual, the no-rebellion constraint for workers is binding because incumbents gain by extracting as much as possible from them. What is novel here is that discouraging rebellion by workers is no longer sufficient for an allocation to survive when investment opportunities are taken: the incumbents must also worry about rebellion from within their own ranks. Ex ante, incumbents would like to establish an allocation encouraging investment by allowing investors to keep a large part of the capital they produce, but there is also the temptation ex post for them to participate in a rebellion that will allow for a new allocation specifying full expropriation. The fact that the effort cost of investment is sunk gives rise to a time-inconsistency problem, which is reflected in the threat of rebellion coming from inside as well as outside the group in power.[46]

Given this time-inconsistency problem, it might be thought impossible to sustain any investment in equilibrium because individuals cannot commit not to rebel. Since the defence of the current allocation relies on those with power, a rebellion backed by all incumbents succeeds without requiring any fighting effort. Were the size p of the group in power equal to its equilibrium size $p^\dagger$ following a rebellion, those in power would be able to change the current allocation through a costless 'suspension of the constitution'. A new allocation could be established that leaves all the current incumbents in power, essentially granting them full discretion ex post to change the allocation of resources. However, when power is shared more broadly and thus $p > p^\dagger$, costless suspension of the constitution is not possible. The equilibrium size of the group in power after the rebellion is smaller than beforehand, so some incumbents must lose their positions. The rebellion launched by insiders is now necessarily a 'coup d'état' that shrinks the incumbent group. Conflict with those incumbents who would lose their positions of power makes this a costly course of action.

The analysis in Proposition 2 confirms that satisfaction of the no-rebellion constraints for both workers and incumbents is equivalent to ensuring power sharing p at the pre-investment stage is large in relation to $p^\dagger$, the equilibrium power sharing after a rebellion at the post-investment stage. The claim in (3.7c) is that credible limits on expropriation require more power sharing than what would be optimal for incumbents after investment decisions have actually been made. Furthermore, as the proportion s of investors rises, the amount of power sharing p that is needed increases. Given the incentive constraint, a higher value of s requires that investors keep more of the capital they produce, which worsens the time-inconsistency problem for incumbents.

The proposition shows that not only is this increase in power sharing *sufficient* for credible protection of property rights; it is also *necessary*. There is no other allocation of power and resources that can both establish credible incentives for investors and survive the power struggle. In particular, it might be thought possible to solve the problem by setting up a different allocation of resources, essentially 'buying off' those who would otherwise rebel. But discouraging rebellion by workers would require transferring resources from incumbents to workers, while discouraging rebellion by incumbents would require transfers in the opposite direction. Transfers away from investors would

[46] The no-rebellion constraint for the incumbents places a lower bound on their payoff U_{p} even though the equilibrium allocation is set up to maximize U_{p} ex ante. The constraint then represents the absence of incentives to deviate from the initial allocation of power and resources through rebellion ex post.

of course destroy the very incentives that must be preserved. The only way to discourage rebellion simultaneously from both inside and outside the group in power is an increase in power sharing. Fundamentally, transfers are a zero-sum game, and can only redistribute disgruntlement with an allocation.[47]

Sharing power among a wider group thus allows incumbents to act as a government committed to rules that would otherwise be time inconsistent. Even though all individuals have the discretion to participate in a rebellion against the allocation of power and resources, overcoming the time-inconsistency problem is feasible. Sharing power thus emerges endogenously as a commitment device. It provides a solution to the classic problem of 'who will guard the guardians?': time-inconsistent rules can be protected from those who hold power (the 'rule of law') when some incumbents fear losing their privileged status if the rules are changed from within.[48]

Broadly speaking, the extra individuals in power required to sustain property rights (the difference between p and $p^{\dagger}$) might be interpreted as a 'parliament', or an 'independent judiciary', or any other group with the power to resist attempts to change the allocation of power and resources, coming especially from others in power. In the model, these extra individuals in power are in no way intrinsically different from other incumbents, and do not have access to any special technology directly protecting property rights.[49] Power sharing enables commitment to otherwise time-inconsistent rules because it makes it costlier for incumbents to replace the current allocation with a new one — with potential differences in how resources and power are distributed. Once power is too concentrated, rules become subject to the whims of those in power, as noted by Montesquieu.

The essence of the argument is that while the model presupposes that an investment-contingent allocation of resources can be established, for any property rights specified by that allocation to be 'rules' in the sense that is commonly understood, they must survive in a world where allocations can always be contested by rebellions. The result here is that property rights can be a feature of the equilibrium allocation only if power is shared among a larger group. The model does not allow potential rebels to commit to creating a new allocation that is not an equilibrium, in particular, one which is not in their interests after the rebellion. Once individuals have incurred the sunk effort costs of investing, those in power would like to sign a 'rebellion contract' where they agree to change the allocation to expropriate capital, but bind themselves not to change the allocation when

[47] On the other hand, the notion of being in power is essentially an ability when fighting occurs to impose costs on others at a lower cost to oneself.

[48] One early historical example that resonates with the finding of a connection between power sharing and the rule of law is provided by Malmendier (2009), who studies the Roman *societas publicanorum*. These were groups (precursors of the modern business corporation) to which the government contracted functions such as tax collection and public works. Their demise occurred with the transition from the Roman republic to the Roman empire. Why? According to Malmendier (2009), one possible explanation is that "the Roman Republic was a system of checks and balances. But the emperors centralized power and could, in principle, bend law and enforcement in their favor." In other words, while power was decentralized, it was possible to have rules that guaranteed the government's adherence to its contract with the *societas publicanorum* and their property rights, presumably because changing the allocation of power and resources would result in some of the individuals in power coming into conflict with their peers, which would be costly. Once power was centralized, protection against expropriation was not possible any longer.

[49] In practice, the roles of these extra individuals in power are specific (legislative, judicial, etc.), but the model suggests that commitment to time-inconsistent rules relies not only on the actual functions of the additional incumbents, but also on power being distributed among a larger group in itself.

it comes to the distribution of power. However, each has an incentive to reduce the extent of power sharing (imposing the loss of status on others within the former incumbent group), so this contract could only be enforced by some exogenous higher authority. In the absence of such a thing, rebels may contest the existing allocation, but cannot commit to what they will then do next.[50]

3.3 The equilibrium allocation and economic efficiency

Economic development ultimately requires rewarding the productive rather than just the strong, and for this to happen, there must be credible protection of investors' property rights. It is an endogenous feature of the model that broader power sharing can sustain such a commitment, but is establishing such an allocation in the interests of those in power?[51]

Proposition 2 shows that the equilibrium allocation can be found by maximizing the incumbent payoff subject to two binding no-rebellion constraints and a binding incentive constraint.[52] The allocation is characterized by power sharing p, and given payoff equalization, the consumption levels C_{p}, C_{w}, and C_{k} of incumbents, workers, and capitalists, respectively. Using the resource constraint and the power-sharing constraint (which combines the two binding no-rebellion constraints), the payoff of incumbents can be written in terms of the fraction s of investment opportunities that are taken, which is linked to consumption levels via the incentive constraint. This payoff is given in equation (3.7d), which is maximized by the following choice of s:

$$s^* = \max\left\{0, \frac{\delta\kappa - (q + 2\delta)\psi}{2(q + 2\delta)(\kappa - \psi)}\right\}. \quad [3.8]$$

As confirmed below in Proposition 3, this characterizes the unique equilibrium of the model.[53] But does s^* correspond to the efficient level of investment?

To study the question of efficiency, the payoffs of all individuals are derived given the measure p of incumbents and the fraction s of investment opportunities that are taken. The values of p and s imply that there are $1 - p - \mu s$ workers, and μs investors with utility $U_{\mathrm{i}}(\theta) = C_{\mathrm{k}} - \theta$ (those for

[50]If it were possible for rebels to commit to restrict reoptimization to certain areas following a rebellion then paradoxically this makes it harder to establish an allocation that gives rise to commitment to time-inconsistent rules. For example, suppose the distribution of power is defined on the first page of the constitution, and limits on expropriation of private property are specified on the second page. If it were feasible somehow to prevent a successful rebellion from rewriting page one of the constitution then this would annihilate the credibility of page two.

[51]Guimaraes and Sheedy (2015) extend the framework here to a world of open economies where the possibility of trade affects the relative price of two goods, which adds an international dimension to the question of whether property rights will be protected. For the reasons highlighted here, production of an investment good requires power sharing, while the other good is modelled as an endowment. The model predicts that trade generates divergence as production of the investment good becomes concentrated in a subset of economies where power is shared broadly, so trade benefits some economies, but harms others.

[52]It is verified in Proposition 3 that the non-negativity constraint $C_{\mathrm{w}} \geq 0$, the bound $p < 1/2$, and the conditions $U_{\mathrm{p}} > U'_{\mathrm{n}}$ and $U_{\mathrm{p}} > U^{\dagger}_{\mathrm{w}}(K)$ ensuring that incumbents who do not rebel will belong to the incumbent army are all satisfied in equilibrium.

[53]Since s is typically small and the fraction of the surplus received by investors lies between 0 and s, investors do not keep much of the proceeds of investment above and beyond what is required to compensate them for their efforts.

whom the realization of θ is no more than $\tilde{\theta}$). The average utility $\bar{U}$ over all individuals is

$$\bar{U} \equiv \int_{\Omega} U(\imath)\mathrm{d}\imath = pU_{\mathrm{p}} + (1 - p - \mu s)U_{\mathrm{w}} + \mu s \mathbb{E}_{\theta}[U_{\mathrm{i}}(\theta)|\theta \le \tilde{\theta}].$$

Average utility can be written in terms of the expected surplus $\mathcal{S}_{\mathrm{i}}(\tilde{\theta})$ from receiving an investment opportunity:

$$\bar{U} = pU_{\mathrm{p}} + (1 - p)U_{\mathrm{w}} + \mu\mathcal{S}_{\mathrm{i}}(\tilde{\theta}), \quad \text{where } \mathcal{S}_{\mathrm{i}}(\tilde{\theta}) \equiv \mathbb{E}_{\theta} \max\{\tilde{\theta} - \theta, 0\}. \qquad [3.9]$$

Since the distribution of θ lies between ψ and κ, it is easy to see that the surplus $\mathcal{S}_{\mathrm{i}}(\tilde{\theta})$ is maximized when $\hat{\tilde{\theta}} = \kappa$, that is, when all investment opportunities are taken ($\hat{s} = 1$). With no restrictions on the distribution of resources, this choice maximizes $\bar{U}$, and is hence the first-best level of investment. It also follows from the resource constraint that the first-best size of the incumbent group is zero ($\hat{p} = 0$) because those in power do not receive the endowment q.

However, the first best is not the most interesting welfare benchmark. A key lesson of the model is that the property rights necessary to provide incentives to invest only survive the power struggle if power is shared among a sufficiently large group. Therefore, in a world where a social planner must respect the constraints imposed by the power struggle and cannot freely choose all aspects of an allocation of power and resources, an increase in power sharing has the benefit of raising investment. This must be set against the opportunity cost of having more individuals in power who are diverted from directly productive occupations.

Taking that trade-off into account, consider the following notion of *constrained efficiency*. Suppose that it were possible for a social planner to choose some level of investment (and thus the variable s) at all pre-investment stages of the power struggle when a new allocation is established. But all other aspects of an allocation are determined endogenously as before, which means they must ensure s is consistent with investors' incentive-compatibility constraints and the no-rebellion constraints arising from the power struggle. The constrained-efficient level of s is what would then be chosen by a benevolent social planner who takes into account the environment in which the equilibrium allocation is determined. The social planner would appreciate that more investment requires greater protection of property rights, and thus more power sharing. The concept of constrained efficiency requires weighing the benefit of more investment against the resource cost of the necessary increase in power sharing.[54] In the public-good example considered in section 2.6, a social planner could not improve upon the efficiency of the allocation by imposing a level of public-good provision different from what prevails in equilibrium. Here, the issue is whether the equilibrium amount of investment coincides with its constrained-efficient level.

To find the constrained-efficient level of investment, the social planner maximizes the average utility $\bar{U}$ of all individuals subject to the binding constraints identified in Proposition 2 (the configuration of binding constraints is the same even with an exogenously imposed level of $s > 0$).

[54]If there were no resource cost of increasing the size of the group in power then the constrained-efficient level of investment would coincide with the first best. Note that in the model with the public-good technology from section 2.6, the first-best and constrained-efficient levels of public-good provision are the same.

As a consequence of (3.7b), workers' and incumbents' payoffs are tied together by $U_{\mathrm{w}} = U_{\mathrm{p}} - \delta$. Since the social planner takes such constraints into account, this relationship is substituted into the expression for $\bar{U}$:

$$\bar{U} = U_{\mathrm{p}} - \delta(1 - p) + \mu \mathcal{S}_{\mathrm{i}}(\tilde{\theta}). \qquad [3.10]$$

There are thus two differences between the expressions for average utility $\bar{U}$ and the incumbents' utility U_{p}. The second term on the right-hand side is related to the distribution of resources between individuals with different levels of power, and the third term reflects the investors' surplus.

The expression for $\bar{U}$ is maximized by noting that the binding constraints imply U_{p} is given by (3.7d), and that these constraints are equivalent to the power-sharing constraint (3.7c) linking p and s, and by making use of the expression for the investors' surplus from (3.9).

Proposition 3 *(i) The unique equilibrium s^* is given by the expression in* (3.8).

(ii) The constrained-efficient level of s (denoted by $s^\diamond$), which maximizes $\bar{U}$ in (3.10)*, is given by the following expression (if the non-negativity constraint on workers' consumption does not bind):*

$$s^\diamond = \max\left\{0, \frac{\delta\kappa - (\mathrm{q} + \delta)\psi}{(2\mathrm{q} + \delta)(\kappa - \psi)}\right\}. \qquad [3.11]$$

(iii) s^* is positive when $\kappa/\psi - 1 > 1 + \mathrm{q}/\delta$, while $\kappa/\psi - 1 > \mathrm{q}/\delta$ is necessary for $s^\diamond > 0$. Whenever $s^\diamond > 0$, it must be the case that $s^* < s^\diamond$.

Proof See appendix A.3. ■

The constrained-efficient level of s is positive ($s^\diamond > 0$) only if the return to investment $\kappa/\psi - 1$ for the most efficient investor is larger than the resource cost q/δ of the increase in power sharing needed to protect that investor's property rights. To understand this expression, the power sharing constraint (3.7c) implies that protecting the most efficient investor from expropriation requires increasing the size of the incumbent group by ψ/δ. For each worker who becomes an incumbent, q units less of output are produced, so the resource cost of protecting the most efficient investor is $\mathrm{q}\psi/\delta$, which corresponds to the required return on investment q/δ.

The condition for investment to occur in equilibrium ($s^* > 0$) is that the return to investment for the most efficient investor is larger than $1 + \mathrm{q}/\delta$. This required return is greater than that for a positive level of investment to be constrained efficient. The reason for the difference is that incumbents take into account the dilution of rents that follows from power sharing in addition to the resource cost. As before, equation (3.7c) shows that power sharing must increase by ψ/δ to protect the most efficient investor. Since incumbent and worker payoffs need to satisfy $U_{\mathrm{p}} = U_{\mathrm{w}} + \delta$ in equilibrium, each extra incumbent is able to lay claim to rents of δ. Therefore, the cost to other incumbents of protecting the first investor is the resource cost plus $\delta\psi/\delta = \psi$, which adds 100% to the required return on investment.

More generally, although the equilibrium level of investment is positive for a range of parameters, the constrained-efficient level of investment is larger because of the presence of two distortions. The first (and more interesting) distortion follows from the distributional consequences of protection against expropriation described above, which appear in the second term on the right-hand side of equation (3.10). Credible commitment to property rights requires sharing power, which in turn requires sharing rents because incumbents are more powerful than other individuals. Incumbents have access to an endogenous commitment device through power sharing that can in principle implement the constrained-efficient level of investment, but the need to avoid rebellions means this entails sharing rents. The cost to incumbents of expanding their number is not simply the lost output from diverting individuals away from directly productive activities.

