Taking Away the Punch Bowl: Monetary Policy and Financial Instability*

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Abstract

In the last decade, the problem of financial instability has grabbed the attention of economists and policymakers. This paper presents a theory of financial crises due to excessively loose monetary policy. Under political pressure to choose a monetary policy that is popular with the average person in the economy, the central bank sets low interest rates, which most of the time delivers rapid asset-price inflation and a build-up of debt, but also occasional financial crises where asset prices collapse and deleveraging occurs. Pareto-inefficient financial instability occurs because the central bank does not have access to individual-specific lump-sum taxes and transfers to sterilize the distributional consequences of changing interest rates. Consequently, removing the punch bowl is difficult because too many individuals have a vested interest in maintaining cheap credit.

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The Federal Reserve, after the recent increase in the discount rate, is in the position of the chaperone who has ordered the punch bowl removed just when the party was really warming up.

William McChesney Martin, *Chairman of the Federal Reserve (1951–1970)*

1 Introduction

Since the financial crisis of 2007–2008, financial instability has been back at the forefront of the attention of economists and policymakers. There has been much debate about the causes of financial crises, which policy responses are appropriate, and whether there are reforms to the financial system or regulatory changes that might reduce the likelihood of crises occurring in the future.

One hypothesis is that monetary policy bears some responsibility for crises. By setting interest rates too low, central banks create asset-price booms and encourage an unsustainable build-up of debt. Once asset prices eventually drop, the presence of large amounts of debt leads to a financial crisis. This view has been articulated by Taylor (2009) in the context of the U.S. monetary policy prior to the 2007–2008 financial crisis, and more generally as the 'BIS view' (Borio and Lowe, 2002, Borio, 2012). However, it is not clear what theoretical mechanism leads monetary policy to have such dramatic real effects on financial instability. If central banks cannot influence long-term real interest rates, which gravitate to a natural level independent of monetary policy, how can they be responsible for booms and busts in asset prices and debt? Moreover, even if there were a link between monetary policy and crises, why would central banks knowingly adopt policies that increase financial risk?

The contribution of this paper is to answer both of these questions by presenting a theory of financial crises caused by central banks under pressure to maintain excessively loose monetary policies. In this theory, monetary policy takes centre stage in explaining financial crises because it has long-lasting real effects on borrowing costs and the volatility of asset prices. By studying the political economy of monetary policy, the theory also explains why central banks will find it difficult to 'take away the punch bowl' and thus find themselves in the position of pursuing policies that raise the risk of financial crises.

In the equilibrium of the model, monetary policy keeps real borrowing costs low for as long as possible. The consequence of this policy is that most of the time there is sustained inflation in house prices and a build-up of debt. However, the policy also results in occasional financial crises: rare events where house prices collapse and significant deleveraging occurs. The occurrence of financial crises and the booms in asset prices and credit that precede them imply a highly inefficient allocation of resources in the economy. Monetary policy is at fault in the sense that it would be possible for the central bank to achieve an efficient allocation of resources by setting higher interest rates to prevent house-price booms and busts — but tight monetary policy will be highly unpopular with many individuals in the economy.

There are two key ingredients of the theory. First, an incomplete markets friction whereby borrowers are restricted to nominal debt contracts. Households who want to borrow to buy houses only have access to standard mortgages. They cannot raise finance by selling equity shares in their houses, or hedge their exposure to house prices by buying or selling derivatives. Markets are also incomplete in that households cannot decouple their enjoyment of housing services from homeownership by making use of a perfect rental market.

The second key ingredient is political economy. The central bank does not exist in a political vacuum — it cannot ignore the distributional impact of its policies on the majority of the population, which could sway policy from a sole focus on the technocratic goal of economic efficiency. The tension between distribution and efficiency stems from the central bank having limited instruments: interest rates affect both the distribution of wealth and the efficiency of the allocation of resources in the economy. The central bank does not have access to individual-specific lump-sum taxes and transfers to sterilize the distributional consequences of changes to interest rates.

The theory is built on a mechanism through which the central bank can have a long-lasting impact on the real cost of borrowing. This is the effect of monetary policy on risk premiums. Even without introducing any frictions, monetary policy can always affect nominal asset prices and thus influence the probability distribution of unexpected nominal changes in asset prices. Comparing these uncertain nominal returns to the perfectly predictable nominal return on nominal bonds, standard asset-pricing theory predicts that monetary policy will be able to affect the risk premium over nominal bonds of real assets such as housing or shares, even though it cannot change the relative returns on two real assets. Combined with the incomplete markets friction whereby borrowers need to obtain credit in the form of nominal debt, monetary policy is therefore able to vary the real cost of borrowing through its impact on the relative returns on nominal and real assets.

The mechanism behind this effect of monetary policy on real borrowing costs is not restricted to a 'short run' defined in terms of sluggish price adjustment or imperfect information. Consequently, monetary policy operating through the risk premium channel can have long-lasting effects on real asset prices and credit. The greater the risk premium, the larger the gap between mortgage rates and the expected capital gains on housing. A lower real cost of borrowing stimulates a build-up of mortgage debt and pushes up real house prices. The impact is magnified by the financial accelerator where increases in house prices improve balance sheets and stimulate further mortgage lending. Note that the ability of monetary policy to affect real borrowing costs through the mechanism here does not mean it can raise or lower real returns simultaneously for all assets: higher real returns for homeowners are associated with lower real returns for bonds held as pensions.

Importantly, the central bank cannot change real interest rates through the risk premium channel without this having consequences for financial stability. The logic behind the risk premium means there must be in equilibrium a negative relationship between the interest rate on bonds and the volatility of asset prices. Lower interest rates can then only be achieved with a concomitant rise in financial risk. The boom in real asset prices and credit resulting from loose monetary policy is not one that can be sustained forever in equilibrium. All else equal, the larger the boom and the longer on average it is sustained, the larger must be the fall in asset prices when the boom unexpectedly comes to an end.

The theory thus explains the occurrence of financial crises if the central bank pursues loose monetary policy for long periods of time. In this case, there must be some rare occasions when there is a large fall in nominal house prices. Since crises have been preceded by a build-up of nominal debt, the financial accelerator mechanism goes into reverse. New lending is reduced as balance sheets deteriorate, and the fall in nominal assets prices during the crisis largely represents a reduction in the relative price of housing, with goods prices and nominal incomes falling much less.

Even if monetary policy could create a risk of financial crises because of the logic above, it is less obvious why policy would be conducted in this way, especially as the central bank acts to solve a Ramsey problem maximizing the average utility of households in the economy. The explanation lies in the distributional effects of monetary policy. Savers lose from low real interest rates, while borrowers gain, both directly and indirectly through the increased real return on their leveraged investment in housing. All else equal, leveraged homeowners dislike the financial instability that results from low interest rates because they are exposed to house-price risk, but all else is not equal due to the negative equilibrium relationship between the interest rate and the volatility of house prices. The model presented in this paper shows how borrowers overall prefer low rates and financial instability when financial risk is a rare event concentrated in the far left tail of the asset-price probability distribution.

The theory of financial crises in this paper does not depend on agents being irrational, or forming expectations with systematic errors. Furthermore, policymakers are benevolent in the sense of maximizing the average utility of people in the economy. In the model, everyone knows that loose monetary policy creates the conditions for financial crises, even if the timing of crises cannot be predicted. Nonetheless, there is sufficient political pressure from borrowers for central banks to maintain low interest rates in spite of the negative consequences.

In explaining why there are low-probability events with large falls in house prices, the theory endogenously generates 'rare events' of the kind that have been emphasized in the literature on asset pricing (see, for example, Barro, 2006). As in that literature, rare events have a large impact on risk premiums. Here, the difference is that the rare events are the consequence of monetary policy trying to generate a housing boom for as long as possible, with the transmission mechanism of monetary policy working through risk premiums. The importance of risk premiums in understanding house price developments is stressed by Favilukis, Ludvigson and Van Nieuwerburgh (2017) in a richer quantitative model, but that paper does not study the consequences of monetary policy for risk premiums.

The paper differs radically from the common approach in models of financial frictions where a linearized model is subject to shocks that generate small fluctuations in the neighbourhood of a steady state (Bernanke, Gertler and Gilchrist, 1999, Kiyotaki and Moore, 1997). In trying to understand financial crises as large discrete changes that hit an economy, the paper's goals have some similarity to Brunnermeier and Sannikov (2014). However, the mechanism for generating financial crises is completely different from that paper. There, the key assumption is that capital can be used most effectively by financially constrained 'experts'. By finding the full non-linear solution for the model's equilibrium, it is shown how an accumulation of small shocks can trigger a crisis episode. Here, financial crises occur in the housing market and are caused by the political economy of monetary policy. These are novel features in a formal model of financial crises.

The model's theoretical prediction of a credit cycle with credit booms preceding financial crises is consistent with the empirical evidence in Schularick and Taylor (2012). There is also some evidence supporting the role of monetary policy in generating credit booms (Jiménez, Ongena, Peydró and Saurina, 2014).

The normative implications of the paper build on a earlier literature that has explored optimal monetary policy with incomplete financial markets (Koenig, 2013, Sheedy, 2014). Here, achieving financial stability requires setting systematically higher interest rates — a central bank that is 'conservative' in the sense of giving greater weight to the interests of creditors rather than debtors.

The modelling strategy in the paper is to present a highly stylized model that is stripped down to the essentials. Everything can be solved analytically without the need for any approximations or numerical methods, including the constrained optimization of social welfare used to determine the central bank's monetary policy. The essential features of the model are heterogeneous agents with risk aversion, a housing market, nominal debt contracts, aggregate risk, and monetary policy set endogenously to solve a Ramsey problem. The simplest model that allows an exposition of the theory has overlapping generations of individuals with stylized hump-shaped profiles of non-financial income and the marginal utility from housing over their three-period lives, a stochastic endowment of consumption goods, and an exogenous supply of housing. This simple model exogenously assumes incomplete markets, but the main conclusions are robust to an extension where there are also markets subject to frictions for renting houses and trading equity shares in houses.

The plan of the paper is as follows. Section 2 sets up a model of a credit economy with housing and incomplete markets. Section 3 analyses the political economy of monetary policy and the equilibrium with financial crises when the central bank maximizes average household utility. Policy implications are studied in section 4, where it is shown that monetary policy can implement a first-best allocation of resources by making financial stability its goal. Section 5 endogenizes incomplete markets by extending the analysis to include additional markets subject to frictions. Finally, section 6 draws some conclusions.

2 A credit economy with housing and incomplete markets

This section lays out a simple model to illustrate the transmission of monetary policy through risk premiums and the political economy of its distributional effects. The model is deliberately kept basic to demonstrate transparently the novel mechanisms at work. A full global non-linear solution method is required because conventional perturbation methods (of any order) cannot be used: financial crises lie outside the radius of convergence of the Taylor series expansions of the model's equations. An advantage of the simple model proposed here is that both the economy's competitive equilibrium and the solution of the policymaker's Ramsey problem can be found analytically.

2.1 The model

Overlapping generations The economy has overlapping generations of individuals who have deterministic lives spanning three discrete time periods. Individuals of different generations are referred to as the 'young', 'middle-aged', and 'old', indexed by $a \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}$. There is a measure-one population of each age group in each time period.

Preferences Individuals born at date t have the following lifetime expected utility function:

$$U_t = \log C_{\mathrm{y},t} + \beta \mathbb{E}_t \left[\log C_{\mathrm{m},t+1} + \Theta(H_{\mathrm{m},t+1} - \underline{H})\right] + \beta^2 \mathbb{E}_t \log C_{\mathrm{o},t+2}, \tag{1}$$

where $C_{a,t}$ denotes per-person consumption of a composite good by individuals of age a at date t. Utility is logarithmic in consumption of goods. The subjective discount factor is β (satisfying $0 < \beta < \infty$), and $\mathbb{E}_t[\cdot]$ denotes expectations conditional on all date-t information, which are formed rationally. There is no altruism across generations.