In contrast to the public-good example of section 2.6, in an environment with investment, the power struggle imposes an endogenous and binding constraint on the set of possible transfers among individuals. This leads to the breakdown of the political analogue of the Coase theorem. Power sharing can give rise to credible commitment, but the association between power and rents places a lower bound on the consumption of each individual in power. The impossibility of sharing power without sharing rents thus drives a wedge between maximizing total output and maximizing an incumbent's payoff.[55]

The second distortion that results in investment being too low is that the equilibrium allocation does not take account of investors' surplus, which corresponds to the third term on the right-hand side of equation (3.10). Since investors' effort costs θ are not public information, it is impossible for an allocation to specify individual consumption levels that are contingent on this information. The no-rebellion constraints for rebel armies including non-marginal investors will therefore be slack, so no benefit accrues to incumbents from increases in such investors' payoffs. This increases the wedge between total output and the payoff of incumbents.[56]

Acemoglu, Johnson and Robinson (2005) present evidence that institutional failures in providing adequate protection of property rights are especially damaging to economic performance. But why should property rights be so susceptible to political failures compared to other aspects of institutions? The model here sheds light on this question by explaining why there is often tenacious opposition by incumbents to institutions that would effectively 'guard the guardians', allowing for credible property rights. For example, in seventeenth-century England, the Glorious Revolution led to power sharing between king and parliament. By accepting the Bill of Rights, King William III conceded that power would be shared. North and Weingast (1989) argue that the Glorious Revolution began an era of secure property rights and put an end to confiscatory government. As a result, the English government was able to borrow much more, and at substantially lower rates. This was certainly

[55]The welfare implications of the power parameter δ are non-trivial here. In the public-good example of section 2.6, a larger δ can only be harmful to workers because it allows more rents to be extracted, resulting in a more unequal distribution of income. In contrast, in an economy with a very small value of δ, there would not be any investment in equilibrium. A larger δ makes it easier for incumbents to remain in power, which directly benefits them, but might also allow them to offer some protection of property rights.

[56]The first and second distortions correspond respectively to the second and third terms in the expression for $\bar{U}$ in (3.10). The effects of each of them on the first-order condition determining s^* are thus seen to operate independently of the other distortion.

in the interests of the king, yet the earlier Stuart kings had staunchly resisted sharing power with parliament. According to the model, secure property rights require just such power sharing to make it costly for the king to rewrite the rules ex post. However, the existence of a parliament with real power implies that rents have to be shared, so even if the total pie becomes larger, with a smaller share, the amount received by the king might end up being lower.

This raises the question of why the calculus of the ruling elite shifted over time from keeping power highly concentrated to broader power sharing between king and parliament (and the separation of powers more generally). The comparative statics of the model provide a tentative answer. The equilibrium level of power sharing p^* is an increasing function of s^*, which in turn depends positively on κ and negatively on q and ψ. If, for example, the technological progress leading up to the industrial revolution can be interpreted as a higher return to investment (captured in the model by a larger κ), this would spur incumbents to share power where previously this had been resisted. The model is thus consistent with the emergence of parliaments and independent courts as a response to new technologies that open up profitable opportunities for investment.

4 Concluding remarks

Research in economics has frequently progressed by focusing on the behaviour of individuals subject to some fundamental constraints or frictions and deriving the resulting implications for the economy. For example, it is often claimed that unemployment, credit rationing, and missing markets ought not to be directly assumed, but instead derived from the likes of search frictions, limited pledgeability, or asymmetric information. This paper proposes a model of the allocation of power and resources that emerges from the power struggle which we think of as an attempt to model politics in that tradition. The building blocks of the model are the basics of preferences, technologies, and a single rebellion mechanism that allows individuals to form groups and fight for power. Those in power establish an allocation and have an advantage in defending the status quo, but the option of rebelling against an allocation is open to everyone on the same terms.

The modelling of conflict abides by the principle that the mechanism for contesting any allocation does not depend on what that allocation prescribes. This is important to ensure results are not due to individuals having access to different mechanisms to change or resist changes to the various aspects of an allocation. In particular, credible commitment does not arise because of an assumption that some aspects of an allocation cannot ever be changed, nor does commitment fail to arise because some aspects of an allocation can always be changed without a contest. The model also adopts the principle of imposing no exogenous restrictions on the transfers that an allocation can prescribe. This is important to ensure commitment neither arises nor fails to arise owing to assumptions on the functional forms of feasible tax or transfer schedules.

The model is used to study an environment where investment is possible but can be expropriated. In order to 'guard the guardians', that is, ensure those in power adhere to time-inconsistent rules, the incumbent group is endogenously enlarged. But the same conflict mechanism that explains how power sharing gives rise to the 'rule of law' also implies that sharing power cannot be done without sharing rents. This imposes endogenous limits on the set of possible allocations and leads to a breakdown of the political analogue of the Coase theorem. In equilibrium, while there is commitment to some protection of individual rights, there is too little power sharing to achieve economic efficiency.

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A Technical appendix

A.1 Proof of Proposition 1

Given the third and fourth equilibrium conditions in Definition 2 and the absence of fundamental state variables at any stage of the power struggle in Figure 1, the allocation of power and resources established after any rebellion must feature some particular amount p' of power sharing and some particular average incumbent payoff U'_{p}. These requirements place no restriction on the identities of the p' individuals who would belong to the subsequent incumbent group $\mathcal{P}'$. The first and second equilibrium conditions in Definition 2 then require that the equilibrium allocation maximizes the average payoff of incumbents subject to the absence of any rational rebellion (the incumbents have no interest in choosing an allocation that triggers a successful rebellion). Given p' and U'_{p} and a current allocation, the criteria in Definition 1 for a rational rebellion can be assessed for each subsequent incumbent group $\mathcal{P}'$ with measure p'. Considering all possible $\mathcal{P}'$ with measure p' leads to a set of no-rebellion constraints, and the average incumbent payoff can be maximized subject to these constraints.

There must be a binding no-rebellion constraint for a rebel army including a positive measure of workers

If not, then there must be a positive measure of workers for whom inclusion in a rebel army (displacing incumbents) strictly reduces the effective fighting strength of the rebel army (effective fighting strength in the sense of the total fighting strength of the rebels plus any loss of strength by the incumbent army when some of those in power join the rebel army). It follows that consumption of these workers can be reduced by some positive amount, allowing the average incumbent payoff to be increased, but without implying that any rebellion would now be rational.

Payoff equalization among workers

Given that a rebel army associated with a binding no-rebellion constraint must include a positive measure of workers and since the size of any incumbent group cannot exceed the number of workers, a rebel army never includes all workers. Now suppose there is payoff inequality among workers. Consider a different allocation where consumption of all workers is equal to the previous average worker payoff. Given weakly concave utility, this modification of the allocation is feasible, and with strictly concave utility, it leaves surplus resources that can now be distributed among the incumbents. Since the previous strongest rebel army including workers would include those with the lowest payoffs, following this change, the maximum fighting strengths of all rebel armies including workers are reduced. This slackens a binding constraint, so payoff inequality among workers is inconsistent with an allocation being optimal. Claim (i) of the proposition is therefore established.

Power determines rents in equilibrium

Let $\bar{U}^*_{\mathrm{p}}$ denote the average incumbent payoff in equilibrium. The claim is that $\bar{U}^*_{\mathrm{p}} = U^*_{\mathrm{w}} + \delta$. To prove this, first suppose $\bar{U}^*_{\mathrm{p}} > U^*_{\mathrm{w}} + \delta$. Since $U'_{\mathrm{p}} = \bar{U}^*_{\mathrm{p}}$ in equilibrium, it would follow that a rebel army including only workers (of which there are enough) would put in fighting effort strictly greater than δp^* (given that the rebel army would have size $p' = p^*$ in equilibrium). This rebellion would be rational, hence it must be that $\bar{U}^*_{\mathrm{p}} \leq U^*_{\mathrm{w}} + \delta$.

Now suppose $\bar{U}^*_{\mathrm{p}} < U^*_{\mathrm{w}} + \delta$. Consider the following modification of the allocation. First, redistribute all consumption equally among incumbents. Given weakly concave utility, the new common payoff U_{p} is at least as large as $\bar{U}^*_{\mathrm{p}}$. Next, reduce consumption of all workers by a positive amount, but small enough so that $\bar{U}^*_{\mathrm{p}} \leq U_{\mathrm{w}} + \delta$ holds, where U_{w} is the new payoff of workers. The reduction in workers' consumption ensures $U_{\mathrm{p}} > \bar{U}^*_{\mathrm{p}}$. Given that this means $U_{\mathrm{p}} > U'_{\mathrm{p}}$, no-one in the group in power is willing to join a rebel army. A rebel army can then only comprise workers, but the modification of the allocation is constructed to ensure such a rebellion would not be rational. The claim is proved.

Payoff equalization among incumbents

Given that $\bar{U}^*_\mathrm{p} = U^*_\mathrm{w} + \delta$ and $U'_\mathrm{p} = \bar{U}^*_\mathrm{p}$ hold in equilibrium, it follows that each worker would not be willing to exert more than δ fighting effort in a rational rebellion. Now suppose there is payoff inequality among a positive measure of incumbents, which means a positive measure r of incumbents receive payoffs strictly below the average $\bar{U}^*_\mathrm{p}$. These are willing to rebel and to put in strictly positive fighting effort. A rebel army including these incumbents and with the remaining places filled by a measure $p^* - r$ of workers would have total fighting strength larger than $\delta(p^* - r)$, which is the fighting strength of the incumbent army, so the rebellion would be rational. Therefore, payoff inequality among incumbents is inconsistent with equilibrium, proving claim (ii) of the proposition. The common incumbent payoff must then satisfy $U^*_\mathrm{p} - U^*_\mathrm{w} = \delta$, confirming claim (iii).

Equivalence to a single no-rebellion constraint

With payoff equalization among all workers and among all incumbents, the choice of an allocation reduces to a choice of power sharing p and a common level of consumption for workers C_w (with consumption of incumbents determined by the resource constraint). An optimal allocation is then the solution of the following maximization problem

$$\max_{p,C_\mathrm{w}} U_\mathrm{p} \;\text{ s.t. }\; \sigma(U'_\mathrm{p} - U_\mathrm{w}) + (1-\sigma)\mathbb{1}[U_\mathrm{p} \leq U'_\mathrm{p}](U'_\mathrm{p} - U_\mathrm{p} + \delta) \leq \delta\frac{p}{p'} \;\text{ for all } \sigma \in \left[\max\left\{0, \frac{p'-p}{p'}\right\}, 1\right], \tag{A.1.1}$$

where σ denotes the fraction of places in the post-rebellion incumbent group assigned to those who are currently workers. The equilibrium allocation is characterized by the values of p^* and C_w that solve the maximization problem (A.1.1) taking p' and U'_p as given, but with $p' = p^*$ and $U'_\mathrm{p} = U^*_\mathrm{p}$ in equilibrium.

Now consider a simpler problem where the no-rebellion constraint from (A.1.1) is imposed only for $\sigma = 1$:

$$\max_{p,C_\mathrm{w}} U_\mathrm{p} \;\text{ subject to }\; U'_\mathrm{p} - U_\mathrm{w} \leq \delta\frac{p}{p'}. \tag{A.1.2}$$

An equilibrium of this alternative problem is defined as a solution $(p^\star, C^\star_\mathrm{w})$ of the maximization problem (A.1.2), taking p' and U'_p as given, but with $p' = p^\star$ and $U'_\mathrm{p} = U^\star_\mathrm{p}$ in equilibrium.

Start by considering an equilibrium $(p^\star, C^\star_\mathrm{w})$ of the simpler problem (A.1.2). Since this allocation must satisfy the no-rebellion constraint from (A.1.2) and $p' = p^\star$ and $U'_\mathrm{p} = U^\star_\mathrm{p}$, it follows that $U^\star_\mathrm{p} - U^\star_\mathrm{w} \leq \delta$. Therefore, for any $\sigma \in [0,1]$:

$$\sigma(U^\star_\mathrm{p} - U^\star_\mathrm{w}) + (1-\sigma)\mathbb{1}[U^\star_\mathrm{p} \leq U^\star_\mathrm{p}](U^\star_\mathrm{p} - U^\star_\mathrm{p} + \delta) = \sigma(U^\star_\mathrm{p} - U^\star_\mathrm{w}) + (1-\sigma)\delta \leq \sigma\delta + (1-\sigma)\delta = \delta = \delta\frac{p^\star}{p^\star}.$$

This demonstrates that $(p^\star, C^\star_\mathrm{w})$ satisfies the no-rebellion constraints of the original problem (A.1.1) when $p' = p^\star$ and $U'_\mathrm{p} = U^\star_\mathrm{p}$. Now take any other allocation (p, C_w) not subject to rebellion in the original problem (A.1.1), again when $p' = p^\star$ and $U'_\mathrm{p} = U^\star_\mathrm{p}$. Evaluating the no-rebellion constraint at $\sigma = 1$ yields $U^\star_\mathrm{p} - U_\mathrm{w} \leq \delta p/p^\star$, which shows that this allocation also satisfies the no-rebellion constraint of the simpler problem (A.1.2). Since $(p^\star, C^\star_\mathrm{w})$ maximizes U_p over all allocations satisfying the constraint in (A.1.2), it must be the case that $U_\mathrm{p} \leq U^\star_\mathrm{p}$ for any allocation (p, C_w) consistent with the constraint in (A.1.1). Therefore, $(p^\star, C^\star_\mathrm{w})$ is also an equilibrium of (A.1.1) as well.

Now consider the converse. Take an equilibrium (p^*, C^*_w) of the original problem (A.1.1). This allocation is clearly consistent with the constraint in (A.1.2) when $p' = p^*$ and $U'_\mathrm{p} = U^*_\mathrm{p}$ because the constraint is a special case of that in (A.1.1) when $\sigma = 1$. Suppose for contradiction that (p^*, C^*_w) is not an equilibrium of the problem (A.1.2). Since it satisfies the no-rebellion constraint, it must therefore be the case that there exists another allocation (p, C_w) such that $U_\mathrm{p} > U^*_\mathrm{p}$ satisfying the no-rebellion constraint in (A.1.2) when $p' = p^*$ and $U'_\mathrm{p} = U^*_\mathrm{p}$. Now take any $\sigma \in [0,1]$ and multiply both sides of the inequality in (A.1.2) by this number to obtain:

$$\sigma(U^*_\mathrm{p} - U_\mathrm{w}) \leq \sigma\delta\frac{p}{p^*} \leq \delta\frac{p}{p^*}.$$

Observe that $(1-\sigma)\mathbb{1}[U_\mathrm{p} \le U_\mathrm{p}^*](U_\mathrm{p}^* - U_\mathrm{p} + \delta) = 0$, so this demonstrates (p, C_w) satisfies (A.1.1) for all $\sigma \in [\max\{0, (p' - p)/p'\}, 1]$. Since this allocation satisfies the no-rebellion constraint in (A.1.1), the resulting incumbent payoff cannot be higher than the incumbent payoff in equilibrium, hence $U_\mathrm{p} \le U_\mathrm{p}^*$. This contradicts the inequality $U_\mathrm{p} > U_\mathrm{p}^*$ obtained earlier, and thus proves (p^*, C_w^*) must be an equilibrium of the simpler problem (A.1.2).

In summary, it has been shown that the set of equilibria of the original problem (A.1.1) is identical to the set of equilibria of the simpler problem (A.1.2). Therefore, there is no loss of generality in imposing (2.6) as the only no-rebellion constraint, confirming claim (iv) of the proposition.

Existence and uniqueness of the equilibrium

With payoff equalization, the resource constraint implies $C_\mathrm{p} = (1-p)(\mathrm{q} - C_\mathrm{w})/p$. Worker and incumbent payoffs are given by $U_\mathrm{w} = u(C_\mathrm{w})$ and $U_\mathrm{p} = u((1-p)(\mathrm{q} - C_\mathrm{w})/p)$ respectively. It has been shown that the equilibrium allocation can be characterized as a solution of (A.1.2). Any equilibrium (p^*, C_w^*) is thus a solution of the maximization problem

$$\max_{p, C_\mathrm{w}} u\left(\frac{(1-p)(\mathrm{q} - C_\mathrm{w})}{p}\right) \text{ subject to } U_\mathrm{p}^* - u(C_\mathrm{w}) \le \delta \frac{p}{p^*}, \qquad \text{[A.1.3]}$$

taking p^* and U_p^* as given, but with $p = p^*$ and $U_\mathrm{p} = U_\mathrm{p}^*$ in equilibrium. The solution of the maximization problem must also respect the bound $p < 1/2$, and the non-negativity constraints on all individuals' consumption, which are equivalent to $0 \le C_\mathrm{w} \le \mathrm{q}$ here.