Utility from housing services $H_{a,t}$, a continuous variable, is additively separable from consumption.[1] The utility function assumes a stylized life-cycle pattern of housing demand that is concentrated in middle age. Individuals have an exogenous need for housing services $\underline{H}$ at all stages of life, and receive utility $\Theta(H_{\mathrm{m},t} - \underline{H})$ from housing services in excess of this minimum when middle aged. The function $\Theta(H)$ is strictly increasing, strictly concave, and satisfies the Inada conditions ($\Theta'(H) > 0$, $\Theta''(H) < 0$, $\lim_{H\to 0} \Theta'(H) = \infty$, and $\lim_{H\to\infty} \Theta'(H) = 0$). Making these assumptions provides a simple reason for houses to be traded between the generations.

Endowments The economy is an endowment economy:

$$Y_t = (1 + g_t) Y_{t-1} \text{ where } g_t \in [\underline{g}, \bar{g}], \quad \text{and } L_t = L. \tag{2}$$

[1] Conditional on additive separability, logarithmic utility in consumption is required for the existence of a balanced growth path (see footnote 7 below). Risk aversion is essential to the model because of the central role of risk premiums.

Real GDP Y_t is an exogenous stochastic supply of non-storable composite goods, and $L_t = L$ is the exogenous fixed supply of housing, which has no maintenance costs and can be interpreted as 'land'. The growth rate g_t of real GDP can be any continuous random variable with bounded support between $\underline{g} > -1$ and $\bar{g} < \infty$. The inelastic supply of land L can be any finite number in excess of the minimum housing needs $3\underline{H}$ of the measure-three population.

The economy's endowments of goods and housing are distributed as follows:

$$y_{\mathrm{y},t} = 0, \quad y_{\mathrm{m},t} = Y_t, \quad y_{\mathrm{o},t} = 0, \quad \text{and } H_{\mathrm{y},t} = H_{\mathrm{o},t-1}, \tag{3}$$

where $y_{a,t}$ denotes the real non-financial income of individuals of age a at time t, which can be interpreted as labour income, assuming labour is supplied inelastically. Non-financial income is assumed to have a stylized life-cycle pattern that is concentrated in middle age. Each middle-aged person receives an equal share of real GDP (there is aggregate risk, but no idiosyncratic risk), while the young and the old receive nothing. Making these assumptions provides a simple reason for individuals to participate in financial markets, namely to borrow to buy houses and consume goods when young, and to save for retirement when old.

All young individuals receive an equal endowment of housing given by the amount held by the previous generation of old at the end of their lives, who pass on this housing to the young as an involuntary bequest.

Since the economy is an endowment economy, the model is silent about the transmission of a financial crisis to the real economy. Instead, the model is used to explore the causes of crises.

Money Money is used as a unit of account, but the economy is 'cash-less' in the sense of having no physical monetary tokens. The nominal price of a unit of goods is denoted by P_t and the nominal price of unit of housing by V_t.

Markets and market incompleteness All markets are perfectly competitive and all prices are fully flexible. The model therefore differs from the 'nominal rigidities' approach to generating real effects of monetary policy. Instead, the crucial friction is market incompleteness. Apart from housing, the only asset or liability is a one-period nominal bond.[2]

Let $B_{a,t}$ denote the quantity of nominal bonds purchased (or issued, if negative) by households of age a at the end of time period t. Each nominal bond is a riskless claim to one monetary unit at time $t+1$. Consequently, there is no option of voluntary default, and the natural borrowing limit is imposed to ensure there is no involuntary default in any state of the world.[3] At time t, the nominal price of a bond is Q_t.

[2]The one period maturity is without loss of generality given the overlapping generations structure.

[3]A collateral constraint where borrowing is limited to a given fraction of housing values is not imposed, but it turns out that borrowing is proportional to housing values in equilibrium even without a binding collateral constraint.

The bond market is assumed to operate without any explicit financial intermediation by banks. However, the overlapping generations structure of the economy naturally restricts new lending to be done only by the current generation of middle aged, who play the role of 'bankers', and whose lending is limited by their net worth.

A key missing market is for securities where payments depend on the realization of house prices (or the exogenous state of the world more generally). Here, house purchases cannot be financed by issuing such securities, but the presence of such a market subject to frictions is considered later in section 5. Utility from housing services can only be obtained through homeownership, and houses must be purchased and held between $t-1$ and t to enjoy utility flows at time t. Another missing market is a perfect rental market that would allow consumption of housing services to be separated from homeownership. The presence of a frictional rental market is considered later in section 5.[4]

The budget identities of the young, middle-aged, and old are respectively:

$$C_{\mathrm{y},t} + \frac{V_t H_{\mathrm{m},t+1}}{P_t} + \frac{Q_t B_{\mathrm{y},t}}{P_t} = \frac{V_t H_{\mathrm{y},t}}{P_t}; \tag{4a}$$

$$C_{\mathrm{m},t} + \frac{V_t H_{\mathrm{o},t+1}}{P_t} + \frac{Q_t B_{\mathrm{m},t}}{P_t} = y_{\mathrm{m},t} + \frac{V_t H_{\mathrm{m},t}}{P_t} + \frac{B_{\mathrm{y},t-1}}{P_t}; \tag{4b}$$

$$C_{\mathrm{o},t} = \frac{B_{\mathrm{m},t-1}}{P_t}. \tag{4c}$$

The young begin with an endowment of housing but no financial assets. Given their life-cycle income in (3), they must borrow ($B_{\mathrm{y},t} < 0$) to purchase the housing they will enjoy when middle aged and to consume. The middle aged earn income, repay their debts, save for retirement ($B_{\mathrm{m},t} > 0$), and then sell some housing to the next generation of young. While old, they consume the value of their financial assets, leaving no bequests ($B_{\mathrm{o},t} = 0$) other than the housing they hold at the end of their lives. Given non-negativity constraints on consumption and the minimum housing need $\underline{H}$, the natural borrowing limit that must hold in all states of the world is:

$$-\frac{B_{\mathrm{y},t-1}}{P_t} \leq y_{\mathrm{m},t} + \frac{V_t (H_{\mathrm{m},t} - \underline{H})}{P_t}. \tag{5}$$

Competitive equilibrium The equilibrium concept is standard competitive equilibrium. Given prices P_t, V_t, and Q_t, individual consumption, housing, and financial asset demands maximize lifetime expected utility (1) subject to budget identities (4) with endowments (2) and (3), the minimum housing need $\underline{H}$, and the natural borrowing limit (5). Given individuals' optimizing behaviour, prices

[4]Even in the absence of a rental market, it is conceivable that individuals might hold housing greater than their consumption of housing services purely as a 'bubble asset' with the only returns coming from capital gains or losses. This possibility is ruled out for now, but it is considered later as a special case of the analysis in section 5.

adjust to ensure that all markets clear:

$$C_{\mathrm{y},t} + C_{\mathrm{m},t} + C_{\mathrm{o},t} = Y_t; \tag{6a}$$

$$H_{\mathrm{y},t} + H_{\mathrm{m},t} + H_{\mathrm{o},t} = L_t; \tag{6b}$$

$$B_{\mathrm{y},t} + B_{\mathrm{m},t} = 0. \tag{6c}$$

Consumption of goods $C_{a,t}$ and holdings of housing $H_{a,t}$ across all individuals must sum to the economy's respective endowments Y_t and L_t. The net supply of nominal bonds is zero: the retirement savings of the middle aged must be matched by the borrowing of the young.

Lifetime utility (1) is a strictly concave function and the constraints in (4) are linear in the choice variables, so first-order conditions are necessary and sufficient for a global maximum. The natural borrowing limit is equivalent to non-negative consumption, and since logarithmic utility in consumption satisfies the Inada conditions, marginal utility is infinite if the natural borrowing limit binds. This means (5) does not need to be imposed as an additional constraint.[5] The equations characterizing the solution of the constrained utility-maximization problem are:

$$\frac{V_t}{P_t C_{\mathrm{y},t}} = \beta \mathbb{E}_t \left[\Theta'(H_{\mathrm{m},t+1} - \underline{H}) + \frac{V_{t+1}}{P_{t+1} C_{\mathrm{m},t+1}} \right], \quad \text{and } H_{\mathrm{o},t+1} = \underline{H}; \tag{7a}$$

$$\frac{Q_t}{P_t C_{\mathrm{y},t}} = \beta \mathbb{E}_t \left[\frac{1}{P_{t+1} C_{\mathrm{m},t+1}} \right], \quad \text{and} \quad \frac{Q_t}{P_t C_{\mathrm{m},t}} = \beta \mathbb{E}_t \left[\frac{1}{P_{t+1} C_{\mathrm{o},t+1}} \right]. \tag{7b}$$

The demand for housing is characterized by the equations in (7a), and borrowing and saving decisions by the Euler equations in (7b).

A competitive equilibrium is a set of prices P_t, V_t, Q_t, and quantities $C_{a,t}$, $H_{a,t}$, $B_{a,t}$ that satisfy (4), (5), (6), and (7), taking as given the exogenous variables $y_{a,t}$, Y_t, and L_t from (2) and (3). There are many possible competitive equilibria because prices P_t, V_t, and Q_t are in terms of a monetary unit of account. It will be seen that equilibria differ not only in nominal prices but also in real quantities because financial markets are restricted to nominal bonds.

Instrument of monetary policy The role of monetary policy is to provide a nominal anchor to the economy by pinning down one nominal price and the associated competitive equilibrium. The instrument of monetary policy is the nominal interest rate i_t paid by the central bank on reserves that are perfect substitutes for risk-free nominal bonds.[6] For both reserves and privately issued nominal

[5]Formally, imposing the constraint (5) would mean adding its Lagrangian multiplier to the marginal utility of consumption in the first-order conditions. The multiplier is only positive when the constraint binds, which occurs only when the marginal utility of consumption is infinite, so this would not change the solution of the first-order conditions.

[6]The nominal interest rate i_t is not subject to the zero lower bound here because there is no physical cash.

bonds to be willingly held, the nominal bond price Q_t must satisfy:

$$Q_t = \frac{1}{1+i_t}. \tag{8}$$

A crucial assumption is that the central bank does not have access to lump-sum taxation or transfers. For consistency with this limit on the central bank's instruments, it manages its balance sheet so that no gains or losses occur in equilibrium, matching the positive issuance of reserves exactly with purchases of risk-free nominal bonds. Reinterpreting $B_{a,t}$ as an individual's net holdings of nominal bonds plus reserves, the budget identities (4) remain valid. Equilibrium in the markets for nominal bonds and reserves requires the no-arbitrage condition (8) and equation (6c), which is unchanged from before because reserves are exactly matched by central-bank bond purchases.

2.2 Properties of a competitive equilibrium

Before considering the choice of monetary policy, some key features of any competitive equilibrium of the model are derived.

Asset returns The main focus of this paper is the returns on housing and bonds. The ex-post nominal returns between $t-1$ and t are denoted by R_t for nominal bonds and $\hat{R}_t$ for housing:

$$R_t = \frac{1}{Q_{t-1}} - 1, \quad \text{and } \hat{R}_t = \pi_t + \frac{Z_t}{V_{t-1}} \text{ with } \pi_t = \frac{V_t - V_{t-1}}{V_{t-1}} \text{ and } Z_t = \Theta'(H_{\text{m},t} - \underline{H})P_t C_{\text{m},t}. \tag{9}$$

The return on housing is the sum of capital gains π_t, that is, house-price inflation $(V_t - V_{t-1})/V_{t-1}$, and an imputed rental yield Z_t/V_{t-1}. The formula for the imputed rent Z_t above is such that the housing return satisfies the asset pricing condition $1 = \beta \mathbb{E}_t[(1+\hat{R}_{t+1})P_t C_{\text{y},t}/P_{t+1}C_{\text{m},t+1})]$ given (7a). The same equation holds for the nominal bond return, $1 = \beta \mathbb{E}_t[(1+R_{t+1})P_t C_{\text{y},t}/P_{t+1}C_{\text{m},t+1})]$, given (7b). The expected excess return ξ_t of housing over nominal bonds is defined by:

$$\xi_t \equiv \frac{\mathbb{E}_t\hat{R}_{t+1} - \mathbb{E}_t R_{t+1}}{1+\mathbb{E}_t R_{t+1}}, \quad \text{which satisfies } \xi_t = -\mathbb{C}\text{ov}_t\left[\hat{R}_{t+1}, \frac{\beta P_t C_{\text{y},t}}{P_{t+1}C_{\text{m},t+1}}\right]. \tag{10}$$

The variable ξ_t is given by the usual conditional covariance formula for a risk premium, noting that it is the covariance between the *nominal* return on housing and the *nominal* stochastic discount factor of buyers of housing because bonds are risk free in nominal terms, but not necessarily in real terms.

Real returns on bonds and housing in terms of consumption goods are denoted by r_t and $\hat{r}_t$ ex post, and expected real returns by ρ_t and $\hat{\rho}_t$ ex ante:

$$r_t \equiv \frac{1+R_t}{1+\gamma_t} - 1, \quad \hat{r}_t \equiv \frac{1+\hat{R}_t}{1+\gamma_t} - 1, \quad \text{with } \gamma_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \quad \rho_t \equiv \mathbb{E}_t r_{t+1}, \quad \hat{\rho}_t = \mathbb{E}_t \hat{r}_{t+1}. \tag{11}$$

The inflation rate for goods prices P_t is denoted by γ_t.

Ratios It is convenient to define variables representing some key ratios that remain stationary:

$$c_{a,t} \equiv \frac{C_{a,t}}{Y_t}, \quad h_t \equiv \frac{V_t(H_{\mathrm{m},t} - \underline{H})}{P_t y_{\mathrm{m},t}}, \quad d_t \equiv -\frac{Q_t B_{\mathrm{y},t}}{P_t Y_t}, \quad \text{and } b_t \equiv -\frac{B_{\mathrm{y},t-1}}{V_t(H_{\mathrm{m},t} - \underline{H})}. \tag{12}$$

Consumption of each age group relative to GDP is $c_{a,t}$, the ratio of the value of house purchases to income is h_t, the debt-to-GDP ratio is d_t, and b_t summarizes the balance sheets of the middle aged, being a ratio of their debt to the value of their housing assets in excess of minimum housing needs.

Equilibrium conditions Goods-market equilibrium (6a) in terms of $c_{a,t}$ from (12) is:

$$c_{\mathrm{y},t} + c_{\mathrm{m},t} + c_{\mathrm{o},t} = 1. \tag{13}$$

Using (2) and (3), the other equilibrium conditions (4), (6b), (6c), (7), and (8) are stated equivalently as the system of equations below in terms of the new variables from (9), (11), and (12):

$$b_t = \frac{(1+r_t)d_{t-1}}{(1+g_t)h_t}, \quad \text{and } \frac{1+\pi_t}{1+\gamma_t} = \frac{(1+g_t)h_t}{h_{t-1}}; \tag{14a}$$

$$c_{\mathrm{y},t} + h_t = d_t, \quad c_{\mathrm{m},t} + d_t = 1 + (1-b_t)h_t, \quad \text{and } c_{\mathrm{o},t} = b_t h_t; \tag{14b}$$

$$\frac{h_t}{c_{\mathrm{y},t}} = \beta\theta + \beta\mathbb{E}_t\left[\frac{h_{t+1}}{c_{\mathrm{m},t+1}}\right], \quad \text{where } \theta \equiv (L - 3\underline{H})\Theta'(L - 3\underline{H}); \tag{14c}$$

$$\frac{1}{c_{\mathrm{y},t}} = \beta\mathbb{E}_t\left[\frac{1+r_{t+1}}{(1+g_{t+1})c_{\mathrm{m},t+1}}\right], \quad \text{and } \frac{1}{c_{\mathrm{m},t}} = \beta\mathbb{E}_t\left[\frac{1+r_{t+1}}{(1+g_{t+1})c_{\mathrm{o},t+1}}\right]; \tag{14d}$$

$$1 + r_t = \frac{1+i_{t-1}}{1+\gamma_t}, \tag{14e}$$

and the natural borrowing limit (5) is equivalent to the following inequality:

$$1 + (1-b_t)h_t \geq 0. \tag{15}$$

The equations in (14a) are accounting identities that link together the newly defined variables. The equations in (14b) are the budget identities (4) of the young, middle aged, and old after imposing housing- and bond-market clearing. Equation (14c) specifies optimal housing choices (7a) in terms of a parameter θ (satisfying $0 < \theta < \infty$) that summarizes housing preferences $\Theta(\cdot)$, needs $\underline{H}$, and availability L. The equations in (14d) are the Euler equations (7b) for saving and borrowing behaviour. Last, (14e) is the Fisher equation defining the ex-post real return on nominal bonds.

There are a total of ten equations in (13) and (14). However, (13) is redundant by Walras' law, being implied by the budget identities (14b), leaving nine independent equations. Real GDP growth g_t is exogenous, and there are ten other variables including the monetary policy instrument i_t. This leaves one degree of freedom that is resolved by adding an equation to describe monetary policy. Different monetary policies imply different paths of nominal prices, and without loss of generality, let these be indexed by the state-contingent path of nominal house-price inflation π_t. The variable π_t can depend on the exogenous state of the world at each date t, where the state space includes both the fundamental g_t, real GDP growth, and also a non-fundamental exogenous random variable ψ_t independent of g_t and continuously distributed, which can be interpreted as a 'sunspot'.

Behaviour of lenders Total lending is measured by the debt-to-GDP ratio d_t. All lending is done by the current middle-aged generation.

Step 1 *The debt accounting identity in* (14a), *the budget identity of the old in* (14b), *and the bond Euler equation of the middle aged in* (14d) *imply* $d_t = \beta c_{\mathrm{m},t}$.

PROOF See appendix A.1. ■

The finding that lending is proportional to the consumption of the middle aged implies that they lend a constant fraction of their net worth. This proportionality is because income and substitution effects cancel out with logarithmic utility in consumption for those who will be retirees in the future.[7] Given net worth, lending does not depend on either the expected real return on bonds $\rho_t = \mathbb{E}_t r_{t+1}$ or uncertainty about the real return r_{t+1}. A fall in ρ_t or a rise in uncertainty about r_{t+1} both discourage lending (substitution effect), but both also make the middle-aged worse off, leading them to consume less and lend more (income effect).

Behaviour of borrowers The young choose consumption and house purchases funded through borrowing. This means they have a portfolio choice problem: how much debt to take on relative to the housing assets they acquire, as measured by the leverage ratio $\lambda_t = -Q_t B_{\mathrm{y},t}/V_t(H_{\mathrm{m},t+1} - \underline{H})$.

Step 2 *The budget identity of the young in* (14b) *and the housing Euler equation* (14c), *combined with lenders' optimal behaviour in Step 1, imply that* $\lambda_t = \lambda$, *where* $\lambda = (1-\delta/\beta)^{-1}$ *for a constant* δ *satisfying* $0 < \delta < \beta$ *that depends only on* β *and* θ. *The leverage ratio* λ *is decreasing in both patience* β *and the housing parameter* θ.

PROOF See appendix A.2. ■

[7] If utility from consumption of goods has the general isoelastic form $(C^{1-\varkappa}-1)/(1-\varkappa)$ instead of $\log C$ when $\varkappa = 1$, but remains additively separable from housing in (1), then the equilibrium conditions (14c) and (14d) are replaced by:

$$\frac{h_t}{c_{\mathrm{y},t}^{\varkappa}} = \beta\theta Y_t^{\varkappa-1} + \beta\mathbb{E}_t\left[\frac{(1+g_{t+1})^{1-\varkappa}h_{t+1}}{c_{\mathrm{m},t+1}^{\varkappa}}\right], \quad \frac{1}{c_{\mathrm{y},t}^{\varkappa}} = \beta\mathbb{E}_t\left[\frac{1+r_{t+1}}{(1+g_{t+1})^{\varkappa}c_{\mathrm{m},t+1}^{\varkappa}}\right], \quad \frac{1}{c_{\mathrm{m},t}^{\varkappa}} = \beta\mathbb{E}_t\left[\frac{1+r_{t+1}}{(1+g_{t+1})^{\varkappa}c_{\mathrm{o},t+1}^{\varkappa}}\right],$$

where $1/\varkappa$ is the elasticity of intertemporal substitution. The other equilibrium conditions in (14) remain unchanged. By solving these equations in the absence of any uncertainty and with non-zero steady-state real growth $g_t = g \neq 0$, the existence of a steady-state equilibrium for the house price-to-income ratio h_t, the debt-to-GDP ratio d_t, and the real interest rate r_t requires $\varkappa = 1$. There is no balanced growth path in the general case $\varkappa \neq 1$.

A constant leverage ratio means that spending on house purchases is proportional to lending. Since the supply of housing is inelastic, a corollary is that the debt-to-GDP ratio d_t determines the house price-income ratio h_t. Lending and house prices are perfectly correlated, but unlike with a collateral constraint, causation goes from lending to house prices. Given the budget identity of the young, a further corollary is that borrowed funds are divided in constant shares between those used to finance house purchases and those used to pay for consumption:

$$h_t = \frac{1}{\lambda} d_t, \quad \text{and } c_{\mathrm{y},t} = \left(1 - \frac{1}{\lambda}\right) d_t. \tag{16}$$

The constant leverage ratio is consistent with optimal behaviour for individuals because housing does not hedge consumption risk in this environment. Since lending is proportional to the consumption of the middle aged (Step 1), a constant leverage ratio makes future house prices h_{t+1} perfectly co-move with lending d_{t+1} and consumption $c_{\mathrm{m},t+1}$. Moreover, with logarithmic utility, the imputed rental value is also proportional to $c_{\mathrm{m},t+1}$ (see 9), which corresponds to the constant term θ in (14c). Consequently, the optimality condition for housing (14c) is satisfied when house prices h_t are proportional to consumption $c_{\mathrm{y},t}$.[8]

Since lending is proportional to the value of houses at the time of purchase, the ex-post level of debt relative to the value of houses at time t varies only with the nominal interest rate i_{t-1} and nominal house-price inflation π_t. Using (14a), the Fisher equation (14e), and (16):

$$b_t = \lambda \frac{1 + i_{t-1}}{1 + \pi_t}. \tag{17}$$

The realized balance sheet of the middle aged improves (lower b_t) with greater house-price inflation π_t or lower borrowing costs i_{t-1}.

Equilibrium in asset markets In addition to borrowers' optimal portfolio choice implying that borrowing and house purchases are proportional, their overall demand for loans can be characterized and combined with the behaviour of lenders to obtain equilibrium in asset markets. This equilibrium is conditional on the as-yet-undetermined path of nominal house-price inflation π_t.

Step 3 *Given any path of nominal house-price inflation, there exists a unique equilibrium of the housing and bond markets (combining Step 1, Step 2, and borrowers' bond Euler equation in 14d):*

$$i_t = \frac{1}{\beta \delta \mathbb{E}_t \left[(1 + \pi_{t+1})^{-1}\right]} - 1, \quad \textit{and } \mathbb{E}_t b_{t+1} = \frac{\lambda}{\beta \delta}. \tag{18}$$

[8] To see that the constant solution of (14c) is the unique solution, observe that if the share of loans going to house purchases were higher then $h_t / c_{\mathrm{y},t}$ would rise, which requires a higher expected value of $h_{t+1}/c_{\mathrm{m},t+1}$ and entails an even higher share of loans for house purchases in the future. Similarly, a lower share would entail an even lower share is expected in the future. Hence any solution of (14c) other than the constant leverage ratio would imply economically meaningless lending shares at some future date.