It has already been seen that the no-rebellion constraint must be binding. Using the specification of the constraint in (A.1.3), the equation for the binding constraint can be solved to obtain C_w as a function of p for given values of p^* and U_p^*:

$$C_\mathrm{w} = u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right). \qquad \text{[A.1.4]}$$

Differentiating shows that the constraint implicitly specifies a negative relationship between C_w and p:

$$\frac{\partial C_\mathrm{w}}{\partial p} = -\frac{\delta}{p^*} \frac{1}{\frac{\partial u\left(u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right)\right)}{\partial C}} < 0. \qquad \text{[A.1.5]}$$

The problem (A.1.3) is equivalent to maximizing $C_\mathrm{p} = (1-p)(\mathrm{q} - C_\mathrm{w})/p$ over values of p after substituting for C_w using equation (A.1.4):

$$\max_p \frac{(1-p)}{p}\left(\mathrm{q} - u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right)\right), \qquad \text{[A.1.6]}$$

subject to $p < 1/2$ and the value of C_w implied by (A.1.4) being such that $0 \le C_\mathrm{w} \le \mathrm{q}$. Taking the derivative of C_p with respect to p and making use of (A.1.5):

$$\frac{\partial C_\mathrm{p}}{\partial p} = \frac{1}{p^2}\left(\frac{p}{p^*} \frac{\delta(1-p)}{\frac{\partial u\left(u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right)\right)}{\partial C}} - \left(\mathrm{q} - u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right)\right)\right). \qquad \text{[A.1.7]}$$

The second derivative is

$$\frac{\partial^2 C_\mathrm{p}}{\partial p^2} = -\frac{2}{\frac{\partial u\left(u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right)\right)}{\partial C}} \frac{\delta p}{p^*} + \frac{\frac{\partial^2 u\left(u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right)\right)}{\partial C^2}}{\left\{\frac{\partial u\left(u^{-1}\left(U_\mathrm{p}^* - \delta \frac{p}{p^*}\right)\right)}{\partial C}\right\}^3} - \frac{2}{p}\frac{\partial C_\mathrm{p}}{\partial p}. \qquad \text{[A.1.8]}$$

In equilibrium it is necessary to have $p = p^*$, in which case the binding no-rebellion constraint (A.1.4) reduces to

$$C_{\mathrm{w}}^* = u^{-1}(U_{\mathrm{p}}^* - \delta),$$

or equivalently

$$u\left(\frac{(1-p^*)(\mathrm{q} - C_{\mathrm{w}}^*)}{p^*}\right) = U_{\mathrm{p}}^* = u(C_{\mathrm{w}}^*) + \delta. \qquad \text{[A.1.9]}$$

This equation is in turn equivalent to

$$p^* u^{-1}(u(C_{\mathrm{w}}^*) + \delta) = (1-p^*)(\mathrm{q} - C_{\mathrm{w}}^*). \qquad \text{[A.1.10]}$$

Evaluating the derivative of C_{p} from (A.1.7) at $p = p^*$:

$$\left.\frac{\partial C_{\mathrm{p}}}{\partial p}\right|_{p=p^*} = \frac{1}{p^{*2}}\left(\frac{\delta(1-p^*)}{\frac{\partial u\left(u^{-1}\left(U_{\mathrm{p}}^*-\delta\right)\right)}{\partial C}} - \left(\mathrm{q} - u^{-1}\left(U_{\mathrm{p}}^* - \delta\right)\right)\right).$$

Since (A.1.9) implies that $u^{-1}(U_{\mathrm{p}}^* - \delta) = C_{\mathrm{w}}^*$, this expression can be simplified as follows:

$$\left.\frac{\partial C_{\mathrm{p}}}{\partial p}\right|_{p=p^*} = \frac{1}{p^{*2}}\left(\frac{\delta(1-p^*)}{\frac{\partial u(C_{\mathrm{w}}^*)}{\partial C}} - (\mathrm{q} - C_{\mathrm{w}}^*)\right). \qquad \text{[A.1.11]}$$

Now define the following functions $\mathcal{T}(p,\varkappa)$ and $\mathcal{F}(p,\varkappa)$, which will represent respectively (with $\varkappa = C_{\mathrm{w}}$) the no-rebellion constraint and the first-order condition in equilibrium:

$$\mathcal{T}(p,\varkappa) \equiv p u^{-1}(u(\varkappa) + \delta) - (1-p)(\mathrm{q} - \varkappa), \quad \text{and} \quad \mathcal{F}(p,\varkappa) \equiv \delta(1-p) - (\mathrm{q} - \varkappa)\frac{\partial u(\varkappa)}{\partial C}. \qquad \text{[A.1.12]}$$

The no-rebellion constraint (A.1.10) holds in equilibrium when $\mathcal{T}(p^*, C_{\mathrm{w}}^*)$ is zero, while the derivative of C_{p} in (A.1.11) is proportional to $\mathcal{F}(p^*, C_{\mathrm{w}}^*)$:

$$\mathcal{T}(p^*, C_{\mathrm{w}}^*) = 0, \quad \text{and} \quad \left.\frac{\partial C_{\mathrm{p}}}{\partial p}\right|_{p=p^*} = \frac{1}{p^{*2}}\frac{1}{\frac{\partial u(C_{\mathrm{w}}^*)}{\partial C}}\mathcal{F}(p^*, C_{\mathrm{w}}^*). \qquad \text{[A.1.13]}$$

The partial derivatives of the function $\mathcal{T}(p,\varkappa)$ from (A.1.12) are:

$$\frac{\partial \mathcal{T}}{\partial p} = u^{-1}(u(\varkappa) + \delta) + (\mathrm{q} - \varkappa), \quad \text{and} \quad \frac{\partial \mathcal{T}}{\partial \varkappa} = p\frac{\frac{\partial u(\varkappa)}{\partial C}}{\frac{\partial u(u^{-1}(u(\varkappa)+\delta))}{\partial C}} + (1-p). \qquad \text{[A.1.14]}$$

Since $u(\cdot)$ is strictly increasing, $\partial u(\varkappa)/\partial C > 0$ and $\partial u(u^{-1}(u(\varkappa)+\delta))/\partial C > 0$, and also $u^{-1}(u(\varkappa)+\delta) > \varkappa$. It follows that both of the partial derivatives above are strictly positive for all $0 \leq p < 1/2$ and $0 \leq \varkappa \leq \mathrm{q}$. The partial derivatives of the function $\mathcal{F}(p,\varkappa)$ from (A.1.12) are:

$$\frac{\partial \mathcal{F}}{\partial p} = -\delta, \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial \varkappa} = \frac{\partial u(\varkappa)}{\partial C} - (\mathrm{q} - \varkappa)\frac{\partial^2 u(\varkappa)}{\partial C^2}. \qquad \text{[A.1.15]}$$

The properties of $u(\cdot)$ ensure that $\partial u(\varkappa)/\partial C > 0$ and $\partial^2 u(\varkappa)/\partial C^2 \leq 0$, so $\mathcal{F}(p,\varkappa)$ is strictly decreasing in p and strictly increasing in $\varkappa$.

Now consider two functions $\mathcal{V}(p)$ and $\mathcal{G}(p)$ defined implicitly by the equations:

$$\mathcal{T}(p, \mathcal{V}(p)) = 0, \quad \text{and} \quad \mathcal{F}(p, \mathcal{G}(p)) = 0. \qquad \text{[A.1.16]}$$

Where these functions are defined, the signs of their partial derivatives can be deduced using (A.1.14) and

(A.1.15):

$$\frac{\partial \mathcal{V}(p)}{\partial p} = -\frac{\partial \mathcal{T}}{\partial p} \bigg/ \frac{\partial \mathcal{T}}{\partial \varkappa} < 0, \quad \text{and} \quad \frac{\partial \mathcal{G}(p)}{\partial p} = -\frac{\partial \mathcal{F}}{\partial p} \bigg/ \frac{\partial \mathcal{F}}{\partial \varkappa} > 0. \qquad \text{[A.1.17]}$$

Observe from (A.1.12) that $\mathcal{F}(1, \varkappa) = -(\mathrm{q} - \varkappa)\partial u(\varkappa)/\partial C$, with the definition (A.1.16) then implying $\mathcal{G}(1) = \mathrm{q}$ given $\partial u(\varkappa)/\partial C > 0$. (A.1.17) shows that $\mathcal{G}(p)$ is strictly increasing in p, so it follows by continuity that either there exists a $\underline{\pi} > 0$ such that $\mathcal{G}(\underline{\pi}) = 0$, or $\mathcal{G}(0) \geq 0$, in which case $\underline{\pi}$ is set to zero. With the resulting $\underline{\pi} \in [0, 1)$, define $\underline{\varkappa} \equiv \mathcal{G}(\underline{\pi})$, noting that this satisfies $0 \leq \underline{\varkappa} < \mathrm{q}$ because $\mathcal{G}(\underline{\pi}) \geq 0$ and $\mathcal{G}(1) = \mathrm{q}$. The function $\mathcal{G}(p)$ is then well defined on the interval $[\underline{\pi}, 1]$ in the sense of returning a value of $\varkappa$ in the interval $[\underline{\varkappa}, \mathrm{q}]$.

It can be seen from (A.1.12) that $\mathcal{T}(0, \varkappa) = -(\mathrm{q} - \varkappa)$, so the definition in (A.1.16) implies $\mathcal{V}(0) = \mathrm{q}$. Note that since the utility function $u(\cdot)$ is strictly increasing, so is its inverse $u^{-1}(\cdot)$. It follows that $u^{-1}(u(\varkappa) + \delta) > \varkappa$ and thus for all $p > 0$:

$$\mathcal{T}(p, \varkappa) > p\varkappa - (1 - p)(\mathrm{q} - \varkappa) = \varkappa - \mathrm{q}(1 - p).$$

Using (A.1.16), this means that $0 = \mathcal{T}(p, \mathcal{V}(p)) > \mathcal{V}(p) - \mathrm{q}(1 - p)$, and hence

$$\mathcal{V}(p) < \mathrm{q}(1 - p), \quad \text{for all } \ p > 0.$$

(A.1.17) shows that $\mathcal{V}(p)$ is strictly decreasing in p, so given $\mathcal{V}(0) = \mathrm{q}$ and the bound above, it follows by continuity that there exists a $\overline{\pi} \in (0, 1)$ such that $\mathcal{V}(\overline{\pi}) = 0$. The function $\mathcal{V}(p)$ is then well defined on the interval $[0, \overline{\pi}]$ in the sense of returning a value of $\varkappa$ in the interval $[0, \mathrm{q}]$.

Let $\mathcal{V}^{-1}(\varkappa)$ denote the inverse function of $\mathcal{V}(p)$, defined on $[0, \mathrm{q}]$. Similarly, $\mathcal{G}^{-1}(\varkappa)$ is the inverse function of $\mathcal{G}(p)$, defined on $[\underline{\varkappa}, \mathrm{q}]$, where $0 \leq \underline{\varkappa} \equiv \mathcal{G}(\underline{\pi}) < \mathrm{q}$. Since $\mathcal{V}(p)$ is strictly decreasing and $\mathcal{G}(p)$ is strictly increasing according to (A.1.17), their inverse functions inherit these properties. Now define the following function $\mathcal{H}(\varkappa)$ on $[\underline{\varkappa}, \mathrm{q}]$:

$$\mathcal{H}(\varkappa) \equiv \mathcal{G}^{-1}(\varkappa) - \mathcal{V}^{-1}(\varkappa), \qquad \text{[A.1.18]}$$

where the properties of $\mathcal{V}(p)$ and $\mathcal{G}(p)$ imply that $\mathcal{H}(\varkappa)$ is strictly increasing in $\varkappa$. Note also that $\mathcal{H}(\mathrm{q}) = 1 - 0 = 1$ since $\mathcal{V}(0) = \mathrm{q}$ and $\mathcal{G}(1) = \mathrm{q}$.

Consider first the case where $\underline{\pi} < \overline{\pi}$. The definition (A.1.18) and $\mathcal{G}(\underline{\pi}) = \underline{\varkappa}$ imply:

$$\mathcal{H}(\underline{\varkappa}) = \underline{\pi} - \mathcal{V}^{-1}(\underline{\varkappa}).$$

The definition of $\underline{\pi}$ was such that $\underline{\pi} = 0$ if $\underline{\varkappa} > 0$, and so $\mathcal{H}(\underline{\varkappa}) = -\mathcal{V}^{-1}(\underline{\varkappa})$. Given that $\underline{\varkappa} < \mathrm{q}$, $\mathcal{V}(0) = \mathrm{q}$, and $\mathcal{V}(p)$ is strictly decreasing, it follows that $\mathcal{V}^{-1}(\underline{\varkappa}) > 0$, hence $\mathcal{H}(\underline{\varkappa}) < 0$. When $\underline{\varkappa} = 0$, note that $\mathcal{V}^{-1}(\underline{\varkappa}) = \overline{\pi}$ since $\mathcal{V}(\overline{\pi}) = 0$. This implies that $\mathcal{H}(\underline{\varkappa}) = \underline{\pi} - \overline{\pi} < 0$ because $\underline{\pi} < \overline{\pi}$ in the case under consideration. Therefore, it has been shown that $\mathcal{H}(\underline{\varkappa}) < 0$ and $\mathcal{H}(\mathrm{q}) > 0$, thus $\mathcal{H}(\varkappa)$ being continuous and strictly increasing proves there exists a unique value $\varkappa^* \in (0, \mathrm{q})$ such that $\mathcal{H}(\varkappa^*) = 0$.

Consider the remaining case where $\underline{\pi} \geq \overline{\pi}$. It is necessary that $\underline{\pi} > 0$ in this case since $\overline{\pi} > 0$, and the definition of $\underline{\pi}$ then guarantees that $\mathcal{G}(\underline{\pi}) = 0$, and $\underline{\varkappa} = 0$ because $\underline{\varkappa} = \mathcal{G}(\underline{\pi})$. Therefore, $\mathcal{H}(\underline{\varkappa}) = \mathcal{H}(0) = \underline{\pi} - \overline{\pi} \geq 0$. Since $\mathcal{H}(\varkappa)$ is continuous and strictly increasing and $\mathcal{H}(\mathrm{q}) > 0$ as before, it follows that either $\varkappa^* = 0$ is the only possible solution of $\mathcal{H}(\varkappa) = 0$ for $\varkappa \in [0, \mathrm{q}]$, or there is no solution of the equation. Whether or not a solution exists, set $\varkappa^* = 0$ in this case, noting that $\mathcal{H}(\varkappa^*) \geq 0$.

Depending on which case above prevails there is either $\mathcal{H}(\varkappa^*) = 0$ or $\mathcal{H}(\varkappa^*) \geq 0$. In both cases, $\varkappa^* < \mathrm{q}$. Define $p^* = \mathcal{V}^{-1}(\varkappa^*)$, and since $\mathcal{V}(p)$ is strictly decreasing and $\mathcal{V}(0) = \mathrm{q}$, it follows that $p^* > 0$. The definition of $\mathcal{V}(p)$ in (A.1.16) also implies that $\mathcal{T}(p^*, \varkappa^*) = 0$. Therefore, using (A.1.12):

$$\mathrm{q} - \varkappa^* = \frac{p^*}{1 - p^*} u^{-1}(u(\varkappa^*) + \delta). \qquad \text{[A.1.19]}$$

Since $\mathcal{H}(\varkappa^*) \geq 0$, equation (A.1.18) implies $\mathcal{G}^{-1}(\varkappa^*) \geq \mathcal{V}^{-1}(\varkappa^*) = p^*$. As has been shown in (A.1.15),

$\mathcal{F}(p, \varkappa)$ is decreasing in p. The definition (A.1.16) implies $\mathcal{F}(\mathcal{G}^{-1}(\varkappa), \varkappa) = 0$, hence it then follows that $\mathcal{F}(p^*, \varkappa^*) \geq 0$. Using (A.1.12):

$$\mathrm{q} - \varkappa^* \leq \frac{\delta(1 - p^*)}{\frac{\partial u(\varkappa^*)}{\partial C}},$$

and combining this with (A.1.19) yields:

$$\frac{p^*}{1 - p^*} u^{-1}(u(\varkappa^*) + \delta) \leq \frac{\delta(1 - p^*)}{\frac{\partial u(\varkappa^*)}{\partial C}}.$$

Therefore, the following inequality must hold:

$$u^{-1}(u(\varkappa^*) + \delta) \leq \frac{(1 - p^*)^2}{p^*} \frac{\delta}{\frac{\partial u(C^*_{\mathrm{w}})}{\partial C}}. \qquad \text{[A.1.20]}$$

Now note that since $u(\cdot)$ is a concave function, its inverse $u^{-1}(\cdot)$ is a convex function, so it is bounded below by its tangent at $u(\varkappa^*)$. Together with $\varkappa^* \geq 0$, this leads to:

$$u^{-1}(u(\varkappa^*) + \delta) \geq u^{-1}(u(\varkappa^*)) + \frac{1}{\frac{\partial u(u^{-1}(u(\varkappa^*)))}{\partial C}} \delta = \varkappa^* + \frac{\delta}{\frac{\partial u(\varkappa^*)}{\partial C}} \geq \frac{\delta}{\frac{\partial u(\varkappa^*)}{\partial C}}.$$

By combining this with the earlier inequality in (A.1.20):

$$\frac{\delta}{\frac{\partial u(\varkappa^*)}{\partial C}} \leq \frac{(1 - p^*)^2}{p^*} \frac{\delta}{\frac{\partial u(\varkappa^*)}{\partial C}}, \quad \text{and hence } \ 1 \leq \frac{(1 - p^*)^2}{p^*}.$$

Therefore, the value of p^* must satisfy the quadratic inequality $\mathcal{B}(p^*) \geq 0$ where:

$$\mathcal{B}(p) \equiv (1 - p)^2 - p = p^2 - 3p + 1.$$

Since $\mathcal{B}(0) > 0$ and $\mathcal{B}(1) < 0$, the quadratic $\mathcal{B}(p)$ has exactly one root $\bar{p} \in (0, 1)$. The product of the roots is positive, so this must be the smallest root, which can then be obtained using the formula:

$$\bar{p} = \frac{3 - \sqrt{5}}{2} = 2 - \left(\frac{1 + \sqrt{5}}{2}\right) = 2 - \varphi, \qquad \text{[A.1.21]}$$

where $\varphi \equiv (1 + \sqrt{5})/2$. As $\mathcal{B}(p^*) \geq 0$, it must be the case that $p^* \leq \bar{p}$, and therefore $p^* \leq 2 - \varphi$.

Given the constraint $p < 1/2$, the search for an equilibrium is restricted to the interval $p \in [0, 1/2]$. The non-negativity constraints on consumption are equivalent to $0 \leq C_{\mathrm{w}} \leq \mathrm{q}$. Equation (A.1.2) shows that the value of C_{w} consistent with the binding no-rebellion constraint is strictly decreasing in p. Since the utility function $u(\cdot)$ is strictly increasing and weakly concave, equations (A.1.7) and (A.1.8) imply that any critical point of the objective function C_{p} must be a local maximum. Therefore, these observations show that the general necessary and sufficient condition for an equilibrium is:

$$\left.\frac{\partial C_{\mathrm{p}}}{\partial p}\right|_{p=p^*} \begin{cases} \leq 0 & \text{if } \ p^* = 0 \ \text{ or } \ C^*_{\mathrm{w}} = \mathrm{q} \\ = 0 & \text{if } \ 0 < p^* < 1/2 \ \text{ and } \ 0 < C^*_{\mathrm{w}} < \mathrm{q} \ . \\ \geq 0 & \text{if } \ p^* = 1/2 \ \text{ or } \ C^*_{\mathrm{w}} = 0 \end{cases} \qquad \text{[A.1.22]}$$

Consider first the possibility of an equilibrium with $p^* = 0$ or $C^*_{\mathrm{w}} = \mathrm{q}$. Equation (A.1.13) and (A.1.16) imply that $C^*_{\mathrm{w}} = \mathcal{V}(p^*)$, and since $\mathcal{V}(0) = \mathrm{q}$ it follows that any such equilibrium must feature $p^* = 0$ and $C^*_{\mathrm{w}} = \mathrm{q}$. Thus, by using (A.1.12), $\mathcal{F}(p^*, C^*_{\mathrm{w}}) = \delta > 0$. From equation (A.1.13) it follows that $\partial C_{\mathrm{p}}/\partial p \to \infty$ at $p^* = 0$. But the first-order condition (A.1.22) would require $\partial C_{\mathrm{p}}/\partial p \leq 0$ for this type of equilibrium. Therefore, there are no equilibria with either $p^* = 0$ or $C^*_{\mathrm{w}} = \mathrm{q}$.

Now consider the possibility of an equilibrium with $p^* = 1/2$. From the relevant first-order condition in (A.1.22) and (A.1.13), such an equilibrium would need to satisfy $\mathcal{F}(p^*, C_{\mathrm{w}}^*) \geq 0$ and $\mathcal{T}(p^*, C_{\mathrm{w}}^*) = 0$. But it has been shown that $p^* \leq \overline{p}$ for any value of p^* consistent with these conditions, where $\overline{p}$ is defined in (A.1.21). It can be seen that $\overline{p} < 1/2$, so there are no equilibria with $p^* = 1/2$.

Next, consider the case of an equilibrium with $0 < p^* < 1/2$ and $0 < C_{\mathrm{w}}^* < \mathrm{q}$. Using (A.1.22) and (A.1.13), the required conditions are $\mathcal{T}(p^*, C_{\mathrm{w}}^*) = 0$ and $\mathcal{F}(p^*, C_{\mathrm{w}}^*) = 0$. From (A.1.16), this is seen to be equivalent to $p^* = \mathcal{V}^{-1}(C_{\mathrm{w}}^*)$ and $p^* = \mathcal{G}^{-1}(C_{\mathrm{w}}^*)$, and to $\mathcal{H}(C_{\mathrm{w}}^*) = 0$ using (A.1.18). In the case where $\underline{\pi} < \overline{\pi}$ such a solution has been shown to exist, and to be unique. There is no solution when $\underline{\pi} \geq \overline{\pi}$. Therefore, an equilibrium of this type exists (and is unique among those of this type) if and only if $\underline{\pi} < \overline{\pi}$.

Finally, consider the case of an equilibrium with $C_{\mathrm{w}}^* = 0$. According to (A.1.13), this must satisfy $\mathcal{T}(p^*, 0) = 0$, and hence $p^* = \mathcal{V}^{-1}(0)$ using (A.1.16). Using (A.1.22) and (A.1.13), the first-order condition in this case requires $\mathcal{F}(p^*, 0) \geq 0$. The definition in (A.1.16) implies $\mathcal{F}(\mathcal{G}^{-1}(0), 0) = 0$, and since (A.1.15) shows $\mathcal{F}(p, \varkappa)$ is strictly decreasing in p, it follows that $\mathcal{F}(\mathcal{V}^{-1}(0), 0) \geq 0$ if and only if $\mathcal{G}^{-1}(0) \geq \mathcal{V}^{-1}(0)$. This is seen to be equivalent to $\mathcal{H}(\varkappa^*) \geq 0$ using (A.1.18). The earlier analysis shows this inequality is satisfied if and only if $\underline{\pi} \geq \overline{\pi}$. Hence a unique equilibrium exists in this case too.

Therefore, irrespective of whether $\underline{\pi} < \overline{\pi}$ or $\underline{\pi} \geq \overline{\pi}$ holds, a unique equilibrium exists. In the case $\underline{\pi} < \overline{\pi}$, the equilibrium features $0 < C_{\mathrm{w}}^* < \mathrm{q}$, so all non-negativity constraints are slack. In the case $\underline{\pi} \geq \overline{\pi}$, the equilibrium features $C_{\mathrm{w}}^* = 0$, so the non-negativity constraint is binding for workers. In all cases, $0 < p^* \leq 2 - \varphi < 1/2$. Claim (v) of the proposition is established, completing the proof.

A.2 Proof of Proposition 2

Consider an equilibrium allocation $\{p^*, C_{\mathrm{p}}^*(\cdot), C_{\mathrm{w}}^*(\cdot), C_{\mathrm{k}}^*(\cdot)\}$ for which a positive fraction s^* of investment opportunities are taken.

Given the third and fourth equilibrium conditions in Definition 2 and the absence of any fundamental state variables at the pre-investment stage of the sequence of events in Figure 3, the allocation of power and resources established after any rebellion at the pre-investment stage must feature some particular amount p' of power sharing and some particular average incumbent payoff U_{p}'. These requirements place no restriction on the identities of the p' individuals who would belong to the subsequent incumbent group $\mathcal{P}'$.

At the post-investment stage of Figure 3, the features of any equilibrium allocation established following a rebellion (which is off the equilibrium path) are characterized in section 3.2.1. Hence, given the third and fourth equilibrium conditions in Definition 2, the allocation of power and resources established after any rebellion at the post-investment stage must feature an amount of power sharing $p^\dagger$ and an incumbent payoff $U_{\mathrm{p}}^\dagger(K)$, where the payoff depends on the aggregate capital stock K, which is a fundamental state variable at this stage. These requirements place no restriction on the identities of the $p^\dagger$ individuals who would belong to the subsequent incumbent group $\mathcal{P}^\dagger$.

The first and second equilibrium conditions in Definition 2 require that the equilibrium allocation maximizes the average payoff of incumbents subject to the absence of any rational rebellion (the incumbents have no interest in choosing an allocation that triggers a successful rebellion). At the pre-investment stage, given p' and U_{p}' and a current allocation, the criteria in Definition 1 for a rational rebellion can be assessed for each subsequent incumbent group $\mathcal{P}'$ with measure p'. At the post-investment stage, given a current allocation, the criteria for a rational rebellion can be assessed for each subsequent incumbent group $\mathcal{P}^\dagger$ with measure $p^\dagger$. Considering all possible $\mathcal{P}'$ with measure p' at the pre-investment stage and all possible $\mathcal{P}^\dagger$ with measure $p^\dagger$ at the post-investment stage leads to a set of no-rebellion constraints, and the average incumbent payoff can be maximized subject to these constraints.

There is a binding no-rebellion constraint with a positive measure of non-incumbents in the rebel army

If not, the power-adjusted fighting effort of all (non-zero measures of) non-incumbents is strictly greater than the power-adjusted fighting effort of incumbents (power-adjusted fighting effort in the sense of the maximum fighting effort of each rebel plus the loss of defensive strength for the incumbent army when a rebel holds a position of power). Hence an allocation can be modified to reduce consumption of those outside the group in power and to distribute it equally among incumbents, which raises incumbent payoffs (the reduction in consumption can be chosen to be small enough so that there is no rational rebellion, noting that the number of incumbents willing to rebel cannot increase, nor can their power-adjusted fighting effort in a rational rebellion).

Payoff equalization for workers

Suppose an allocation features dispersion in worker consumption $C^*_{\mathrm{w}}(\imath)$ for a positive measure of workers. Let $\gamma(\imath) = C_{\mathrm{k}}(\imath) - C_{\mathrm{w}}(\imath)$. From (3.6), at the pre-investment stage, those outside the incumbent group have (expected) payoffs $U_{\mathrm{n}}(\imath) = C_{\mathrm{w}}(\imath) + \alpha \mathbb{E}_\theta \max\{\gamma(\imath), 0\}$, while at the post-investment stage, workers have payoffs $U_{\mathrm{w}}(\imath) = C_{\mathrm{w}}(\imath)$ and capitalists $U_{\mathrm{k}}(\imath) = C_{\mathrm{w}}(\imath) + \gamma(\imath)$. Now consider a modification of the allocation where consumption for every worker is equalized at C_{w}, which is equal to the average of $C^*_{\mathrm{w}}(\imath)$, while $\gamma(\imath)$ and all incumbent payoffs remain unchanged. That reduces payoff inequality among non-incumbents, but does not affect incentives to invest because $\gamma(\imath)$ is unchanged. Since there are more non-incumbents than places in a possible rational rebellion (at any stage of the power struggle), the maximum fighting effort from a rebel army including any positive measure of non-incumbents is now strictly lower, while fighting effort from incumbents in a rebel army is unaffected. Since it is known that at least one no-rebellion constraint involving a positive measure of non-incumbents is binding, this argument shows that this binding constraint can be slackened without violating any other constraint. Thus, it must be the case that $C^*_{\mathrm{w}}(\imath) = C^*_{\mathrm{w}}$ in equilibrium. This confirms the first part of claim (i) of the proposition.

No capitalists in a rebel army with a binding no-rebellion constraint

Given the incentive constraint and given payoff equalization among workers, capitalists ex post must receive a higher payoff than workers. This means capitalists' power-adjusted fighting effort in a rebel army is less than workers (their power is the same), and since there is never a shortage of workers, capitalists will not feature in a rebel army associated with a binding no-rebellion constraint.

Consumption equalization for capitalists

Since capitalists will not belong to the rebel army with the greatest fighting strength, ex-post inequality among capitalists' payoffs does not increase incentives for rebellion. The question is then how dispersion in the consumption of capitalists affects incentives for investment. An individual with an investment opportunity will invest and become a capitalist if $C_{\mathrm{k}}(\imath) - C_{\mathrm{w}} \geq \theta$. Since $\theta \geq \psi$, a capitalist's consumption cannot be smaller than $C_{\mathrm{w}} + \psi$, and since $\theta \leq \kappa$, a capitalist having consumption larger than $C_{\mathrm{w}} + \kappa$ is costly for incumbents without bringing any benefits. Hence $C_{\mathrm{k}}(\imath) \in [C_{\mathrm{w}} + \psi, C_{\mathrm{w}} + \kappa]$. Using the effort distribution in (3.1), since the utility function $u(\cdot)$ is linear, the probability an individual will choose to take an investment opportunity is $(C_{\mathrm{k}}(\imath) - C_{\mathrm{w}} - \psi)/(\kappa - \psi)$. Averaging that across individuals implies the fraction s of investment opportunities that are taken is given by $(\bar{C}_{\mathrm{k}} - C_{\mathrm{w}} - \psi)/(\kappa - \psi)$, where $\bar{C}_{\mathrm{k}}$ is the average consumption of a capitalist. This shows that given linearity of $u(\cdot)$, dispersion in capitalists' consumption does not affect investment, thus it does not affect any aggregate variable. Hence there is no loss of generality in assuming all capitalists receive consumption C_{k}. Using $s = (C_{\mathrm{k}} - C_{\mathrm{w}} - \psi)/(\kappa - \psi)$ yields the expression in (3.7a). This confirms the claims in part (ii) of the proposition.

It follows that all individuals outside the group in power have the same expected payoff:

$$U_{\mathrm{n}} = C_{\mathrm{w}} + \alpha \mathcal{S}_{\mathrm{i}}(\tilde{\theta}), \quad \text{where } \mathcal{S}_{\mathrm{i}}(\tilde{\theta}) \equiv \mathbb{E}_\theta \max\{\tilde{\theta} - \theta, 0\}, \tag{A.2.1}$$

where $\mathcal{S}_{\mathrm{i}}(\tilde{\theta})$ is the expected surplus from receiving an investment opportunity. The average incumbent payoff $\bar{U}_{\mathrm{p}}$ obtained from the budget constraint can then be written as:

$$\bar{U}_{\mathrm{p}} = \frac{(1-p)\,(\mathrm{q} - U_{\mathrm{w}}) + \mu(\kappa - \tilde{\theta})s}{p}. \tag{A.2.2}$$

The time inconsistency problem

In the equilibrium allocation following a post-investment rebellion, an incumbent payoff is $U_{\text{p}}^{\dagger}(K^*)$, where $K^* = \mu\kappa s^*$ is the capital stock if a fraction s^* of investment opportunities are taken up. This payoff is the solution of the problem of maximizing the average incumbent payoff subject to the post-investment no-rebellion constraints. Any equilibrium allocation established at this stage features payoff equalization among incumbents and among non-incumbents. Now consider a hypothetical problem: maximizing the average incumbent payoff subject to the post-investment stage no-rebellion constraints (with rebels expecting payoffs consistent with the equilibrium) and an additional constraint (but one not imposed on any subsequent incumbents): a measure μs^* of non-incumbents have to receive consumption $\tilde{\theta} > 0$ above the average of the remaining non-incumbents. The maximized average incumbent payoff $\tilde{\bar{U}}_{\text{p}}$ in this hypothetical problem must be strictly less than $U_{\text{p}}^{\dagger}(K)$ (to see this, note that removing this constraint allows resources to be transferred to the incumbents).