The coefficient δ is such that $\delta < 1$ and $0 < \beta\delta < 1$, and the combined coefficient $\beta\delta$ is increasing in patience β and decreasing in the housing parameter θ. The equilibrium returns on bonds and houses from (9) and the expected excess housing return (10) satisfy:

$$R_t = i_{t-1}, \quad \hat{R}_t = \frac{1+\pi_t}{\beta\delta} - 1, \quad \text{and } \xi_t = \mathbb{E}_t[1+\pi_{t+1}]\mathbb{E}_t[(1+\pi_{t+1})^{-1}] - 1. \tag{19}$$

The housing risk premium ξ_t is strictly positive for any non-degenerate probability distribution of house-price inflation π_{t+1} (conditional on date-t information) and increases with a mean-preserving spread of π_{t+1}; it would be zero if π_{t+1} were perfectly predictable at date t.

PROOF See appendix A.3. ■

The nominal return R_t on bonds is the predetermined interest rate i_{t-1}, which is the sense in which bonds are risk-free in nominal terms. The predetermined nominal payments on bonds are uncorrelated with any unpredictable changes in homeowners' consumption expenditure, which means that taking on nominal mortgage debt to buy houses exposes homeowners to greater financial risk. The expected return on housing must therefore be greater than the return on bonds for individuals to be willing to borrow money to buy houses.

Owing to the comovement between house prices and the consumption of homeowners, the probability distribution of nominal house-price inflation π_t is a sufficient statistic for the risk borne by borrowers (and the overall housing return, comprising imputed rents as well as capital gains or losses, is proportional to house-price inflation). The housing risk premium ξ_t can thus be interpreted as the compensation to borrowers for absorbing house-price risk while promising to make fixed nominal debt repayments. The risk premium increases with uncertainty about nominal house-price inflation, and is only zero in the special case where house-price inflation is perfectly predictable and houses become equivalent to bonds as financial assets.

Given a state-contingent path of nominal house-price inflation π_t, the equilibrium interest rate is (18).[9] Every sequence of probability distributions of π_t corresponds to a different competitive equilibrium. These different state-contingent paths of the nominal variable π_t can be interpreted as different monetary policy regimes because π_t cannot be determined without reference to monetary policy. Notice that nominal interest rates i_t and expected inflation $\mathbb{E}_t\pi_{t+1}$ generally do not move one-for-one across monetary policy regimes because of differences in housing risk premiums ξ_t:

$$\frac{1+i_t}{1+\mathbb{E}_t\pi_{t+1}} = \frac{1}{\beta\delta(1+\xi_t)}. \tag{20}$$

[9] In an infinitely-lived representative-agent version of the model with discount factor $\beta < 1$, logarithmic utility in consumption, and additively separable utility from housing, equation (18) would hold with $\delta = 1$. Thus, the coefficient $\delta < 1$, implying a higher interest rate all else equal, captures the effect of the overlapping generations structure where the young must borrow to buy houses. This effect would be present even if markets were complete (see section 5).

This logic points to a portfolio balance channel of monetary policy transmission through risk premiums even in the absence of any unconventional balance-sheet policies by the central bank.[10]

Financial conditions The balance sheets of the middle aged depend on interest rates i_t and house-price inflation π_t. The following measure ϕ_t of financial conditions is a sufficient statistic for the state of balance sheets, which determines how much lending can be done by the middle aged:

$$\phi_t = 1 - \beta\delta \frac{1+i_{t-1}}{1+\pi_t}, \quad \text{implying } \; b_t = \frac{\lambda}{\beta\delta}(1-\phi_t). \tag{21}$$

Financial conditions ϕ_t are increasing in asset prices and decreasing in interest rates. Equation (18) shows that interest rates settle in equilibrium at the point where the expected value of the balance-sheet variable b_t is $\lambda/\beta\delta$, meaning that the expected ratio of debt obligations to assets held is always constant. Every competitive equilibrium thus has expected financial conditions one period ahead equal to a constant, and ϕ_t is defined in such a way in (21) that this expected neutral value is zero without loss of generality. Formally, ϕ_t must be a martingale difference sequence in equilibrium, with realized values determined by the state-contingent path of nominal house-price inflation:

$$\mathbb{E}_{t-1}\phi_t = 0 \text{ with } \phi_t \in [-\infty, 1], \quad \text{where } \; \phi_t = -\frac{(1+\pi_t)^{-1} - \mathbb{E}_{t-1}\left[(1+\pi_t)^{-1}\right]}{\mathbb{E}_{t-1}\left[(1+\pi_t)^{-1}\right]}. \tag{22}$$

The choice of monetary policy regime can generate any probability distribution of financial conditions satisfying $\mathbb{E}_{t-1}\phi_t = 0$ with support $[-\infty, 1]$ as a competitive equilibrium. Knowing this probability distribution is sufficient to calculate the housing risk premium ξ_t, which is given by $\xi_t = \mathbb{E}_t[\phi_{t+1}/(1-\phi_{t+1})]$. This is the expectation of a convex function of financial conditions, which increases with any spread of financial conditions ϕ_{t+1} around its mean value of zero.

The relative price of housing The characterization of a competitive equilibrium is completed by imposing the remaining equilibrium condition, the budget identity of the middle aged, to determine the relative price of housing V_t/P_t and thus the house price-to-income ratio h_t. Step 1 shows that the lending done by the middle aged is proportional to their consumption. Together with their budget identity in (14b), lending as measured by the debt-to-GDP ratio d_t is proportional to the net worth of middle-aged lenders as a fraction of GDP, denoted by $n_t = (y_{\mathrm{m},t} + V_t(H_{\mathrm{m},t} - \underline{H})/P_t + B_{\mathrm{y},t-1}/P_t)/Y_t$:

$$d_t = \frac{\beta}{1+\beta} n_t, \quad \text{where } \; n_t = 1 + (1-b_t)h_t. \tag{23}$$

[10]It is analogous to the more-familiar idea that monetary policy affects the inflation risk premium between nominal and inflation-indexed bonds. Here, the same logic applies to the difference in return between houses and nominal bonds, though quantitatively the housing risk premium is likely to be larger than the inflation risk premium because house prices are more volatile than goods prices, both empirically and in the model here.

With house prices determined by lending (see 16), but also affecting lenders' net worth in (23) and thus feeding back into lending, the equilibrium value of h_t is the fixed point of the equation:

$$h_t = \frac{\beta}{\lambda(1+\beta)}\left(1+\left(1-\frac{\lambda}{\beta\delta}(1-\phi_t)\right)h_t\right). \tag{24}$$

The solution for h_t is a function of financial conditions ϕ_t and parameters. An improvement in financial conditions ϕ_t lowers b_t (see 21) and improves net worth n_t (see 23) by a fraction $\lambda/\beta\delta$ of h_t. A unit increase in net worth n_t raises h_t by $\beta/\lambda(1+\beta)$, so the direct effect of a unit increase in ϕ_t is an increase of h_t by $1/\delta(1+\beta)$ percent. However, there is also a feedback effect from h_t to n_t and then to d_t and h_t. Taking account of this effect in equation (24), the semi-elasticity of h_t with respect to financial conditions evaluated at the mean value of ϕ_t is:

$$\alpha = \left.\frac{\partial \log h_t}{\partial \phi_t}\right|_{\phi_t=\mathbb{E}\phi_t} = \frac{\frac{\beta}{\lambda(1+\beta)}\frac{\lambda}{\beta\delta}}{1-\frac{\beta}{\lambda(1+\beta)}\left(1-\frac{\lambda}{\beta\delta}\right)}.$$

The term α captures how much a shock to financial conditions working through the balance sheets of lenders (the 'financial accelerator') affects real lending and the relative price of housing.

Step 4 *For a given path of financial conditions ϕ_t consistent with (22), there is a unique competitive equilibrium with house price-income ratio $h_t = \alpha\beta\delta/\lambda(1-\alpha\phi_t)$, where $\alpha = 1/(1+\delta+\delta^2)$. The value of α lies between $1/3$ and 1, and is increasing in the housing parameter θ and the distance of the discount factor β from 1. Equilibrium net worth is $n_t = \alpha(1+\beta)\delta/(1-\alpha\phi_t)$ and the debt-to-GDP ratio is $d_t = \alpha\beta\delta/(1-\alpha\phi_t)$.*

PROOF See appendix A.4. ■

Using (23), the natural borrowing constraint (15) is equivalent to lenders never becoming bankrupt (non-negative net worth $n_t \geq 0$). This condition is always met in equilibrium, so (15) holds.

An improvement in financial conditions raises the equilibrium house price-income ratio, debt-to-GDP ratio, and the net worth of lenders, with the semi-elasticities all given by α. From the definition in (12), a larger value of h_t means a higher relative price of housing. The positive relationship between h_t and financial conditions ϕ_t implies that variation in nominal house prices reflects changes in the relative price of housing in same direction, with correspondingly smaller variation in nominal goods prices. Given nominal house-price inflation π_t and real GDP growth g_t, the rate of nominal goods-price inflation γ_t implied by Step 4 can be obtained from (11), (12), and (21):

$$\gamma_t = \frac{((1-\alpha)(1+\pi_t)+\alpha\beta\delta(1+i_{t-1}))(1+\pi_{t-1})}{((1-\alpha)(1+\pi_{t-1})+\alpha\beta\delta(1+i_{t-2}))(1+g_t)} - 1. \tag{25}$$

The larger is α, the less responsive are nominal goods prices to a shock to nominal house prices.

Note that there is an important asymmetry in how financial conditions affect h_t, which shows up in the link (25) between house-price inflation π_t and goods-price inflation γ_t. The ratio h_t is bounded above no matter how expansionary are financial conditions: extremely high realizations of house-price inflation are associated with extremely high realizations of goods-price inflation. However, there is in principle no limit to how small h_t could become for low realizations of financial conditions that drive lenders close to bankruptcy: a collapse in nominal house prices thus means a collapse in the relative price of housing and *not* a similar collapse in goods prices.

Real returns on assets Conditional on the monetary policy regime and the distribution of financial conditions ϕ_t it implies, the results of Step 3 and Step 4 characterize the equilibrium values of all prices, and from this the real returns on bonds and housing are calculated and the associated levels of consumption of each generation. The ex-post real returns on bonds r_{t+1} and housing $\hat{r}_{t+1}$ measured relative to the economy's growth rate g_{t+1} follow from (14a), (14e), Step 3, (21), and Step 4:

$$\frac{1+r_{t+1}}{1+g_{t+1}} = \frac{(1+i_t)h_{t+1}}{(1+\pi_{t+1})h_t} = \frac{(1-\alpha\phi_t)(1-\phi_{t+1})}{\beta\delta(1-\alpha\phi_{t+1})}; \tag{26a}$$

$$\frac{1+\hat{r}_{t+1}}{1+g_{t+1}} = \frac{h_{t+1}}{\beta\delta h_t} = \frac{(1-\alpha\phi_t)}{\beta\delta(1-\alpha\phi_{t+1})}. \tag{26b}$$

Result 1 *Take any distribution of financial conditions ϕ_{t+1} consistent with a competitive equilibrium (22). A weighted average of the expected real returns on housing and bonds depends only on expected real GDP growth and parameters, that is, $(1-\alpha)\mathbb{E}[(1+\hat{\rho}_t)/(1+\mathbb{E}_t g_{t+1})]+\alpha\mathbb{E}[(1+\rho_t)/(1+\mathbb{E}_t g_{t+1})] = 1/\beta\delta$. For any non-degenerate distribution of ϕ_{t+1}, the expected real return on housing is greater than the expected real real return on bonds, that is, $\mathbb{E}[(1+\hat{r}_{t+1})/(1+g_{t+1})] > 1/\beta\delta > \mathbb{E}[(1+r_{t+1})/(1+g_{t+1})]$ (these two terms have a weighted average equal to $1/\beta\delta$, and are identical if ϕ_{t+1} has a degenerate distribution). A spread of financial conditions ϕ_{t+1} increases the expected housing return $\mathbb{E}_t[(1+\hat{r}_{t+1})/(1+g_{t+1})]$ and lowers the expected bond return $\mathbb{E}_t[(1+r_{t+1})/(1+g_{t+1})]$.*

PROOF See appendix A.5. ■

These findings show that a monetary policy regime with a greater housing risk premium, and thus a larger gap between the expected returns on housing and bonds, must correspond in equilibrium to both a lower expected real bond return and a higher expected real housing return.

The behaviour of the relative price of housing is crucial in understanding these claims. If h_t were fixed, it can be seen from (26a) that the real bond return would be proportional to $(1+i_t)/(1+\pi_{t+1})$, the expected value of which is a constant $1/\beta\delta$ using (18). This reflects the result that whatever happens to nominal house price risk, the nominal interest rate adjusts so that the expected ratio of debt repayments to housing values remains stable ($\mathbb{E}_t b_{t+1} = \lambda/\beta\delta$). But the relative price h_t is not constant, and when nominal interest rates fall owing to a greater risk of extreme realizations of house price inflation, this does not translate into an equivalent increase in the risk of extreme realizations of goods-price inflation (see 25). Specifically, the nominal interest rate might be low and the housing risk premium high because leveraged homeowners are worried about a very large drop in house prices. However, if that event were to occur, it would not mean a huge drop in goods prices, which is what would be needed to avoid lower expected real bond returns.

To understand why a greater housing risk premium means both a fall in real bond returns *and* a rise in real housing returns, it is important to appreciate that monetary policy in the environment studied here has no special ability to raise or lower real returns across all assets. The overall real return on assets is determined by fundamentals (real GDP growth and preferences) because real returns ultimately translate into consumption. If everyone's real return were to increase or decrease, aggregate consumption demand would be inconsistent with the economy's supply of goods. Note (14a) and (14b) imply $c_{\mathrm{o},t} = ((1+r_t)/(1+g_t))d_{t-1}$; Step 1, (16), (26b), and the expression for λ from Step 2 imply $c_{\mathrm{m},t} = \delta\lambda((1+\hat{r}_t)/(1+g_t))h_{t-1}$ and $c_{\mathrm{y},t} = \delta^2\lambda((1+\hat{r}_t)/(1+g_t))h_{t-1}$. Given parameters, past outcomes, and current fundamentals g_t, goods-market equilibrium (13) requires that any increase in the real housing return $\hat{r}_t$ must be matched by a decrease in the real bond return r_t.

Each generation's consumption associated with real returns (26a) and (26b) is found by using the formulas above together with Step 4:

$$c_{\mathrm{y},t} = \frac{\alpha\delta^2}{1-\alpha\phi_t}, \quad c_{\mathrm{m},t} = \frac{\alpha\delta}{1-\alpha\phi_t}, \quad \text{and} \quad c_{\mathrm{o},t} = \frac{\alpha(1-\phi_t)}{1-\alpha\phi_t}. \tag{27}$$

Young and middle-aged consumption are strictly convex functions of financial conditions ϕ_t, while the consumption of the old is a strictly concave function. For any non-degenerate distribution of ϕ_t, Jensen's inequality implies $\mathbb{E}c_{\mathrm{y},t} > c^*_{\mathrm{y}}$, $\mathbb{E}c_{\mathrm{m},t} > c^*_{\mathrm{m}}$, and $\mathbb{E}c_{\mathrm{o},t} < c^*_{\mathrm{o}}$, where $c^*_{\mathrm{y}} = \alpha\delta^2$, $c^*_{\mathrm{m}} = \alpha\delta$, and $c^*_{\mathrm{o}} = \alpha$ are the values of these consumption ratios if there is no uncertainty about ϕ_t.

Long-run real effects of monetary policy The results above show that different monetary policy regimes imply different average age-specific levels of consumption. This reflects the fact that there is a range of housing risk premiums across policy regimes and thus different average levels of real bond and housing returns. Low real bond returns hit the pension savings of the old, while high housing returns boost the portfolios of the middle aged. Since the young depend on loans from the middle aged, their consumption moves in the same direction.

A corollary of Result 1 is that there is no meaningful notion in the incomplete-markets economy of a 'natural rate of interest' that is invariant to monetary policy.[11] The market-clearing real interest rate (and its long-run average) is dependent on the monetary policy regime in place.

[11] Since there is no nominal rigidity in the economy considered here, the concept of the natural rate of interest coincides with the actual real interest rate.

3 Financial crises and monetary policy

Having set up the model and explained the channel through which monetary policy has real effects on the economy, this section develops the key claim that the interaction between incomplete markets and the political economy of monetary policy creates the conditions for financial crises to occur.

3.1 The political economy of monetary policy

Distributional effects of monetary policy The first step is to understand how the choice of monetary policy regime affects different groups in the economy. Individuals are borrowers and savers at different points in their lifetimes, and so may benefit from a policy regime at some ages and lose out at others. To be precise about this, define continuation expected utilities $U_{\mathrm{a},t}$ for each age group a:

$$U_{\mathrm{y},t} = U_t, \quad U_{\mathrm{m},t} = \log C_{\mathrm{m},t} + \Theta(H_{\mathrm{m},t} - \underline{H}) + \beta \mathbb{E}_t \log C_{\mathrm{o},t+1}, \quad \text{and} \quad U_{\mathrm{o},t} = \log C_{\mathrm{o},t}, \tag{28}$$

which include all current and expected future terms from the lifetime utility function (1).

Result 2 *For all competitive equilibria, individuals fall into one of two groups in a particular period of life: a group labelled 'savers' who lose from lower real bond returns in that period, or a group labelled 'borrowers' who gain from lower real interest rates on bonds and higher real returns on housing, even though this is associated with greater uncertainty about financial conditions. Formally, the expected continuation utilities* (28) *satisfy* $\mathbb{E}_{t-1} U_{\mathrm{y},t} = w_{\mathrm{B},t} + \beta \mathbb{E}_t w_{\mathrm{B},t+1} + \beta^2 \mathbb{E}_t w_{\mathrm{S},t+2} + \textit{t.i.p.}$, $\mathbb{E}_{t-1} U_{\mathrm{m},t} = w_{\mathrm{B},t} + \beta \mathbb{E}_t w_{\mathrm{S},t+1} + \textit{t.i.p.}$, *and* $\mathbb{E}_{t-1} U_{\mathrm{o},t} = w_{\mathrm{S},t} + \textit{t.i.p.}$, *where* $w_{\mathrm{S},t} = \mathbb{E}_{t-1}[\log(1-\phi_t) - \log(1-\alpha\phi_t)]$ *and* $w_{\mathrm{B},t} = -\mathbb{E}_{t-1}[\log(1-\alpha\phi_t)]$ *denote respectively the welfare of the groups of savers* (S) *and borrowers* (B)*, and t.i.p. refers to terms independent of the monetary policy regime. Distributions of* ϕ_t *with lower expected real bond returns imply lower* $w_{\mathrm{S},t}$ *and higher* $w_{\mathrm{B},t}$.

PROOF See appendix A.6. ■

As shown in Result 1, a change to the monetary policy regime can lower the expected real return between $t-1$ and t on bonds and raise the expected real return on housing if it is associated with a spread of financial conditions ϕ_t. It is already known (see 27) that this lowers the expected consumption of the old. It is also bad for the old because they are risk averse and the spread of financial conditions increases uncertainty about asset returns.

For the middle-aged, lower expected real bond returns and higher expected real housing returns are good for their portfolios on average, raising their average level of consumption. However, because of the associated spread of financial conditions, their leveraged position in housing exposes them to greater risk coming from shocks to house prices, which they dislike owing to risk aversion. In utility terms, it turns out that the level effect on average returns always dominates the risk effect in the environment considered here.[12] The middle aged therefore favour monetary policy regimes with cheap credit and high average housing returns, even though they dislike risk and know such policy regimes must feature greater risk in equilibrium.

Since the young are dependent on loans from the current middle-aged generation (the ‘bankers’ in the economy), their welfare depends on the financial health of that generation. Thus, the interests of the young and middle aged are aligned and both favour monetary policy regimes with low bond returns and high housing returns. This group is referred to as ‘borrowers’ as a shorthand, even though it includes those taking out new loans (the young) as well as those repaying past loans (the middle aged). Opposed to them are the old, referred to as ‘savers’, who would like high bond returns.

Social welfare function A key idea of this paper is that monetary policy is not arbitrary — to be chosen, a monetary policy regime must be in the interests of some group of individuals. The social welfare function W_t aggregates individuals’ preferences over outcomes from date t onwards:

$$W_t = \mathbb{E}_{t-1}\left[\Omega_{t-2|t}U_{\text{o},t} + \Omega_{t-1|t}U_{\text{m},t} + \sum_{\ell=0}^{\infty}\Omega_{t+\ell|t}U_{\text{y},t+\ell}\right], \tag{29}$$

where $\Omega_{t+\ell|t}$ denotes the weight assigned from date t to the expected continuation utility (28) of individuals born at date $t+\ell$ ($\ell = -2,-1,0,1,\ldots$). The social welfare function is based on ex-ante expectations of utility, hence the presence of the conditional expectation operator $\mathbb{E}_{t-1}[\cdot]$. The weights $\Omega_{t+\ell|t}$ must be positive and non-stochastic conditional on information known at $t-1$, ensuring all individuals receive some weight and are treated as autonomous during their lifetimes.

Ramsey problem The monetary policy regime is chosen with social welfare as the objective, conditional on some weights assigned to different generations. Monetary policy is the solution of a Ramsey problem in that the central bank can only choose allocations that are competitive equilibria of the economy for some policy regime, that is, those satisfying the equilibrium conditions (14) for some path of nominal house-price inflation $\{\pi_t\}$. Formally, state-contingent π_t are chosen from some date s for k time periods to make social welfare W_s (29) as high as possible, where k denotes the length of time for which there is commitment to these choices made at date s:

$$\sup_{\{\pi_t\}_{t=s}^{s+k-1}} W_s \;\text{ subject to }\; (14) \;\text{ for all } t, \quad \text{and } \{\pi_t\}_{t=s+k}^{\infty} \text{ taken as given.} \tag{30}$$

The constrained optimization problem is stated in terms of finding the supremum of social welfare rather than the maximum because the central bank is choosing among nominal prices that are competitive equilibria, which cannot be restricted to a compact set. The state-contingent path of inflation

[12] This might be surprising because the difference in housing and bond returns is simply the risk premium that compensates buyers of housing for greater risk at the margin. However, those real returns in general equilibrium depend on the financial accelerator channel analysed in Step 4, which gives rise to pecuniary externalities.

π_t beyond the period of commitment $t = s, \ldots, s+k-1$ is taken as given (except in the case of perpetual commitment, $k = \infty$). This is a Markovian restriction that means the future path is assumed to depend on the exogenous state of the world rather than on histories of endogenous variables.

Past values of π_t are predetermined in the problem (30), but monetary policy must respect equilibrium conditions at all dates t, even those outside the period of commitment. This means that unlike 'discretionary policymaking', it is not allowed here to choose policy regimes ex post that violate equilibrium conditions related to past expectations of current policy. There is always some minimum degree of commitment assumed in (30), though it may not be perpetual commitment.