Now let $\bar{U}_{\text{p}}^*$ denote the equilibrium average incumbent payoff. This is the solution to a maximization problem subject to all of the constraints described in the hypothetical problem above. Note that the capital stock is the same, thus the budget constraint is the same, the extra transfer $\tilde{\theta}$ is required by the incentive constraint for investors, and the no-rebellion constraints in the hypothetical problem are the same as the post-investment stage no-rebellion constraints in the actual problem. In addition, the actual problem also includes the pre-investment stage no-rebellion constraints. It follows that the maximized average incumbent payoff subject to these constraints cannot be more than in the hypothetical problem, hence $\bar{U}_{\text{p}}^* \leq \tilde{\bar{U}}_{\text{p}} < U_{\text{p}}^{\dagger}(K)$.

At least one no-rebellion constraint including only non-incumbents must bind

Since there is no shortage of either non-incumbents at the pre-investment stage, or workers at the post-investment stage, the following no-rebellion constraints must hold:

$$U_{\text{p}}' - U_{\text{n}} \leq \delta \frac{p}{p'}, \quad \text{and} \tag{A.2.3a}$$

$$U_{\text{p}}^{\dagger}(K) - U_{\text{w}} \leq \delta \frac{p}{p^{\dagger}}, \tag{A.2.3b}$$

where $p' = p^*$ and $U_{\text{p}}' = \bar{U}_{\text{p}}^*$ in equilibrium.

Now suppose for contradiction that neither (A.2.3a) nor (A.2.3b) binds in equilibrium:

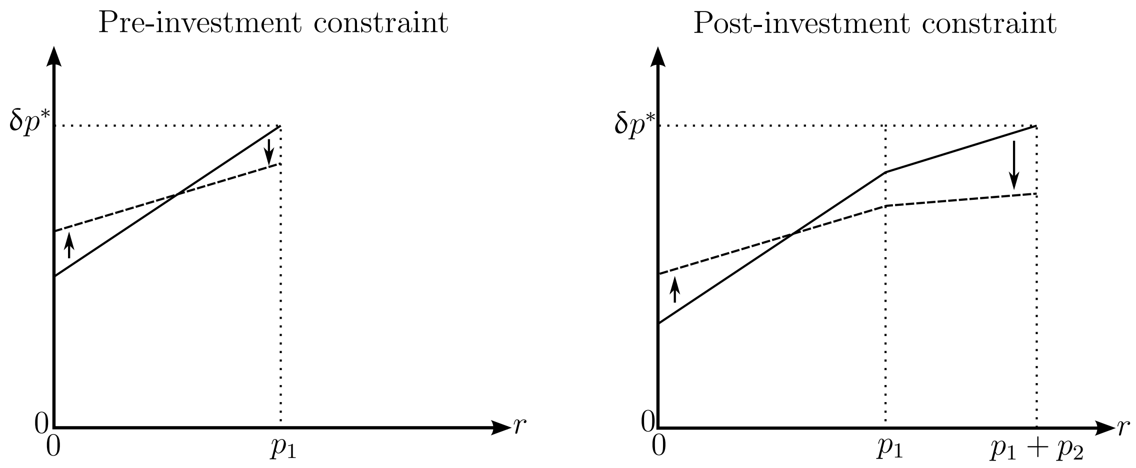
$$U_{\text{p}}' - U_{\text{n}} < \delta \frac{p}{p'}, \quad \text{and} \;\; U_{\text{p}}^{\dagger}(K) - U_{\text{w}} < \delta \frac{p}{p^{\dagger}}.$$

Given whatever consumption distribution is part of the conjectured equilibrium allocation, divide the group in power into three sets. Group 1 would be willing to rebel at either the pre- or post-investment stage, i.e. $U_{\text{p}}(\imath) \leq U_{\text{p}}' = \bar{U}_{\text{p}}^* < U_{\text{p}}^{\dagger}(K)$. Group 2 would be willing to rebel at the post-investment stage, but not at the pre-investment stage, i.e. $\bar{U}_{\text{p}}^* = U_{\text{p}}' < U_{\text{p}}(\imath) \leq U_{\text{p}}^{\dagger}(K)$. Group 3 would not rebel at either stage, i.e. $U_{\text{p}}(\imath) > U_{\text{p}}^{\dagger}(K)$. In equilibrium, group 1 must contain some positive measure of incumbents, though group 2 and/or group 3 could be empty.

Now modify the allocation by taking the consumption assigned to groups 1 and 2 and redistributing within groups (but not across groups, and group 3 is left untouched) so that all individuals within group 1 have the same payoff and all individuals within group 2 have the same payoff. This redistribution does not directly reduce the incumbents' objective function. Nor does it change the numbers of incumbents who would be willing to rebel at either of the stages. Furthermore, it cannot increase the fighting effort that can be obtained from the strongest rebel army conditional on including a given measure of incumbents.

Let p_1 denote the measure of incumbents in group 1, and p_2 the measure in group 2. Following the redistribution, the incumbent payoff distribution has two point masses at $U_{\text{p},1}$ and $U_{\text{p},2}$, and potentially a general distribution above $U_{\text{p}}^{\dagger}(K)$. Figure 4 depicts total effective fighting strength as a function of the measure r of incumbents included in a rebel army, for both pre- and post-investment stages (total effective fighting strength is the total fighting effort of the rebels plus any loss of defensive strength for the incumbent army when those in power join the rebel army). These functions are piecewise linear, bounded above by

Figure 4: *Total effective fighting strength*



δp^*, and given the supposition, both functions are strictly below δp^* at $r = 0$. The first function is defined only for $r \in [0, p_1]$, and the second is defined for $r \in [0, p_1 + p_2]$.

Consider a further modification of the allocation. Reduce consumption of all non-incumbents by an amount ϵ, distributing the revenue equally among incumbents in group 1. The ϵ is chosen so that both no-rebellion constraints remain slack at $r = 0$. The group-1 incumbents obtain an increment $\epsilon(1 - p^*)/p_1$ to consumption and utility. The functions in Figure 4 remain piecewise linear following this. At $r = p_1$, the effect on the total effective fighting strength by group-1 incumbents in the rebel army is reduced by $(1 - p^*)\epsilon$, while the effect on workers' total fighting effort is $(p^* - p_1)\epsilon$. The latter is smaller, hence the net effect is negative. Since the function corresponding to the pre-investment no-rebellion constraint is linear, that no-rebellion constraint must be slack. For the function corresponding to the post-investment constraint, note that the slope between p_1 and $p_1 + p_2$ is also reduced because the gap between U_{w} and $U_{\mathrm{p},2}$ is increased (the slope is equal to $\delta + U_{\mathrm{w}} - U_{\mathrm{p},2}$). Hence, the post-investment constraint is now also slack. Since the average incumbent payoff has been increased by this modification of the allocation, the initial supposition must be false.

Power sharing must be larger than it would be in equilibrium following a rebellion at the post-investment stage

Suppose not for contradiction, that is, $p^* \leq p^\dagger$. It is known that one of the constraints (A.2.3a) and (A.2.3b) must be binding in equilibrium. Consider the case where it is (A.2.3a). This means that in equilibrium ($p' = p^*$):

$$\bar{U}_{\mathrm{p}}^* - U_{\mathrm{n}}^* = \delta.$$

Using this equation and (A.2.1), it follows that:

$$U_{\mathrm{p}}^\dagger(K) - U_{\mathrm{w}}^* = (U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}}^*) + \alpha \mathcal{S}_{\mathrm{i}}(\tilde{\theta}) + \delta > \delta \geq \delta \frac{p^*}{p^\dagger},$$

where the first inequality uses the time inconsistency result and the second uses the conjecture $p^* \leq p^\dagger$. It follows that the no-rebellion constraint (A.2.3b) would be violated in this case.

Therefore, if $p^* \leq p^\dagger$ then it must be the case that (A.2.3b) binds:

$$U_{\mathrm{p}}^\dagger(K) - U_{\mathrm{w}}^* = \delta \frac{p^*}{p^\dagger}.$$

This represents a rebellion where $p^\dagger$ workers each put in $\delta p^*/p^\dagger$ units of fighting effort, bringing the total fighting effort exactly to δp^*. However, it is known that $\bar{U}_{\mathrm{p}}^* < U_{\mathrm{p}}^{\dagger}(K)$. Obviously, since $\bar{U}_{\mathrm{p}}^*$ is an average, there must be a positive measure of incumbents receiving this average payoff or a lower one. These incumbents have a strict preference to rebel at the post-investment stage. If they joined a rebellion, their power-adjusted fighting effort (taking account of the loss of defensive strength for the incumbent army) would be δ plus some strictly positive number. Hence substituting a positive measure of these incumbents into the rebel army replaces workers exerting power-adjusted fighting effort $\delta p^*/p^{\dagger} \leq \delta$ with incumbents with power-adjusted fighting effort strictly larger than δ. Since the no-rebellion constraint holds exactly with only workers, it is now violated once the rebel army includes a positive measure of incumbents, so a rational rebellion would occur. Therefore, contradictions are obtained in all cases supposing $p^* \leq p^{\dagger}$, hence it is shown that an equilibrium allocation must feature $p^* > p^{\dagger}$.

The pre-investment stage constraint for a rebel army with only non-incumbents cannot bind on its own

Consider the possibility of an equilibrium with $s > 0$ in which the pre-investment stage constraint (A.2.3a) is the only effective binding no-rebellion constraint. With $p' = p^*$ and $U_{\mathrm{p}}' = U_{\mathrm{p}}^*$, the binding constraint (A.2.3a) implies

$$U_{\mathrm{p}}^* - U_{\mathrm{n}} = \delta \frac{p}{p^*}.$$

Combining this with the expression for the non-incumbent expected payoff U_{n} from (A.2.1) (using $U_{\mathrm{w}} = C_{\mathrm{w}}$) yields:

$$U_{\mathrm{w}} = U_{\mathrm{p}}^* - \delta \frac{p}{p^*} - \alpha \mathcal{S}_{\mathrm{i}}(\tilde{\theta}) = U_{\mathrm{p}}^* - \delta \frac{p}{p^*} - \frac{\mu}{1-p} \mathcal{S}_{\mathrm{i}}(\tilde{\theta}), \tag{A.2.4}$$

where the formula for the probability α of receiving an investment opportunity is taken from (3.6). Using the expression for $\bar{U}_{\mathrm{p}}$ from (A.2.2) and the formula for U_{w} yields:

$$\bar{U}_{\mathrm{p}} = \frac{(1-p)\left(\mathrm{q} + \delta \frac{p}{p^*} - U_{\mathrm{p}}^*\right) + \mu(\kappa - \tilde{\theta})s + \mu \mathcal{S}_{\mathrm{i}}(\tilde{\theta})}{p}. \tag{A.2.5}$$

The first-order condition for maximizing $\bar{U}_{\mathrm{p}}$ with respect to p (taking p^* and U_{p}^* as given) is:

$$\frac{1}{p}\left((1-p)\left(\mathrm{q} + \delta \frac{p}{p^*} - U_{\mathrm{p}}^*\right) + \mu(\kappa - \tilde{\theta})s + \mu \mathcal{S}_{\mathrm{i}}(\tilde{\theta})\right) = (1-p)\frac{\delta}{p^*} - \left(\mathrm{q} + \delta \frac{p}{p^*} - U_{\mathrm{p}}^*\right),$$

and after imposing the equilibrium conditions $p = p^*$ and $\bar{U}_{\mathrm{p}} = U_{\mathrm{p}}^*$ and making use of (A.2.5):

$$U_{\mathrm{p}}^* = \delta \frac{(1-p^*)}{p^*} - \left(\mathrm{q} + \delta - U_{\mathrm{p}}^*\right).$$

Solving this equation for p^* yields:

$$p^* = \frac{\delta}{\mathrm{q} + 2\delta} = p^{\dagger},$$

given the expression for $p^{\dagger}$ in (3.5). This contradicts the requirement that $p^* > p^{\dagger}$ in equilibrium, so this constraint cannot bind on its own.

A binding pre-investment stage constraint for non-incumbents requires payoff equalization for the incumbents and implies that other pre-investment stage constraints may be ignored

If constraint (A.2.3a) is binding in equilibrium then:

$$\bar{U}_{\mathrm{p}}^* - U_{\mathrm{n}}^* = \delta.$$

This represents a rebel army where a measure p^* of non-incumbents each exert δ fighting effort. Now suppose there is payoff inequality among a positive measure of incumbents in this case. Since $\bar{U}_{\mathrm{p}}^*$ is the average payoff, this means there is a positive measure of incumbents with payoffs strictly lower than $\bar{U}_{\mathrm{p}}^*$. As $U_{\mathrm{p}}' = \bar{U}_{\mathrm{p}}^*$ holds, it follows that this positive measure of incumbents would be willing to join a rebel army at the pre-investment stage and exert a strictly positive amount of fighting effort. Hence they would contribute power-adjusted fighting effort strictly larger than δ. This positive measure in combination with non-incumbents exerting effort δ would lead to a total power-adjusted fighting effort of the rebel army greater than δp^*, and hence a rational rebellion. So in the case under consideration, there must be complete payoff equalization among the incumbents.

Following the argument in Proposition 1, the set of equilibria imposing only the non-incumbents' pre-investment stage no-rebellion constraint is the same as the set of equilibria imposing all combinations of pre-investment stage no-rebellion constraints.

The post-investment stage constraint involving only workers cannot bind on its own

The expression for the average incumbent payoff from (A.2.2) can be rearranged as follows:

$$\bar{U}_{\mathrm{p}} = \frac{(1-p)\left(\mathrm{q} + \delta\frac{p}{p^\dagger} - U_{\mathrm{p}}^\dagger(K)\right) + \mu(\kappa - \tilde{\theta})s}{p} - \left(\frac{1-p}{p}\right)\left(U_{\mathrm{w}} - U_{\mathrm{p}}^\dagger(K) + \delta\frac{p}{p^\dagger}\right),$$

from which an expression for $U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}}$ can be obtained:

$$U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}} = \frac{1}{p}\left(U_{\mathrm{p}}^\dagger(K) - \left((1-p)\left(\mathrm{q} + \delta\frac{p}{p^\dagger}\right) + \mu(\kappa - \tilde{\theta})s\right)\right) + \left(\frac{1-p}{p}\right)\left(U_{\mathrm{w}} - U_{\mathrm{p}}^\dagger(K) + \delta\frac{p}{p^\dagger}\right). \quad \text{[A.2.6]}$$

Substituting for $U_{\mathrm{p}}^\dagger(K)$ from (3.5) and rearranging the first term in brackets yields:

$$\begin{aligned}
U_{\mathrm{p}}^\dagger(K) - \left((1-p)\left(\mathrm{q} + \delta\frac{p}{p^\dagger}\right) + \mu(\kappa - \tilde{\theta})s\right) &= \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} + K - (1-p)\left(\mathrm{q} + \delta\frac{p}{p^\dagger}\right) - \mu\kappa s + \mu\tilde{\theta}s \\
&= (\mathrm{q}+\delta)(1-p^\dagger) + \mu\kappa s - \mathrm{q}(1-p) - \frac{\delta}{p^\dagger}p + \frac{\delta}{p^\dagger}p^2 - \mu\kappa s + \mu\tilde{\theta}s \\
&= \frac{\delta}{p^\dagger}\left(p^2 - p + \frac{\mathrm{q}p^\dagger}{\delta}p + \frac{p^\dagger}{\delta}\left((\mathrm{q}+\delta)(1-p^\dagger) - \mathrm{q}\right)\right) + \mu\tilde{\theta}s \\
&= \frac{\delta}{p^\dagger}\left(p^2 - \left(1 - \frac{\mathrm{q}}{\mathrm{q}+2\delta}\right)p + \frac{p^\dagger}{\delta}\left(\frac{(\mathrm{q}+\delta)^2 - \mathrm{q}(\mathrm{q}+2\delta)}{\mathrm{q}+2\delta}\right)\right) + \mu\tilde{\theta}s \\
&= \frac{\delta}{p^\dagger}\left(p^2 - 2\left(\frac{\delta}{\mathrm{q}+2\delta}\right)p + p^\dagger\left(\frac{\delta}{\mathrm{q}+2\delta}\right)\right) + \mu\tilde{\theta}s \\
&= \frac{\delta}{p^\dagger}\left(p^2 - 2p^\dagger p + {p^\dagger}^2\right) + \mu\tilde{\theta}s = \frac{\delta}{p^\dagger}\left(p - p^\dagger\right)^2 + \mu\tilde{\theta}s.
\end{aligned} \quad \text{[A.2.7]}$$