Step 5 *Existence of a solution of the Ramsey problem requires $\sum_{t=s}^{\infty} \Omega_{t|s} < \infty$ and is guaranteed by $\sum_{t=s}^{\infty} t\Omega_{t|s} < \infty$. The solution can be found by solving the following simpler problem to determine the state-contingent value of financial conditions ϕ_t at each date $t = s, \ldots, s+k-1$:*

$$\sup_{\phi_t} \left((1-\omega_{t|s}) w_{\mathrm{B},t} + \omega_{t|s} w_{\mathrm{S},t} \right) \text{ subject to } \mathbb{E}_{t-1}\phi_t = 0 \text{ and } \phi_t \in [-\infty, 1], \tag{31}$$

where $\omega_{t|s} = \beta^{\min\{2,t-s\}}\Omega_{t-2|s}/(\Omega_{t|s} + \beta^{\min\{1,t-s\}}\Omega_{t-1|s} + \beta^{\min\{2,t-s\}}\Omega_{t-2|s})$ is a number strictly between 0 and 1 that gives the relative weight on the welfare of savers at date t. The weighted average of borrowers' and savers' welfare $w_t = (1-\omega_{t|s})w_{\mathrm{B},t} + \omega_{t|s}w_{\mathrm{S},t} = \mathbb{E}_{t-1}\left[\omega_{t|s}\log(1-\phi_t) - \log(1-\alpha\phi_t)\right]$ depends only on the probability distribution of financial conditions ϕ_t. Social welfare is given by $W_s = \sum_{t=s}^{\infty} \Delta_{t|s}\mathbb{E}_{s-1}w_t + \text{t.i.p.}$, where $\Delta_{t|s} = \Omega_{t|s} + \beta^{\min\{1,t-s\}}\Omega_{t-1|s} + \beta^{\min\{2,t-s\}}\Omega_{t-2|s}$.

PROOF See appendix A.7. ■

Solving the Ramsey problem turns out to be simpler than it might first appear. The solution can be broken down into a series of independent problems for the monetary policy regime in place at each date t — even when commitment extends over several periods. Each date's policy regime is selected with a weighted average of the expected utilities $w_{\mathrm{B},t}$ and $w_{\mathrm{S},t}$ of the groups of borrowers and savers at that date as the objective. This simplification is possible because each period's expected utilities depend solely on the probability distribution of financial conditions ϕ_t at that date (see Result 2), and the only requirement for financial conditions to be a competitive equilibrium is $\mathbb{E}_{t-1}\phi_t = 0$. There are no intertemporal constraints linking the values of financial conditions at different dates, and thus no difference between a multi-period commitment and a series of one-period commitments, unless the relative weight on savers $\omega_{t|s}$ changes with the starting dates s of the commitments.

Democracy The political pressure that shapes the monetary policy regime takes the form of 'democracy', interpreted as assigning an equal weight to all individuals currently alive, and with no commitment to the policy regime in place in the future. Formally:

$$\Omega_{t|t-2} = \Omega_{t|t-1} = \Omega_{t|t} \text{ with any } \Omega_{t+\ell|t} > 0 \text{ such that } \sum_{\ell=0}^{\infty} \ell\Omega_{t+\ell|t} < \infty, \quad \text{and } k = 1. \tag{32}$$

This implies $\omega_{t|t} = 1/3$ for all t irrespective of the weights assigned to unborn future generations, which can be chosen to be any positive numbers such that the social welfare is well defined.[13]

The baseline assumption makes the unweighted mean welfare of all individuals currently alive the objective of the monetary policy regime. It is also possible to consider the case where the welfare of the median individual is the objective, an application of the median voter theorem which has stronger political-economy foundations. That extension is taken up in section 5.

3.2 Financial crises

The solution of the Ramsey problem with a democratic social welfare function turns out to have a very surprising feature. The monetary policy regime adopted leads financial conditions ϕ_t to have a two-point discrete probability distribution. This means there are endogenously two regimes for financial conditions — boom and bust — with stochastic transitions between the two over time.

Regime switching Since $\mathbb{E}_{t-1}\phi_t = 0$ and $\phi_t \in [-\infty, 1]$ must hold in any competitive equilibrium, the general form of a two-point discrete distribution of financial conditions is:

$$\phi_t = \begin{cases} \bar{\phi}_t & \text{with probability } 1-\varepsilon_t \\ \underline{\phi}_t & \text{with probability } \varepsilon_t \end{cases}, \quad \text{where } \bar{\phi}_t = \frac{x_t}{1+x_t} \text{ and } \underline{\phi}_t = -\frac{(1-\varepsilon_t)x_t}{\varepsilon_t(1+x_t)}. \tag{33}$$

The terms $x_t \geq 0$ and $0 < \varepsilon_t < 1$, which are known conditional on information available at date $t-1$, describe the probability distribution of financial conditions at date t. Which of the two values of ϕ_t is drawn depends on the history of the exogenous state of the world $\{g_t, \psi_t, g_{t-1}, \psi_{t-1}, \ldots\}$ in such a way that the event $\phi_t = \underline{\phi}_t$ has probability ε_t conditional on what is known at date $t-1$.

For any $x_t > 0$, there are two possible realizations of ϕ_t: a boom $\bar{\phi}_t > 0$ (probability $1-\varepsilon_t$) with a high realization of house-price inflation π_t relative to the interest rate i_{t-1}, and a bust $\underline{\phi}_t < 0$ (probability ε_t) with a low realization of π_t relative to i_{t-1} (see 21). The term x_t measures how expansionary are financial conditions in the boom relative to average, and thus how much the two regimes differ (in the special case $x_t = 0$ there is a single regime with stable financial conditions). As shown in Step 4, these fluctuations in financial conditions have real consequences. In a boom, the debt-to-GDP ratio is $\bar{d}_t = \alpha\beta\delta(1+x_t)/(1+(1-\alpha)x_t)$ and the house price-income ratio is $\bar{h}_t = \alpha\beta\delta(1+x_t)/\lambda(1+(1-\alpha)x_t)$, both of which are larger than their corresponding values $\underline{d}_t$ and $\underline{h}_t$ in a bust. Both $\bar{d}_t$ and $\bar{h}_t$ increase with the magnitude x_t of the boom.

Rare events Not only does the Ramsey problem solution endogenously generate two regimes, it also features a stark asymmetry: the bust regime is a rare event with a small but positive probability

[13] It is not possible to give all current and future individuals equal weight and solve the Ramsey problem with full commitment because the social welfare function would not be well defined.

ε_t. For a given level of expansion x_t in the boom, the low probability of a bust means that when one does happen, the consequences are severe. Since ϕ_t must have mean zero in equilibrium, a smaller value of ε_t requires a more negative $\underline{\phi}_t$ in the bust regime (see 33). As ε_t shrinks towards zero, a bust features an increasingly large drop in house-price inflation π_t, the house price-income ratio h_t, and the debt-to-GDP ratio d_t. The bust regime can thus be interpreted as a financial crisis, and the other normal regime (probability near one) with expansionary financial conditions as a long credit boom.

Result 3 *The solution of the Ramsey problem* (30) *with democratic social welfare* (32) *has rare but severe financial crises occurring in equilibrium. Formally, financial conditions ϕ_t has a two-point probability distribution* (33) *with $x_t = x = (3\alpha - 1)/(1-\alpha) > 0$ and ε_t small but positive (the supremum is approached as ε_t becomes arbitrarily small), with any mapping from the exogenous state of the world to the realization of ϕ_t consistent with probability ε_t of a financial crisis.*

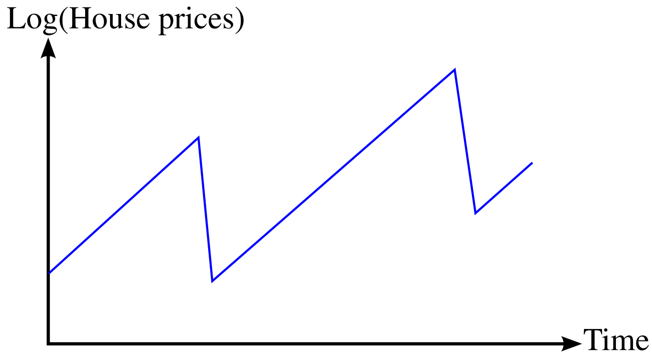
PROOF See appendix A.8. ■

The equilibrium behaviour of house-price inflation implied by this result is seen from $1+\pi_t = (1+\mathbb{E}_{t-1}\pi_t)/(1+\xi_{t-1})(1-\phi_t)$ using (18) and (21). When ε_t is small, the probability distribution (33) implies $\xi_{t-1} = x_t$, and since the equilibrium value of x_t is constant, it follows that $1+\pi_t = (1+\mathbb{E}_{t-1}\pi_t)/(1+x)(1-\phi_t)$. The small but positive value of ε_t together with the positive constant x in (33) means the equilibrium features long periods of tranquillity with steady growth of house prices in line with expectations that are very occasionally interrupted by sudden reversals where house prices drop sharply. A typical realization of the path of nominal house-price inflation for small but positive ε_t and a constant average rate of nominal house-price inflation is shown in Figure 1.

Interestingly, the equilibrium has a probability distribution with the type of 'rare events' that have been emphasized in the literature on asset pricing (see, for example, Barro, 2006). Unlike many models studying imperfections in financial markets (Bernanke, Gertler and Gilchrist, 1999, Kiyotaki and Moore, 1997), the economy undergoes occasional large discrete jumps rather than continuous small fluctuations in the neighbourhood of a steady state. It is important to emphasize that the probability distribution (33) is endogenously generated by the interaction of incomplete markets and the political economy of monetary policy. Its features do not derive from the probability distributions of the exogenous variables g_t and ψ_t.

Fat tails, negative skewness, and excess kurtosis The monetary policy regime solving the Ramsey problem gives rise to a unique equilibrium for the probability distribution of financial conditions ϕ_t with long periods of expansion punctuated by rare and dramatic collapses in asset prices. The probability distribution of financial conditions in (33) has a 'fat tail' in the sense that second- and higher-order moments of the distribution of ϕ_t such as $\mathbb{E}[\phi_t^2]$ are unbounded as ε_t becomes small. The distribution of ϕ_t has negative skewness as $\mathbb{E}[\phi_t^3]/\mathbb{E}[\phi_t^2]^{3/2}$ is negative for small ε_t, and also ex-

Figure 1: *Equilibrium with financial crises*



Notes: The graph depicts a typical realization of the path of (log) nominal house prices given the equilibrium characterized in Result 3, assuming the average rate of nominal house-price inflation is constant.

cess kurtosis as $\mathbb{E}[\phi_t^4]/\mathbb{E}[\phi_t^2]^2$ exceeds 3 for small ε_t.[14] This negative skewness and excess kurtosis are also features of the associated distribution of house-price inflation π_t.

Bubbles and sunspots Informally, the episodes of boom and bust in house prices could be described as 'bubbles', though logically they are completely different from the usual notion of rational bubbles. Unlike models with rational bubbles, there is no multiplicity of equilibria encompassing cases with no bubbles and cases of bubbles with various stochastic properties — here, the monetary policy regime solving the Ramsey problem has a unique probability distribution of asset prices. Also unlike rational bubbles, here, new bubbles can form in equilibrium after past bubbles have burst, and there is no requirement for the economy to be dynamically inefficient to sustain bubbles.[15]

Although the probability distribution of financial conditions is uniquely determined, there is one sense in which there is a role for multiple equilibria: the exogenous trigger for switching between the credit boom and financial crisis regimes in (33) is not uniquely determined. The solution of the Ramsey problem can have regime shifts caused by any exogenous variable as long as the implied probability distribution of financial conditions is consistent with Result 3. As a result, credit booms could turn to financial crises either because of shocks to the economy's fundamentals g_t (real GDP growth) or because of sunspots ψ_t. Intuitively, monetary policy is creating the conditions for financial crises to occur with some positive probability, but this leaves open precisely which circumstances will lead to the onset of a crisis.