Now consider the possibility of an equilibrium featuring $s > 0$ with workers' post-investment constraint (A.2.3b) as the only effective binding no-rebellion constraint. When the inequality in (A.2.3b) binds:

$$U_{\mathrm{w}} = U_{\mathrm{p}}^\dagger(K) - \delta\frac{p}{p^\dagger}, \quad \text{[A.2.8]}$$

which is substituted into the expression for the average incumbent payoff (A.2.2) to obtain:

$$\bar{U}_{\mathrm{p}} = \frac{(1-p)\left(\mathrm{q} + \delta\frac{p}{p^\dagger} - U_{\mathrm{p}}^\dagger(K)\right) + \mu(\kappa - \tilde{\theta})s}{p}. \quad \text{[A.2.9]}$$

Taking the derivative of $\bar{U}_{\mathrm{p}}$ with respect to p:

$$\frac{\partial \bar{U}_{\mathrm{p}}}{\partial p} = \frac{1}{p}\left((1-p)\frac{\delta}{p^\dagger} - \left(\mathrm{q} + \delta\frac{p}{p^\dagger} - U_{\mathrm{p}}^\dagger(K)\right)\right) - \frac{1}{p^2}\left((1-p)\left(\mathrm{q} + \delta\frac{p}{p^\dagger} - U_{\mathrm{p}}^\dagger(K)\right) + \mu(\kappa - \tilde{\theta})s\right),$$

and by using the expressions for $\bar{U}_{\mathrm{p}}$ and $p^\dagger$ from (A.2.9) and (3.5) respectively:

$$\frac{\partial \bar{U}_{\mathrm{p}}}{\partial p} = \frac{1}{p}\left(\frac{\delta}{p^\dagger} - \mathrm{q} - 2\delta\frac{p}{p^\dagger} + (U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}})\right) = \frac{1}{p}\left(2\frac{\delta}{p^\dagger}(p^\dagger - p) + (U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}})\right). \qquad [A.2.10]$$

Exploiting the fact that (A.2.8) holds in this case, equations (A.2.6) and (A.2.7) imply that:

$$U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}} = \frac{1}{p}\left(\frac{\delta}{p^\dagger}(p-p^\dagger)^2 + \mu\tilde{\theta}s\right), \qquad [A.2.11]$$

and by substituting this into (A.2.10):

$$\frac{\partial \bar{U}_{\mathrm{p}}}{\partial p} = \frac{1}{p^2}\left(2\frac{\delta}{p^\dagger}(p^\dagger - p)p + \frac{\delta}{p^\dagger}(p-p^\dagger)^2 + \mu\tilde{\theta}s\right) = \frac{\delta}{p^2 p^\dagger}\left(p^\dagger\left(p^\dagger + \frac{\mu\tilde{\theta}s}{\delta}\right) - p^2\right). \qquad [A.2.12]$$

Now make the following definition of the function $\pi(s)$:

$$\pi(s) \equiv p^\dagger + \frac{\mu\tilde{\theta}s}{\delta}. \qquad [A.2.13]$$

The restrictions in (3.3) imply that $\pi(s) < 1/2$ for all $s \in [0,1]$. Using the definition of $\pi(s)$ in (A.2.13) and equation (A.2.12), the derivative of $\bar{U}_{\mathrm{p}}$ can be written as:

$$\frac{\partial \bar{U}_{\mathrm{p}}}{\partial p} = \frac{\delta}{p^2 p^\dagger}\left(p^\dagger\pi(s) - p^2\right). \qquad [A.2.14]$$

It follows that $\bar{U}_{\mathrm{p}}$ is strictly increasing in p for $p < \sqrt{p^\dagger}\sqrt{\pi(s)}$, and strictly decreasing for $p > \sqrt{p^\dagger}\sqrt{\pi(s)}$. The first-order condition for maximizing the expression for $\bar{U}_{\mathrm{p}}$ in (A.2.9) incorporating the binding constraint is therefore $p^* = \sqrt{p^\dagger}\sqrt{\pi(s)}$. It can be seen from (A.2.13) that $p^\dagger < \pi(s)$ for any $s > 0$, so it follows that $p^\dagger < p^* < \pi(s)$.

In the case under consideration where (A.2.3b) is binding, equation (A.2.11) implies the inequality

$$U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}} \leq \delta\left(\frac{p - p^\dagger}{p^\dagger}\right) \qquad [A.2.15]$$

is equivalent to the following:

$$\frac{\delta}{p^\dagger}(p-p^\dagger)^2 + \mu s\tilde{\theta} \leq \frac{\delta}{p^\dagger}(p-p^\dagger)p.$$

Simplification of the terms appearing in the inequality above shows that it is in turn equivalent to $p \geq \pi(s)$, with $\pi(s)$ as defined in (A.2.13). But the first-order condition $p^* = \sqrt{p^\dagger}\sqrt{\pi(s)}$ implies $p^* < \pi(s)$ since $\pi(s) > p^\dagger$ for $s > 0$. Therefore:

$$U_{\mathrm{p}}^\dagger(K) - \bar{U}_{\mathrm{p}}^* > \delta\left(\frac{p^* - p^\dagger}{p^\dagger}\right).$$

The constraint (A.2.3b) that is assumed to bind is associated with a rebel army where $p^\dagger$ workers would each exert fighting effort $\delta p^*/p^\dagger$, which adds up to the total fighting strength δp^* of the incumbent army. Now note that whatever the distribution of incumbent payoffs, since $\bar{U}_{\mathrm{p}}^*$ is the average, there must be a strictly positive measure of incumbents receiving a payoff of $\bar{U}_{\mathrm{p}}^*$ or less. Combined with the inequality derived above, it follows that this positive measure of incumbents is willing to rebel, and the inequality reveals that each of that group would be willing to put in an amount of fighting effort strictly larger than $\delta p^*/p^\dagger - \delta$. Since each has power δ, all would put in power-adjusted fighting effort strictly greater than $\delta p^*/p^\dagger$. Now taking some of these incumbents and combining them with workers leads to a rebel army with total power-adjusted fighting strength more than δp^*, so the rebellion would be rational. Hence it is not possible that (A.2.3b) is the only binding constraint.

The pre-investment stage constraint with only non-incumbents cannot bind in conjunction with a post-investment constraint involving a positive number of incumbents

It has been shown that if (A.2.3a) binds in equilibrium then all incumbents must receive the same payoff. Let U_p^* denote this common payoff. Given that all workers receive the same payoff and the no-rebellion constraint (A.2.3b) must hold, and since $p^* > p^\dagger$, if a post-investment stage no-rebellion constraint is binding with a positive measure of incumbents, it must be that the no-rebellion constraint including only incumbents is binding (the argument is that total effective fighting strength is linear in the composition of the rebel army):

$$U_\mathrm{p}^\dagger(K) - U_\mathrm{p}^* = \delta\left(\frac{p^* - p^\dagger}{p^\dagger}\right).$$

Using equation (3.6) and the binding constraint (A.2.3a), it follows that $U_\mathrm{w}^* = U_\mathrm{p}^* - \alpha\mathcal{S}_\mathrm{i}(\tilde{\theta}) - \delta$. The equation above implies $U_\mathrm{p}^\dagger(K) = U_\mathrm{p}^* + \delta p^*/p^\dagger - \delta$, and putting these two results together yields:

$$U_\mathrm{p}^\dagger(K) - U_\mathrm{w}^* = \delta\frac{p^*}{p^\dagger} + \alpha\mathcal{S}_\mathrm{i}(\tilde{\theta}),$$

but since $\alpha\mathcal{S}_\mathrm{i}(\tilde{\theta}) > 0$, this violates (A.2.3b). Therefore, this combination of binding constraints is inconsistent with equilibrium.

The pre-investment stage constraint with only non-incumbents cannot bind in conjunction with the post-investment stage constraint with only workers

Since the pre-investment stage no-rebellion constraint for non-incumbents is binding, there must be complete payoff equalization among incumbents. Furthermore, when this constraint binds, an expression for $\bar{U}_\mathrm{p}$ is given in (A.2.5). After imposing the equilibrium conditions $p = p^*$ and $U_\mathrm{p}^* = \bar{U}_\mathrm{p}$ in that equation:

$$U_\mathrm{p}^* = (\mathrm{q} + \delta)(1 - p^*) + \mu(\kappa - \tilde{\theta})s + \mu\mathcal{S}_\mathrm{i}(\tilde{\theta}).$$

It has already been shown that when (A.2.3a) binds, equation (A.2.4) for U_w follows. By substituting the equation above into (A.2.4):

$$U_\mathrm{w}^* = \mathrm{q}(1 - p^*) - \delta p^* + \mu(\kappa - \tilde{\theta})s - \mu\frac{p^*}{1 - p^*}\mathcal{S}_\mathrm{i}(\tilde{\theta}).$$

This can be substituted into (A.2.3b) to obtain a condition for p^* to satisfy the post-investment no-rebellion constraint for workers:

$$U_\mathrm{p}^\dagger(K) - \mathrm{q}(1 - p^*) + \delta p^* - \mu(\kappa - \tilde{\theta})s + \mu\frac{p^*}{1 - p^*}\mathcal{S}_\mathrm{i}(\tilde{\theta}) \leq \delta\frac{p^*}{p^\dagger}.$$

Using the expressions for $U_\mathrm{p}^\dagger(K)$ and $p^\dagger$ from (3.5), and $K = \mu\kappa s$ from (3.7a), it can be seen that $U_\mathrm{p}^\dagger(K) - \mathrm{q} = \delta p^\dagger + \mu\kappa s$. By substituting this into the inequality above it follows that it is equivalent to

$$\delta p^\dagger + (\mathrm{q} + \delta)p^* + \mu\tilde{\theta}s + \mu\frac{p^*}{1 - p^*}\mathcal{S}_\mathrm{i}(\tilde{\theta}) \leq \frac{\delta}{p^\dagger}p^*.$$

Noting that (3.5) implies $\delta/p^\dagger = \mathrm{q} + 2\delta$, the inequality above can be rearranged to yield

$$\left(1 - \frac{\mu}{1-p^*}\frac{\mathcal{S}_\mathrm{i}(\tilde{\theta})}{\delta}\right)p^* \geq \pi(s), \tag{A.2.16a}$$

where $\pi(s)$ is defined in (A.2.13).

Now start from the case where the post-investment no-rebellion constraint for workers (A.2.3b) is binding at $p = p^*$. The condition needed for an equilibrium ($p' = p^*$ and $U'_\mathrm{p} = U^*_\mathrm{p}$) of this type to satisfy the pre-investment stage no-rebellion constraint (A.2.3a) is

$$U^*_\mathrm{p} - U^*_\mathrm{n} \leq \delta.$$

The expected payoff U_n for non-incumbents is given in equation (A.2.1). Using the formula for α from (3.6), it follows that $U^*_\mathrm{n} = U^*_\mathrm{w} + (\mu/(1-p^*))\mathcal{S}_\mathrm{i}(\tilde{\theta})$. Substituting this into the inequality above shows that (A.2.3a) is satisfied if and only if

$$U^*_\mathrm{p} - U^*_\mathrm{w} - \frac{\mu}{1-p^*}\mathcal{S}_\mathrm{i}(\tilde{\theta}) \leq \delta.$$

A binding post-investment constraint (A.2.3b) for workers implies that $U^*_\mathrm{w} = U^\dagger_\mathrm{p}(K) - \delta p^*/p^\dagger$, which can be substituted into the condition above to obtain:

$$\delta\frac{p^*}{p^\dagger} - \delta - \frac{\mu}{1-p^*}\mathcal{S}_\mathrm{i}(\tilde{\theta}) \leq U^\dagger_\mathrm{p}(K) - U^*_\mathrm{p}.$$

Equation (A.2.8) — the binding version of (A.2.3b) — has already been shown to imply the expression for $U^\dagger_\mathrm{p}(K) - \bar{U}_\mathrm{p}$ in (A.2.11). Since $\bar{U}^*_\mathrm{p} = U^*_\mathrm{p}$, this can be used to deduce that the inequality above is equivalent to:

$$\frac{\delta}{p^\dagger}(p^* - p^\dagger)p^* - \mu\frac{p^*}{1-p^*}\mathcal{S}_\mathrm{i}(\tilde{\theta}) \leq \frac{\delta}{p^\dagger}(p^* - p^\dagger)^2 + \mu\tilde{\theta}s.$$

After some rearrangement, this condition can be written as

$$p^* - p^\dagger - \mu\frac{p^*}{1-p^*}\frac{\mathcal{S}_\mathrm{i}(\tilde{\theta})}{\delta} \leq \frac{\mu\tilde{\theta}s}{\delta},$$

which can be stated in terms of the function $\pi(s)$ from (A.2.13):

$$\left(1 - \frac{\mu}{1-p^*}\frac{\mathcal{S}_\mathrm{i}(\tilde{\theta})}{\delta}\right)p^* \leq \pi(s). \tag{A.2.16b}$$

Note that if both constraints (A.2.3a) and (A.2.3b) are binding in equilibrium then both (A.2.16a) and (A.2.16b) must hold, hence:

$$\left(1 - \frac{\mu}{1-p^*}\frac{\mathcal{S}_\mathrm{i}(\tilde{\theta})}{\delta}\right)p^* = \pi(s). \tag{A.2.17}$$

The expected surplus of those receiving an investment opportunity is $\mathcal{S}_\mathrm{i}(\tilde{\theta})$, which is defined in equation (A.2.1). The probability distribution of the effort cost θ in (3.1) has density function $1/(\kappa - \psi)$ on support $[\psi, \kappa]$, so an explicit expression for the surplus is:

$$\mathcal{S}_\mathrm{i}(\tilde{\theta}) = \int_{\theta=\psi}^{\tilde{\theta}} \frac{\tilde{\theta} - \theta}{\kappa - \psi}\mathrm{d}\theta = \frac{1}{2}\frac{(\tilde{\theta} - \psi)^2}{\kappa - \psi}. \tag{A.2.18}$$

The parameter restrictions in (3.3) require $\kappa < \delta$. Together with $\psi \leq \tilde{\theta} \leq \kappa$ and $0 < \psi < \kappa$, this implies

$$\mathcal{S}_{\mathrm{i}}(\tilde{\theta}) \leq \frac{1}{2}(\kappa - \psi) < \frac{\kappa}{2} < \frac{\delta}{2}. \tag{A.2.19}$$

Now define the function $\mathcal{M}(p)$ as follows and calculate its derivative:

$$\mathcal{M}(p) \equiv \left(1 - \frac{\mu}{1-p}\frac{\mathcal{S}_{\mathrm{i}}(\tilde{\theta})}{\delta}\right)p, \quad \text{and} \quad \mathcal{M}'(p) = 1 - \frac{1}{1-p}\frac{\mu}{1-p}\frac{\mathcal{S}_{\mathrm{i}}(\tilde{\theta})}{\delta}. \tag{A.2.20}$$

Given the parameter restrictions in (3.3), the formula $\alpha = \mu/(1-p)$ from (3.6) always returns a well-defined probability for $p \leq 1/2$. Given that $p \leq 1/2$ implies $1/(1-p) \leq 2$ and (A.2.19) implies $\mathcal{S}_{\mathrm{i}}(\tilde{\theta})/\delta < 1/2$, it follows from (A.2.20) that $\mathcal{M}'(p) > 0$ for all $p \in [0, 1/2]$. Noting that $\mathcal{M}(0) = 0$ and $\pi(s) > 0$ according to (A.2.13), the equation $\mathcal{M}(p^*) = \pi(s)$ has at most one solution p^* satisfying $0 < p^* < 1/2$. When such a solution exists, it follows from (A.2.20) that $p^* > \pi(s)$ because $\mathcal{S}_{\mathrm{i}}(\tilde{\theta})/\delta < 1/2$ and $\alpha = \mu/(1-p^*) \leq 1$. Observe that the equation $\mathcal{M}(p^*) = \pi(s)$ is equivalent to the condition in (A.2.17) for both constraints to be binding.

In the case where no solution $p^* \in (0, 1/2)$ exists, set $p^* = 1/2$, and as $\mathcal{M}(0) = 0$ and $\mathcal{M}'(p) > 0$ it must be the case that $\mathcal{M}(p^*) = \mathcal{M}(1/2) < \pi(s)$. Since it has been shown that $\pi(s) < 1/2$ for the function $\pi(s)$ defined in (A.2.13), it follows that $p^* > \pi(s)$. Note that this case is consistent with (A.2.16b), but not (A.2.16a), which means that it could only occur in conjunction with the post-investment no-rebellion constraint for workers being binding, while the pre-investment no-rebellion constraint is slack.

Now consider a modification of the allocation in either of the conjectured equilibria described above, namely power sharing satisfying $0 < p^* < 1/2$ and both no-rebellion constraints binding, or $p^* = 1/2$ with the pre-investment no-rebellion constraint slack. The workers' post-investment no-rebellion constraint binds in both cases. The modification of the allocation involves changing power sharing p, while consumption of workers is adjusted so that the post-investment no-rebellion constraint for workers continues to bind, in which case the payoff of a worker is given in equation (A.2.8). The condition for the new choice of p to be consistent with the pre-investment stage constraint (with $p' = p^*$ and $U'_{\mathrm{p}} = U^*_{\mathrm{p}}$ taken as given at their conjectured equilibrium values) is:

$$U^*_{\mathrm{p}} - U_{\mathrm{n}} \leq \delta \frac{p}{p^*}, \tag{A.2.21}$$

where the expected non-incumbent payoff is obtained using equations (A.2.1) and (A.2.8):

$$U_{\mathrm{n}} = U_{\mathrm{w}} + \frac{\mu}{1-p}\mathcal{S}_{\mathrm{i}}(\tilde{\theta}) = U^{\dagger}_{\mathrm{p}}(K) - \delta\frac{p}{p^{\dagger}} + \frac{\mu}{1-p}\mathcal{S}_{\mathrm{i}}(\tilde{\theta}).$$

Substituting this expression into (A.2.21) shows that the feasibility of the modification requires:

$$U^*_{\mathrm{p}} - U^{\dagger}_{\mathrm{p}}(K) + \delta\frac{p}{p^{\dagger}} - \frac{\mu}{1-p}\mathcal{S}_{\mathrm{i}}(\tilde{\theta}) \leq \delta\frac{p}{p^*},$$

which can be rearranged as follows:

$$\left(\frac{1}{p^{\dagger}} - \frac{1}{p^*}\right)p - \frac{\mu}{1-p}\frac{\mathcal{S}_{\mathrm{i}}(\tilde{\theta})}{\delta} \leq \frac{U^{\dagger}_{\mathrm{p}}(K) - U^*_{\mathrm{p}}}{\delta}. \tag{A.2.22}$$

This inequality is known to be satisfied at $p = p^*$ because p^* satisfies the pre-investment no-rebellion constraint.

Now define the function $\mathcal{J}(p)$, and note the following expression for its derivative:

$$\mathcal{J}(p) \equiv \left(\frac{1}{p^{\dagger}} - \frac{1}{p^*}\right)p - \frac{\mu}{1-p}\frac{\mathcal{S}_{\mathrm{i}}(\tilde{\theta})}{\delta}, \quad \text{and} \quad \mathcal{J}'(p) = \frac{1}{p^{\dagger}} - \frac{1}{p^*} - \frac{\mu}{(1-p)^2}\frac{\mathcal{S}_{\mathrm{i}}(\tilde{\theta})}{\delta}. \tag{A.2.23}$$

Evaluating the derivative at $p = p^*$:

$$\mathcal{J}'(p^*) = \frac{1}{p^\dagger p^*}\left(p^* - p^\dagger - \frac{\mu \mathcal{S}_\mathrm{i}(\tilde{\theta})}{\delta}\frac{p^\dagger}{1-p^*}\frac{p^*}{1-p^*}\right) > \frac{1}{p^\dagger p^*}\left(\pi(s) - p^\dagger - \frac{\mu \mathcal{S}_\mathrm{i}(\tilde{\theta})}{\delta}\frac{p^\dagger}{1-p^*}\frac{p^*}{1-p^*}\right),$$

where the inequality above follows from $p^* > \pi(s)$, which is true in all cases under consideration. Since (A.2.13) implies $\pi(s) - p^\dagger = \mu\tilde{\theta}s/\delta$, the inequality above leads to:

$$\mathcal{J}'(p^*) > \frac{\mu}{\delta}\frac{1}{p^\dagger p^*}\left(\tilde{\theta}s - \frac{p^\dagger}{1-p^*}\frac{p^*}{1-p^*}\mathcal{S}_\mathrm{i}(\tilde{\theta})\right). \quad \text{[A.2.24]}$$

Note that $s = \mathbb{P}_\theta[\theta \leq \tilde{\theta}]$ from (3.7a), and use the definition of $\mathcal{S}_\mathrm{i}(\tilde{\theta})$ in (A.2.1) to deduce

$$\tilde{\theta}s - \mathcal{S}_\mathrm{i}(\tilde{\theta}) = \mathbb{E}_\theta\tilde{\theta}\mathbb{1}[\theta \leq \tilde{\theta}] - \mathbb{E}_\theta \max\{\tilde{\theta} - \theta, 0\} = \mathbb{E}_\theta\theta\mathbb{1}[\theta \leq \tilde{\theta}] \geq 0,$$

and hence $\mathcal{S}_\mathrm{i}(\tilde{\theta}) \leq \tilde{\theta}s$. Substituting this into (A.2.24) demonstrates that

$$\mathcal{J}'(p^*) > \frac{\mu}{\delta}\frac{\tilde{\theta}s}{p^\dagger p^*}\left(1 - \frac{p^\dagger}{1-p^*}\frac{p^*}{1-p^*}\right).$$

Since $p^\dagger < p^* \leq 1/2$, it follows that $p^\dagger/(1-p^*) < 1$ and $p^*/(1-p^*) \leq 1$, and thus $\mathcal{J}'(p^*) > 0$.

The pre-investment no-rebellion constraint (A.2.21) is equivalent to (A.2.22), which by comparison with the definition of $\mathcal{J}(p)$ in (A.2.23) is in turn equivalent to:

$$\mathcal{J}(p) \leq \frac{U_\mathrm{p}^\dagger(K) - U_\mathrm{p}^*}{\delta}.$$

It is known that $\mathcal{J}(p)$ is increasing in p and the above inequality is satisfied at $p = p^*$. Observing that the right-hand side is unaffected by movements in p away from the conjectured equilibrium p^*, it then follows that it is possible to reduce p below p^* and still satisfy the pre-investment no-rebellion constraint.

Now consider the incumbents' post-investment stage no-rebellion constraint. Since incumbent payoffs must be equalized, the inequality in (A.2.15) is equivalent to that no-rebellion constraint. Furthermore, it has already been shown that when the post-investment stage no-rebellion constraint binds for workers, (A.2.15) is equivalent to $p \geq \pi(s)$.

Therefore, starting from the conjectured equilibrium at $p = p^* > \pi(s)$, it is feasible to reduce p by some positive amount and continue to satisfy all no-rebellion constraints. Moreover, it has been shown that when the workers' binding post-investment constraint is used to determine C_w, the derivative of the incumbents' objective function $\bar{U}_\mathrm{p}$ with respect to p is given by (A.2.14) and is thus strictly decreasing for $p > \sqrt{p^\dagger}\sqrt{\pi(s)}$. Since $p^* > \sqrt{p^\dagger}\sqrt{\pi(s)}$, the proposed modification of the allocation is not subject to any rational rebellion and is payoff-improving for the incumbents. Hence, there is no equilibrium with this configuration of binding constraints.

The post-investment stage no-rebellion constraint involving only workers cannot bind in conjunction with a pre-investment stage constraint involving some incumbents

Suppose that the pre-investment stage constraints are ignored. This leads to the choice of $p^* = \sqrt{p^\dagger}\sqrt{\pi(s)}$ as seen earlier. Given that there is no violation of (A.2.3a), if the resulting average incumbent payoff $\bar{U}_\mathrm{p}^*$ were equally distributed among incumbents then the pre-investment stage no-rebellion constraint would be satisfied for all possible configurations of the rebel army (since $U_\mathrm{p}^* = U_\mathrm{p}'$). This equal distribution of payoffs has no negative consequences either directly for the incumbent objective function or for the conjectured binding constraints, so $p^* = \sqrt{p^\dagger}\sqrt{\pi(s)}$ would also be the equilibrium taking into account other pre-investment stage no-rebellion constraints. However, this has been seen to lead to a violation of a post-investment stage no-rebellion constraint involving incumbents (the proof that this violation occurs requires no assumptions on the distribution of incumbent payoffs, so choosing a non-equal distribution would not help in avoiding the violation). Therefore this case does not correspond to the equilibrium combination of binding no-rebellion constraints.

The post-investment no-rebellion constraint involving only workers binds in conjunction with a post-investment constraint involving incumbents. In this case, there must be complete payoff equalization among the group in power. All other no-rebellion constraints will hold.

Since some no-rebellion constraints must bind, the only remaining cases are ones where (A.2.3b) binds along with a post-investment stage no-rebellion constraint involving incumbents.

The binding version of (A.2.3b) is:

$$U_{\mathrm{p}}^{\dagger}(K) - U_{\mathrm{w}} = \delta \frac{p}{p^{\dagger}}.$$

Since there is no shortage of workers to fill a rebel army, a post-investment stage no-rebellion constraint with a positive measure of incumbents can bind if and only if

$$U_{\mathrm{p}}^{\dagger}(K) - \underline{U}_{\mathrm{p}} = \delta \left(\frac{p - p^{\dagger}}{p^{\dagger}} \right),$$

where $\underline{U}_{\mathrm{p}}$ is the minimum among the payoffs received by any positive measure of incumbents. Note that the right-hand side must be strictly positive because $p > p^{\dagger}$. The argument is that if the left-hand side were greater than the right-hand side, the no-rebellion constraint would be violated because the constraint for workers is exactly binding with each exerting fighting effort $\delta p/p^{\dagger}$, so if a positive measure of incumbents were to contribute power-adjusted fighting effort greater than $\delta p/p^{\dagger}$ per person then a rational rebellion would occur. On the other hand, if the left-hand side were strictly greater than the right-hand side then all positive measures of incumbents who are willing to rebel would contribute power-adjusted fighting effort strictly less than $\delta p/p^{\dagger}$. Since the no-rebellion constraint holds with equality for workers who each exert effort $\delta p/p^{\dagger}$, this means the no-rebellion constraint involving any positive measure of incumbents would be slack, contrary to the supposition.

Now suppose there is payoff inequality among a positive measure of incumbents. It follows that $\underline{U}_{\mathrm{p}} < \bar{U}_{\mathrm{p}}$ since $\bar{U}_{\mathrm{p}}$ is the average payoff, given the definition of $\underline{U}_{\mathrm{p}}$. Equalization of payoffs would imply all incumbents receive $\bar{U}_{\mathrm{p}}$, which given the inequality above, is such that:

$$U_{\mathrm{p}}^{\dagger}(K) - \bar{U}_{\mathrm{p}} < \delta \left(\frac{p - p^{\dagger}}{p^{\dagger}} \right).$$

But this implies every incumbent would exert power-adjusted fighting effort strictly less than $\delta p/p^{\dagger}$ if included in a rebel army. Since the no-rebellion constraint for workers is binding with fighting effort $\delta p/p^{\dagger}$, this means the post-investment stage no-rebellion constraint is now slack if any positive measure of incumbents is included. The equalization of payoffs has no direct effect on the incumbents' objective function and is seen to slacken a constraint that is binding, hence payoff inequality cannot occur in equilibrium.

Now suppose the post-investment no-rebellion constraint is binding for some incumbents given that there is payoff equalization among all positive measures of incumbents. Since the post-investment constraint for workers is binding, and as $p > p^{\dagger}$ implies there is no shortage of incumbents to fill a rebel army (with $U_{\mathrm{p}} < U_{\mathrm{p}}^{\dagger}(K)$ ensuring all are willing to rebel), because of the linearity of total fighting strength in the composition of the rebel army this case is equivalent to the no-rebellion constraint binding for a rebel army including only incumbents:

$$U_{\mathrm{p}}^{\dagger}(K) - U_{\mathrm{p}} = \delta \left(\frac{p - p^{\dagger}}{p^{\dagger}} \right). \qquad \text{[A.2.25]}$$

Given that the post-investment no-rebellion constraint for workers is binding, the inequality (A.2.15) is equivalent to $p \geq \pi(s)$. Therefore, it must be the case that $p = \pi(s)$.

Now consider an equilibrium p^* and s^* where these values maximize U_{p} subject to the power-sharing constraint $p = \pi(s)$ (with consumption of workers determined by their binding no-rebellion constraint). Note that both binding no-rebellion constraints imply $U^*_{\mathrm{p}} = U^*_{\mathrm{w}} + \delta$. Using (3.6), it follows that $U^*_{\mathrm{n}} = U^*_{\mathrm{p}} - \delta + \alpha\mathcal{S}_{\mathrm{i}}(\tilde{\theta})$, and therefore:

$$U^*_{\mathrm{p}} - U^*_{\mathrm{n}} = \delta - \alpha\mathcal{S}_{\mathrm{i}}(\tilde{\theta}).$$

Since $\alpha\mathcal{S}_{\mathrm{i}}(\tilde{\theta}) \geq 0$, it follows that $U^*_{\mathrm{p}} - U^*_{\mathrm{n}} \leq \delta$, so (A.2.3a) is satisfied. As all incumbents receive the same payoff $U^*_{\mathrm{p}} = U'_{\mathrm{p}}$, all compositions of rebel armies at the pre-investment stage must satisfy their no-rebellion constraints. Therefore this case is the combination of binding no-rebellion constraints in equilibrium when $s > 0$. This establishes the claims in parts (iii) and (iv) of the proposition, and the second part of the claim in (i).

The incumbents' payoff given the binding constraints

Given that both (A.2.8) and (A.2.25) hold, it follows that

$$U_{\mathrm{w}} = U_{\mathrm{p}} - \delta.$$

Substituting this expression for U_{w} into the incumbents' objective function $\bar{U}_{\mathrm{p}}$ from equation (A.2.2), and noting that payoff equalization implies $\bar{U}_{\mathrm{p}} = U_{\mathrm{p}}$:

$$pU_{\mathrm{p}} = (1-p)(\mathrm{q} - (U_{\mathrm{p}} - \delta)) + \mu(\kappa - \tilde{\theta})s,$$

and hence by rearranging the above to find an expression for U_{p}:

$$U_{\mathrm{p}} = (\mathrm{q} + \delta)(1-p) + \mu(\kappa - \tilde{\theta})s.$$

Now substituting for $p = \pi(s)$ using the formula for $\pi(s)$ from (A.2.13):

$$U_{\mathrm{p}} = (\mathrm{q} + \delta)(1-p^\dagger) + \mu(\kappa - \tilde{\theta})s - \frac{\mu\tilde{\theta}s}{\delta}.$$

Simplifying this expression yields

$$U_{\mathrm{p}} = (\mathrm{q} + \delta)(1-p^\dagger) + \mu\left(\kappa - \frac{\tilde{\theta}}{p^\dagger}\right)s,$$

and by substituting the formula for $p^\dagger$ from (3.5), the expression for U_{p} in (3.7d) is obtained. This confirms the claim in part (v) of the proposition and completes the proof.