[14] For given $\varepsilon_t > 0$, these moments are $\mathbb{E}[\phi_t^2] = (1/\varepsilon_t - 1)(x_t/(1+x_t))^2$, $\mathbb{E}[\phi_t^3]/\mathbb{E}[\phi_t^2]^{3/2} = -(1-2\varepsilon_t)\sqrt{1-\varepsilon_t}/\sqrt{\varepsilon_t}$, and $\mathbb{E}[\phi_t^4]/\mathbb{E}[\phi_t^2]^2 = (1-3\varepsilon_t(1-\varepsilon_t))/\varepsilon_t(1-\varepsilon_t)$.

[15] The limited role of dynamic efficiency or inefficiency is discussed further in section 4.

The role of monetary policy What is the central bank doing to generate an equilibrium where there are booms and busts in asset prices? Given that (33) implies $\xi_{t-1} = x_t$ for small ε_t, the observed path of nominal interest rates associated with the equilibrium (33) follows from equation (18):

$$i_t = \frac{1+\mathbb{E}\pi_{t+1}}{\beta\delta(1+x_{t+1})} - 1. \tag{34}$$

The central bank would be seen to adjust interest rates i_t one-for-one in response to predictable changes in nominal asset prices $\mathbb{E}_t\pi_{t+1}$, but in no sense would monetary policy be seen actively to 'pop' bubbles. The only other variable appearing in (34) is x_{t+1}, measuring how expansionary financial conditions are when the economy is in the credit boom state. According to Result 3, the Ramsey solution has x_{t+1} constant over time at x. The effect of the positive value of x in (34) is that the central bank sets a lower nominal interest rate relative to expected asset-price inflation $\mathbb{E}_t\pi_{t+1}$. It is this *systematic* low level of interest rates that brings about the equilibrium (33) with financial crises, not any cyclical variation of monetary policy, nor the occurrence of monetary policy shocks.

Pecuniary externalities and risk taking by policymakers While (34) shows *what* the central bank does, there remains the question of *why*. Since all individuals in the economy are risk averse, why does the central bank act in a way that increases financial risk, especially tail risk? The answer lies in the effect of the monetary policy (34) on asset returns. The equilibrium probability distribution of financial conditions (33) with small ε_t has housing risk premium $\xi_{t-1} = x_t$, which means a wedge between the expected returns on bonds and housing. Using (19), the small but positive ε_t and constant x_t implied by Result 3 lead to the following expected real bond and housing returns:

$$\mathbb{E}\left[\frac{1+r_t}{1+g_t}\right] = \frac{1}{(1+(1-\alpha)x)\beta\delta}, \quad \text{and } \mathbb{E}\left[\frac{1+\hat{r}_t}{1+g_t}\right] = \frac{1+x}{(1+(1-\alpha)x)\beta\delta}. \tag{35}$$

A positive value of x implies a lower expected bond return and a higher expected housing return, which both raise the expected consumption of those who borrow to buy houses. This is what explains why borrowers can gain from loose monetary policy, even though it exposes them to greater financial risk. Moreover, Result 2 shows that for utility, the effects on expected asset returns dominate the effects on risk. Consequently, policymakers face pressure to pursue risky policies even though individuals are prudent and risk averse in their own behaviour. With lending to the young dependent on the financial health of the middle-aged generation, this creates a powerful constituency in favour of maintaining cheap credit, in spite of its foreseeable negative effects on financial stability.

While the difference in returns between houses and bonds is equal in equilibrium to a risk premium that compensates borrowers for the risk of buying houses, the discussion of Result 1 explains why the risk premium has an associated general-equilibrium effect that depresses the expected real return on bonds, and which is not internalized by individuals. That pecuniary externality works through the relative price of housing, which falls sharply with nominal house prices because nominal debt obligations are predetermined and the lending that supports the relative price of housing depends on the net worth of homeowners. This is the financial accelerator mechanism shown in Step 4. The pecuniary externality has its strongest effect on the relative price of housing for extreme left-tail realizations of nominal house prices, explaining why monetary policy is conducted in a way that leaves risk in the left tail of the asset-price distribution while avoiding fluctuations during normal times. Note also the value of x in Result 3 is increasing in the strength α of the financial accelerator.

Numerical methods While there is no reason to think that a similar logic would not apply in more complicated models, it is likely there are serious computational challenges in finding equilibria using numerical methods. Here, the equilibrium can be derived analytically, but it is interesting to note that it *cannot* be found by using perturbation methods, no matter what order of Taylor polynomial is used. The reason is that equilibrium consumption $c_{a,t}$ from (27) and the implied level of utility $\log c_{a,t}$ do not converge to their Taylor series expansions in financial conditions ϕ_t over the support of the equilibrium probability distribution of ϕ_t. Intuitively, financial crises lie outside the radius of convergence of the Taylor series around the model's non-stochastic steady state.[16] While other numerical methods might in principle work, the computational challenges would be non-trivial.[17]

3.3 The implementation of monetary policy

Equation (34) completely characterizes the model's implications for the observable actions of the central bank. However, since monetary policy is conducted by setting the nominal interest rate, it might be asked how it would ensure the solution of the Ramsey problem (33) is the unique equilibrium of the economy. The issue is analogous to the one that arises in New Keynesian models when monetary policy is modelled as a Taylor rule. While this paper focuses on what is the solution of the Ramsey problem rather than how it is implemented, it is nonetheless possible to address the question of implementation by using an interest-rate feedback rule, similar to the typical approach in New Keynesian models.

Consider the following interest-rate feedback rule that responds to expected house-price inflation $\mathbb{E}_t \pi_{t+1}$ and current financial conditions ϕ_t:

$$i_t = \frac{(1+\mathbb{E}_t \pi_{t+1})\zeta\left(\max\left\{\phi_t - \frac{x}{1+x}, 0\right\}\right)}{\beta\delta\left(1+\frac{(1-\varepsilon)x^2}{x+\varepsilon}\right)} - 1, \tag{36}$$

[16] Since $\mathbb{E}_{t-1}\phi_t = 0$, the non-stochastic steady state of the model must be $\phi_t = 0$. From equation (27), $c_{a,t}$ and $\log c_{a,t}$ are infinitely differentiable functions of financial conditions for all $\phi_t \in (-\infty, 1)$, but their Taylor series at 0 do not converge for $\phi_t < -1/\alpha$. For small ε_t, the negative value of ϕ_t from (33) lies in this range, and therefore Taylor polynomials of whatever order will fail to approximate the equilibrium of the economy.

[17] The utility-maximization problems of individuals require solving a portfolio choice problem because of the presence of housing as an asset. On top of this, finding the equilibrium requires solving a Ramsey problem where the social welfare function is not globally concave. Finally, the extent of aggregate risk is non-negligible because the equilibrium features rare events with large aggregate fluctuations.

where $x \geq 0$ and $0 < \varepsilon < 1$ are constants and $\zeta(\cdot)$ is a strictly increasing function with $\zeta(0) = 1$. Different from (34), equation (36) has the central bank raise the nominal interest rate i_t if financial conditions ϕ_t exceed their value in the credit-boom state of (33). If financial conditions are anywhere below that value, i_t responds only to expected house-price inflation $\mathbb{E}_t \pi_{t+1}$.

Step 6 *As ε becomes small,* (33) *is the unique i.i.d. probability distribution of financial conditions ϕ_t consistent with the equilibrium conditions* (22) *and the interest-rate rule* (36).

PROOF See appendix A.9. ■

The interest-rate rule (36) thus ensures the Ramsey solution (33) is the unique i.i.d. equilibrium for financial conditions ϕ_t.[18]

4 Policy implications

The previous section showed that financial crises occur in equilibrium in an incomplete-markets economy where the monetary policy regime is chosen to maximize a democratic unweighted social welfare function. But for this to be a theory of financial crises caused by loose monetary policy, it must be shown that such crises would not occur if monetary policy were conducted differently, in particular, if monetary policy were systematically tighter. Furthermore, since financial crises occur even though the policy regime maximizes a social welfare function, it might be wondered in what sense such crises are actually a bad thing. To address that point, it will be seen that financial crises never result in a first-best (Pareto efficient) allocation of resources, while having sufficiently tight monetary policy to rule out crises does achieve a first-best allocation.

4.1 Inefficiency and welfare costs of financial instability

The social planner benchmark Before analysing alternative monetary policies, it is helpful to consider a hypothetical social planner. The social planner can directly specify a state-contingent allocation of consumption and housing across all individuals subject only to the economy's resource constraints. The social planner effectively has a complete set of policy instruments including lump-sum taxes and transfers, unlike a central banker who operates by setting the nominal interest rate in a market economy.

The resource constraints are that total consumption of goods $C_{\mathrm{y},t} + C_{\mathrm{m},t} + C_{\mathrm{o},t}$ and housing $H_{\mathrm{y},t} + H_{\mathrm{m},t} + H_{\mathrm{o},t}$ must respectively add up to the economy's supplies of goods and housing Y_t and L_t, as given by the endowments in (2). Mathematically, these are the same as the market-clearing

[18] Financial conditions ϕ_t must always be a martingale difference sequence ($\mathbb{E}_{t-1}\phi_t = 0$) in equilibrium (see 22), but strengthening that requirement to i.i.d. in Step 6 rules out equilibria with conditional heteroscedasticity, or other conditional higher moments. Whether such equilibria exist subject to (36) is a question left for future work.

conditions for goods and housing from (6a) and (6b). An allocation of resources from some date s onwards is Pareto efficient if subject only to resource constraints, no individual can be given a higher ex-ante utility (meaning expected utility conditional on date $s-1$ information) without some other individual obtaining a lower ex-ante utility. The allocations satisfying this condition are first best.

Step 7 *Any Pareto-efficient allocation of consumption and housing (from s onwards) must feature $H_{\mathrm{y},t} = \underline{H}$, $H_{\mathrm{m},t} = L - 2\underline{H}$, and $H_{\mathrm{o},t} = \underline{H}$, and have consumption/GDP ratios $c_{a,t}$ that satisfy feasibility (13), risk sharing $c_{\mathrm{y},t}/\mathbb{E}_{s-1}c_{\mathrm{y},t} = c_{\mathrm{m},t}/\mathbb{E}_{s-1}c_{\mathrm{m},t} = c_{\mathrm{o},t}/\mathbb{E}_{s-1}c_{\mathrm{o},t}$ for all $t \geq s$, consumption smoothing $\mathbb{E}_t c_{\mathrm{m},t+1}/c_{\mathrm{y},t} = \mathbb{E}_t c_{\mathrm{o},t+1}/c_{\mathrm{m},t}$ for all $t \geq s$, and dynamic efficiency $\liminf_{t\to\infty} \mathbb{E}_t[c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1}] \leq 1/\beta$ and $\liminf_{t\to\infty} \mathbb{E}_t[c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}] \leq 1/\beta$.*

Conversely, any allocation satisfying these requirements (with $\limsup_{t\to\infty}$ and strict inequality in the dynamic efficiency condition) is Pareto efficient and maximizes the social welfare function W_s in (29) subject to resource constraints for a sequence of weights $\{\Omega_{t|s}\}$ where $\sup W_s$ exists.

PROOF See appendix A.10. ■

The requirements for Pareto efficiency are standard. Individuals must share consumption risk in that any shock to consumption is proportionately distributed across everyone alive at a given date. Individuals with overlapping lifetimes must smooth consumption as much as possible in that their expected rates of consumption growth are equalized. As the economy has overlapping generations, there is also the requirement of dynamic efficiency. This is equivalent to a long-run upper bound on the ratio of consumption between younger and older individuals. If that is not satisfied, resources can be shifted from younger to older individuals at each date to leave everyone better off.

Tests for efficiency in a market economy The three requirements for Pareto efficiency are stated directly in terms of allocations in Step 7. In the incomplete-markets economy, these conditions can be translated into equivalent tests involving inflation and interest rates.