A.3 Proof of Proposition 3

First, note that the relationship between s and $\tilde{\theta}$ in (3.7a) implies

$$\tilde{\theta} = \psi + (\kappa - \psi)s. \qquad [A.3.1]$$

Proposition 2 shows that the two constraints in (3.7b) are binding. By subtracting these equations from one another, it follows that:

$$U_{\mathrm{w}} = U_{\mathrm{p}} - \delta. \qquad [A.3.2]$$

The level of investment in equilibrium

By substituting the formula for $\tilde{\theta}$ from (A.3.1) into the expression for the incumbents' payoff in (3.7d):

$$U_{\mathrm{p}} = \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} + \mu\left(\kappa - \left(\frac{\mathrm{q}+2\delta}{\delta}\right)\psi - \left(\frac{\mathrm{q}+2\delta}{\delta}\right)(\kappa-\psi)s\right)s.$$

The derivative with respect to s is

$$\frac{\partial U_{\mathrm{p}}}{\partial s} = \mu\left(\kappa - \left(\frac{\mathrm{q}+2\delta}{\delta}\right)\psi - 2\left(\frac{\mathrm{q}+2\delta}{\delta}\right)(\kappa-\psi)s\right).$$

Setting the derivative to zero and solving for s yields

$$s = \frac{\delta\kappa - (\mathrm{q}+2\delta)\psi}{2(\mathrm{q}+2\delta)(\kappa-\psi)} = \frac{1}{2}\frac{\delta\kappa - (\mathrm{q}+2\delta)\psi}{(\mathrm{q}+2\delta)\kappa - (\mathrm{q}+2\delta)\psi}. \tag{A.3.3}$$

Since $\mathrm{q}+2\delta > \delta$, this expression can never be more than 1, but could be negative. Given that s is restricted to $s \in [0,1]$, the value of s that maximizes U_{p} is

$$s^* = \max\left\{0, \frac{\delta\kappa - (\mathrm{q}+2\delta)\psi}{2(\mathrm{q}+2\delta)(\kappa-\psi)}\right\}. \tag{A.3.4}$$

This is the solution given in (3.11). It can be seen that s^* is positive whenever $\delta\kappa > (\mathrm{q}+2\delta)\psi$, which is equivalent to $\kappa/\psi - 1 > 1 + \mathrm{q}/\delta$.

To confirm that this is indeed an equilibrium, it is necessary to check whether several auxiliary conditions are satisfied. First, there is the condition from (2.5b) that defending the existing allocation is in the interests of those incumbents who do not belong to a rebel army. This requires $U_{\mathrm{p}}^* > U_{\mathrm{n}}' = U_{\mathrm{n}}^*$ at the pre-investment stage, and $U_{\mathrm{p}}^* > U_{\mathrm{w}}^\dagger(K)$ at the post-investment stage.

Note that (3.6) implies the following expression for U_{n}^*:

$$U_{\mathrm{n}}^* = U_{\mathrm{w}}^* + \alpha\mathcal{S}_{\mathrm{i}}(\tilde{\theta}^*).$$

Given that $U_{\mathrm{p}}^* = U_{\mathrm{w}}^* + \delta$ according to (A.3.2), the condition $U_{\mathrm{p}}^* > U_{\mathrm{n}}^*$ is equivalent to

$$\alpha\mathcal{S}_{\mathrm{i}}(\tilde{\theta}^*) < \delta. \tag{A.3.5}$$

Using the distribution of the effort cost θ from (3.1) and (3.7a), it is known that $0 < \psi < \kappa$ and $\psi \leq \tilde{\theta}^* \leq \kappa$. Hence it must be the case that $\tilde{\theta}^* - \theta < \kappa$ for all $\theta \in [\psi, \kappa]$, and the definition of $\mathcal{S}_{\mathrm{i}}(\tilde{\theta})$ in (A.2.1) can then be used to deduce $\mathcal{S}_{\mathrm{i}}(\tilde{\theta}^*) < \kappa$. Since α is a probability and $\kappa < \delta$ according to the parameter restrictions in (3.3), the condition (A.3.5) is verified.

Using the expression for U_{p} from (3.7d), the expression for $U_{\mathrm{w}}^\dagger(K)$ from (3.5), and the formula for the capital stock K from (3.7a), the condition $U_{\mathrm{p}}^* > U_{\mathrm{w}}^\dagger(K)$ is equivalent to:

$$\frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} + \mu\left(\kappa - \left(\frac{\mathrm{q}+2\delta}{\delta}\right)\tilde{\theta}^*\right)s^* > \frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} - \delta + \mu\kappa s^*.$$

After cancelling terms and rearranging, this requirement reduces to:

$$\mu\left(\frac{\mathrm{q}+2\delta}{\delta}\right)\tilde{\theta}^* s^* < \delta. \tag{A.3.6}$$

Consider an equilibrium where $s^* > 0$, in which case equation (A.3.4) implies

$$(\kappa-\psi)s^* = \frac{\delta\kappa - (\mathrm{q}+2\delta)\psi}{2(\mathrm{q}+2\delta)},$$

and where it must be the case that $\delta\kappa > (q+2\delta)\psi$. Substituting this into the formula for $\tilde{\theta}$ in (A.3.1):

$$\tilde{\theta}^* = \psi + \frac{\delta\kappa - (q+2\delta)\psi}{2(q+2\delta)} = \frac{1}{2}\left(\left(\frac{\delta}{q+2\delta}\right)\kappa + \psi\right).$$

Given that $\psi < \kappa$, it must be the case that $\psi < (\delta/(q+2\delta))\kappa$, and so the expression above for $\tilde{\theta}^*$ implies:

$$\tilde{\theta}^* < \left(\frac{\delta}{q+2\delta}\right)\kappa.$$

Since $\mu \leq 1$ and $s^* \leq 1$, and using the parameter restriction $\kappa < \delta$ from (3.3), the inequality above implies that (A.3.6) must hold, demonstrating that $U_p^* > U_w^\dagger(K)$.

As the utility function is linear, the link between worker and incumbent payoffs in (A.3.2) implies $C_p^* = C_w^* + \delta$. Therefore, all non-negativity constraints on consumption will hold if $C_w^* \geq 0$, which is equivalent to $U_w^* \geq 0$. Observe first that $s = 0$ is always a feasible choice for the incumbents in maximizing U_p, so if $s^* > 0$, it follows from the expression in (3.7d) that:

$$\mu\left(\kappa - \left(\frac{q+2\delta}{\delta}\right)\tilde{\theta}^*\right)s^* \geq 0. \tag{A.3.7}$$

Substituting the expression for U_p from (3.7d) into (A.3.2) yields:

$$U_w^* = \left(\frac{(q+\delta)^2}{q+2\delta} - \delta\right) + \mu\left(\kappa - \left(\frac{q+2\delta}{\delta}\right)\tilde{\theta}^*\right)s^*, \tag{A.3.8}$$

and since (A.3.7) shows the second term is non-negative, a sufficient condition for $U_w^* \geq 0$ is

$$\frac{(q+\delta)^2}{q+2\delta} \geq \delta. \tag{A.3.9}$$

The parameter restriction $\delta/q \leq \varphi$ from (3.3) implies that this inequality holds, so it is confirmed that $C_w^* \geq 0$. Therefore, the solution in (A.3.4) is shown to be the unique equilibrium. This confirms the claims in (i) and the first part of (iii) in the proposition.

The constrained efficient level of investment

Using the relationship between p and s from (3.7c), it follows that:

$$\delta(1-p) = \delta\left(1 - p^\dagger - \frac{\mu\tilde{\theta}s}{\delta}\right) = \delta(1-p^\dagger) - \mu\tilde{\theta}s = \frac{\delta(q+\delta)}{q+2\delta} - \mu\tilde{\theta}s,$$

where the formula for $p^\dagger$ from (3.5) is also substituted into the above expression. The definition of the average payoff $\bar{U}$ from (3.9) is equivalent to (3.10), and the expression for $(1-\delta)p$ above can be used to obtain:

$$\bar{U} = \frac{(q+\delta)^2}{q+2\delta} - \frac{\delta(q+\delta)}{q+2\delta} + \mu\left(\kappa - \left(\frac{q+\delta}{\delta}\right)\tilde{\theta}\right)s + \mu\tilde{\theta}s + \mu\mathcal{S}_i(\tilde{\theta}).$$

After simplification, this expression for $\bar{U}$ reduces to:

$$\bar{U} = \frac{q(q+\delta)}{q+2\delta} + \mu\left(\kappa - \left(\frac{q+\delta}{\delta}\right)\tilde{\theta}\right)s + \mu\mathcal{S}_i(\tilde{\theta}).$$

Equation (A.3.1) gives a relationship between $\tilde{\theta}$ and s, which can also be substituted into the above:

$$\bar{U} = \frac{q(q+\delta)}{q+2\delta} + \mu\left(\kappa - \left(\frac{q+\delta}{\delta}\right) - \left(\frac{q+\delta}{\delta}\right)(\kappa-\psi)s\right)s + \mu\mathcal{S}_i(\tilde{\theta}). \tag{A.3.10}$$

Using (3.1) and (A.2.1), an explicit expression for the expected surplus $\mathcal{S}_{\mathrm{i}}(\tilde{\theta})$ from receiving an investment opportunity is given by:

$$\mathcal{S}_{\mathrm{i}}(\tilde{\theta}) = \int_{\theta=\psi}^{\tilde{\theta}} \frac{\tilde{\theta}-\theta}{\kappa-\psi} \mathrm{d}\theta = \frac{1}{2}\frac{(\tilde{\theta}-\psi)^2}{\kappa-\psi} = \frac{1}{2}(\kappa-\psi)s^2,$$

where equation (3.7a) has been used to write this solely in terms of s. This is then substituted into (A.3.10) to obtain an expression for $\bar{U}$ in terms of s:

$$\bar{U} = \frac{\mathrm{q}(\mathrm{q}+\delta)}{\mathrm{q}+2\delta} + \mu\left(\kappa - \left(\frac{\mathrm{q}+\delta}{\delta}\right) - \left(\frac{2\mathrm{q}+\delta}{2\delta}\right)(\kappa-\psi)s\right)s. \quad \text{[A.3.11]}$$

Using (A.3.11), the derivative of $\bar{U}$ with respect to s is:

$$\frac{\partial \bar{U}}{\partial s} = \mu\left(\kappa - \left(\frac{\mathrm{q}+\delta}{\delta}\right) - \left(\frac{2\mathrm{q}+\delta}{\delta}\right)(\kappa-\psi)s\right).$$

Setting the derivative to zero and solving for s yields:

$$s = \frac{\delta\kappa - (\mathrm{q}+\delta)\psi}{(2\mathrm{q}+\delta)(\kappa-\psi)} = \frac{\delta\kappa-(\mathrm{q}+\delta)\psi}{(\mathrm{q}+\delta)\kappa - (\mathrm{q}+\delta)\psi + \mathrm{q}(\kappa-\psi)}. \quad \text{[A.3.12]}$$

Since $\mathrm{q}+\delta > \delta$ and $\kappa > \psi$, this expression can never be greater than 1, but it could be negative. Therefore, if no other required conditions are violated, the constrained efficient level of s is

$$s^{\diamond} = \max\left\{0, \frac{\delta\kappa-(\mathrm{q}+\delta)\psi}{(2\mathrm{q}+\delta)(\kappa-\psi)}\right\}. \quad \text{[A.3.13]}$$

This is the expression for $s^{\diamond}$ from (3.11). It is positive whenever $\delta\kappa > (\mathrm{q}+\delta)\psi$, which is equivalent to $\kappa/\psi - 1 > \mathrm{q}/\delta$.

The auxiliary conditions to verify are $U_{\mathrm{p}}^{\diamond} > U_{\mathrm{n}}^{\diamond}$, $U_{\mathrm{p}}^{\diamond} > U_{\mathrm{w}}^{\dagger}(K)$, and $C_{\mathrm{w}}^{\diamond} \geq 0$. Given that there is the same configuration of binding no-rebellion constraints, the analysis leading to (A.3.5) also shows that $U_{\mathrm{p}}^{\diamond} > U_{\mathrm{n}}^{\diamond}$ is equivalent to $\alpha\mathcal{S}_{\mathrm{i}}(\tilde{\theta}^{\diamond}) < \delta$. Under the parameter restrictions from (3.3), this condition is necessarily satisfied.

Next, consider the requirement $U_{\mathrm{p}}^{\diamond} > U_{\mathrm{w}}^{\dagger}(K)$. Again, given that the configuration of binding no-rebellion constraints is the same, the analysis leading to (A.3.6) also applies in this case, so the condition is equivalent to

$$\mu\left(\frac{\mathrm{q}+2\delta}{\delta}\right)\tilde{\theta}^{\diamond}s^{\diamond} < \delta. \quad \text{[A.3.14]}$$

Consider a case where $s^{\diamond} > 0$. Using (A.3.13), it must then follow that

$$(\kappa-\psi)s^{\diamond} = \frac{\delta\kappa-(\mathrm{q}+\delta)\psi}{(2\mathrm{q}+\delta)},$$

noting $\delta\kappa > (\mathrm{q}+\delta)\psi$ is necessary. Substituting this into (A.3.1) yields:

$$\tilde{\theta}^{\diamond} = \psi + \frac{\delta\kappa-(\mathrm{q}+\delta)\psi}{(2\mathrm{q}+\delta)} = \frac{\mathrm{q}\psi+\delta\kappa}{2\mathrm{q}+\delta}. \quad \text{[A.3.15]}$$

Since $\psi < (\delta/(\mathrm{q}+\delta))\kappa$ in this case, it follows that

$$\tilde{\theta}^{\diamond} < \frac{\mathrm{q}\left(\frac{\delta}{\mathrm{q}+\delta}\right)\kappa + \delta\kappa}{2\mathrm{q}+\delta} = \frac{\delta(2\mathrm{q}+\delta)\kappa}{(\mathrm{q}+\delta)(2\mathrm{q}+\delta)} = \frac{\delta}{\mathrm{q}+\delta}\kappa,$$

which implies:

$$\left(\frac{\mathrm{q}+2\delta}{\delta}\right)\tilde{\theta}^{\diamond} < \frac{\mathrm{q}+2\delta}{\mathrm{q}+\delta}\kappa.$$

Using the parameter restrictions in (3.3) and the inequality above:

$$\mu\left(\frac{\mathrm{q}+2\delta}{\delta}\right)\tilde{\theta}^{\diamond} < \left(\frac{\mathrm{q}}{2(\mathrm{q}+2\delta)}\right)\left(\frac{\mathrm{q}+2\delta}{\mathrm{q}+\delta}\right)\kappa = \frac{1}{2}\frac{\mathrm{q}}{\mathrm{q}+\delta}\kappa < \kappa < \delta.$$

This demonstrates that (A.3.14) holds, which confirms that $U_{\mathrm{p}}^{\diamond} > U_{\mathrm{w}}^{\dagger}(K)$.

Finally, consider the non-negativity constraint $C_{\mathrm{w}}^{\diamond} \geq 0$ for workers, which is equivalent to $U_{\mathrm{w}}^{\diamond} \geq 0$ given the utility function. Substituting the formula for U_{p} from (3.7d) into (A.3.2), and using (A.3.15) to obtain an expression for $\tilde{\theta}^{\diamond}$:

$$U_{\mathrm{w}}^{\diamond} = \left(\frac{(\mathrm{q}+\delta)^2}{\mathrm{q}+2\delta} - \delta\right) + \frac{\mu}{\delta(\mathrm{q}+2\delta)}\left(\delta(\mathrm{q}-\delta)\kappa - \mathrm{q}(\mathrm{q}+2\delta)\psi\right)s^{\diamond}. \tag{A.3.16}$$

The sign of this expression is ambiguous for general parameters satisfying the restrictions in (3.3), so the non-negativity constraint for workers could be binding. When this expression is non-negative, the constrained efficient level of s is indeed given by the formula in (A.3.13) since all other auxiliary conditions are satisfied. More generally, since the non-negativity constraint is satisfied at $s = 0$, the possibility that it might be binding in equilibrium implies that $\kappa/\psi - 1 > \mathrm{q}/\delta$ is only a necessary condition for $s^{\diamond} > 0$.

Consider a case with $s^{\diamond} > 0$. In the situation where the non-negativity constraint is not binding, the value of $s^{\diamond}$ is given by (A.3.13). Comparison with the expression for s^* in (A.3.4) shows that $s^* < s^{\diamond}$. Now suppose the non-negativity constraint is binding. Since the non-negativity constraint is known to be satisfied at $s = 0$ and $s = s^*$, it follows that constrained efficient value of s must be strictly larger than s^*. Therefore, it is shown that $s^* < s^{\diamond}$ whenever $s^{\diamond} > 0$. This confirms the claims in (ii) and the second part of (iii), which completes the proof.