Step 8 *The risk sharing requirement for efficiency holds if and only if nominal house-price inflation is perfectly predictable, that is, $\pi_t = \mathbb{E}_{t-1}\pi_t$ with probability 1. The consumption smoothing requirement holds if and only if the interest rate i_t is sufficiently far above expected house-price inflation $\mathbb{E}_t\pi_{t+1}$, that is, $i_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta - 1$. The dynamic efficiency conditions hold if and only if the interest rate is above the expected rate of house-price inflation in the long run, that is, $\limsup_{t\to\infty}(i_t - \mathbb{E}_t\pi_{t+1}) \geq 0$.*

Risk sharing always holds between the young and middle aged, and holds between the old and other age groups if and only if $c_{\mathrm{m},t}/\mathbb{E}_{t-1}c_{\mathrm{m},t} = c_{\mathrm{o},t}/\mathbb{E}_{t-1}c_{\mathrm{o},t}$. Consumption smoothing holds if and only if risk sharing does, and when it fails must result in $\mathbb{E}_t c_{\mathrm{m},t+1}/c_{\mathrm{y},t} > \mathbb{E}_t c_{\mathrm{o},t+1}/c_{\mathrm{m},t}$. Dynamic efficiency is implied by either risk sharing or consumption smoothing.

PROOF See appendix A.11. ■

Intuitively, risk sharing requires that nominal house-price inflation is perfectly predictable because otherwise any shocks to nominal house prices would fall disproportionately on leveraged home-owners who have fixed nominal debt. Consumption smoothing requires nominal interest rates sufficiently far above house-price inflation because otherwise investment returns on pensions would not keep pace with the returns of homeowners who are able to finance house purchases with cheap mortgages, leading to a divergence between the consumption growth of the different age groups. Similarly, dynamic efficiency requires the nominal mortgage rate to exceed the expected rate of nominal house-price inflation in the long run because otherwise the consumption of the old relative to the middle aged would fall to a point where transfers from the middle aged to the old could improve the welfare of each generation.

Financial stability Although markets are incomplete, there is nothing inevitable about the inefficiency of the competitive equilibrium. It turns out there is a monetary policy regime that supports an efficient allocation of resources, which can be interpreted as the central bank pursuing the goal of financial stability by setting interest rates sufficiently high to stabilize house-price inflation.

Result 4 *There are monetary policy regimes where nominal house-price inflation is predictable, that is, $\pi_t = \mathbb{E}_{t-1}\pi_t$ with probability 1. Such a monetary policy regime requires a nominal interest rate $i_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta - 1$, which is higher for any $\mathbb{E}_t\pi_{t+1}$ than a monetary policy regime associated with financial crises (compare 34). A monetary policy regime of this type results in financial stability in the sense that the house price-income ratio h_t and the debt-to-GDP ratio d_t remain stable at values $h^* = \alpha\beta\delta/\lambda$ and $d^* = \alpha\beta\delta$, which are lower than both the credit-boom values $\bar{h}$ and $\bar{d}$ and the long-run averages $\mathbb{E}h_t$ and $\mathbb{E}d_t$ of these variables under a monetary policy regime associated with financial crises (33). The equilibrium with financial stability is a first-best allocation.*

PROOF See appendix A.12. ■

Predictable house prices promote risk sharing across the generations, and higher interest rates raise the expected consumption growth of the old to the level enjoyed by the young and middle aged.

Result 4 shows that financial stability can be expressed in three exactly equivalent ways. First, in avoiding unanticipated swings in nominal house-price inflation. This objective is within the domain of monetary policy because the goal is predictability of nominal house prices, a nominal variable, and it can be achieved using the monetary policy instrument the central bank has available by setting sufficiently high interest rates.

Financial stability can also be expressed in real terms as stabilizing the ratio of house prices to income at a lower level than what occurs during a credit boom and what occurs on average in the equilibrium with financial crises. Finally, financial stability also means avoiding credit booms and busts by stabilizing the ratio of lending to GDP, again at a lower level than what prevails during a credit boom and on average across the boom and bust states of an equilibrium with financial crises. While these 'real' interpretations of financial stability are typically seen as outside the domain of monetary policy and being the responsibility of macroprudential or other regulatory policy, the analysis shows that it is possible to view financial instability instead as a monetary phenomenon.[19]

Welfare costs of excessively loose monetary policy While the equilibrium where monetary policy pursues financial stability results in a first-best allocation, the equilibrium with financial crises that is the solution of the Ramsey problem with democratic social welfare is never first best.

Result 5 *An equilibrium with financial crises (33 with positive x and ε) is Pareto inefficient, always violating the risk-sharing and consumption smoothing conditions. For small ε, the equilibrium is also dynamically inefficient if x is larger than $(\beta\delta)^{-1}-1$, which may or may not hold for the value of x in Result 3 depending on parameters.*

PROOF See appendix A.13. ■

An equilibrium with financial crises always fails some, and possibly all, of the requirements for Pareto efficiency. The unpredictability of house-price inflation violates the sharing of risk between leveraged homeowners and pensioners. The low level of interest rates compared to expected house-price inflation drives a wedge between the expected consumption growth of the two groups, and may even result in a dynamically inefficient equilibrium.

4.2 Achieving financial stability

Sterilizing the distributional consequences of monetary policy To understand how the conflict between efficiency and distribution is the source of the monetary policy that causes financial crises, this section hypothetically introduces age-specific lump-sum transfers that can be used to sterilize the distributional consequences of a change to monetary policy.

The per-person net lump-sum transfers to the young, middle-aged, and the old are respectively $T_{\mathrm{y},t}$, $T_{\mathrm{m},t}$, and $T_{\mathrm{o},t}$ (negative values are taxes). The budget identities (4) are modified by adding the transfers to the right-hand side. Since the lump-sum transfers are taken as given by individuals, all earlier first-order conditions are unchanged. The fiscal budget constraint for the transfers is:

$$T_{\mathrm{y},t}+T_{\mathrm{m},t}+T_{\mathrm{o},t}=0. \tag{37}$$

[19] It is important to note that the analysis does *not* predict that stabilizing nominal house-price inflation would simply cause an extreme amount of volatility in nominal goods prices instead. As financial stability leads to stable ratios of lending and house prices to income, an immediate implication is that the volatility of goods-price inflation would only be the same as the volatility of the real GDP growth rate. Although the financial instability due to unpredictability in nominal house prices is a monetary phenomenon, the financial accelerator effect that played a key role in the analysis predicts nominal house-price fluctuations are associated with movements of the relative price of housing in the same direction.

This budget constraint rules out the fiscal authority issuing or holding bonds, which as will be seen is without loss of generality.

Without any restrictions on the state-contingency of the transfers $T_{y,t}$, $T_{m,t}$, and $T_{o,t}$ beyond the budget constraint (37), the policymaker will be able to engineer any state-contingent consumption allocation satisfying the resource constraints. This is irrespective of what is done with monetary policy. The policymaker would essentially have sufficiently powerful instruments to be able to achieve the same as a social planner, and monetary policy would become irrelevant.

To avoid this vacuous outcome, the state-contingency of the transfers is restricted. Starting from a date s, the transfer $T_{a,t}$ for age group a at date t must be proportional to some predetermined multiple $\tau_{a,t}$ of GDP Y_t, that is, $\tau_{a,t} = \mathbb{E}_{s-1}\tau_{a,t}$ with probability 1. The transfer to the young is also restricted to be zero ($T_{y,t} = \tau_{y,t} = 0$), which turns out to be without loss of generality.

Result 6 *Without transfers, the financial stability equilibrium is the only first-best allocation that can be implemented using monetary policy. If in addition to its conventional monetary policy instrument the central bank has access to individual-specific (but not state-contingent) lump-sum taxes and transfers then it chooses a monetary policy with financial stability and predictable nominal house prices for all social welfare weights $\{\Omega_t\}$ (and hence all values of ω_t). Lump-sum taxes and transfers can be used to ensure that moving from financial instability to financial stability is Pareto improving.*

PROOF See appendix A.14. ■

As described earlier, while financial stability is efficient, achieving it requires systematically tighter interest-rate policy, resulting in less lending relative to the size of the economy, and lower house prices relative to incomes. The political economy analysis of Result 2 indicates these effects will be bad for borrowers, who will therefore lose from moves towards financial stability. While transfers from the savers who gain to the borrowers who lose could in principle make everyone better off overall, the central bank lacks access to the lump-sum taxes and transfers needed to make this happen. Taking away the punch bowl is good for efficiency, but will leave borrowers disgruntled as the central bank cannot compensate them for the higher interest rates they will face. Inefficient financial instability can therefore persist because there are too many individuals with a vested interest in maintaining cheap credit.

5 Endogenizing incomplete markets

This paper has explored how systematically loose monetary policy creates the conditions for financial crises to occur, and how the political economy of monetary policy explains why there is pressure on central banks to adopt such risky monetary policies. The analysis was done in the context of an economy with incomplete markets, where all lending must take the form of nominal debt contracts, and where housing services can only be received through homeownership. However, this exogenously assumes that trade cannot ever take place in other markets.

5.1 Frictionless complete markets

Begin by considering a frictionless complete-markets benchmark. The model is the same as that of section 2 except all individuals can trade securities with payoffs contingent on any state of the world (Arrow-Debreu securities). Individuals trade in contingent securities markets sequentially during their lives, excluding participation by individuals before birth.[20]

Formally, let $A_{a,t}$ denote the per-person portfolio of securities making payoffs (denominated in terms of goods without loss of generality) at time t to age a individuals conditional on the realization of a specific state of the world. This portfolio is chosen at time $t-1$. The prices of securities (in terms of goods) relative to the probabilities of the future states (conditional on the current state) are M_{t+1}, so the cost of a portfolio A_{t+1} is $\mathbb{E}_t[M_{t+1}A_{t+1}]$ at time t. The new budget constraints, first-order conditions, and market-clearing conditions can be found in the proof of the result below, which characterizes the unique equilibrium for real variables in the complete-markets economy.

Step 9 *With complete financial markets, there is a unique equilibrium for all real variables that is independent of monetary policy. The equilibrium is $c_{\mathrm{y},t} = c^*_{\mathrm{y}}$, $c_{\mathrm{m},t} = c^*_{\mathrm{m}}$, $c_{\mathrm{o},t} = c^*_{\mathrm{o}}$ and $h_t = h^*$ in all states and at all times. The equilibrium is Pareto efficient and coincides with the equilibrium with financial stability from Result 4. The nominal interest rate i_t, house-price inflation π_t, and the housing risk premium ξ_t continue to satisfy equations (18) and (19).*

PROOF See appendix A.15. ■

With complete markets, monetary policy has no impact on the real equilibrium of the economy. While monetary policy can still affect the risk premium of real assets over nominal bonds, this is irrelevant to any real decisions because individuals can conduct the trade they desire in financial markets by buying or selling packages of contingent securities. Hence, irrespective of monetary policy, the equilibrium of the economy is Pareto efficient because individuals can directly achieve full risk sharing and consumption smoothing by choosing appropriate long or short positions in each contingent security.[21] The outcome is the same as the 'financial stability' equilibrium of the incomplete-markets economy where the central bank stabilizes nominal house-price inflation. Financial crises cannot occur here with complete markets because the mechanism of the central bank holding down real borrowing costs owing to political pressure does not operate.

[20]This turns out to be without loss of generality here.

[21]It turns out that the equilibrium is ex-ante efficient even in respect of newly born individuals who did not participate in financial markets before they were born. This is the sense in which excluding participation by individuals before they are born is without loss of generality. The complete-markets equilibrium must also avoid dynamic inefficiency to be Pareto efficient. Even though the model has overlapping generations of individuals, the equilibrium with complete markets is always dynamically efficient because of the existence of housing as a physical asset that does not depreciate.

5.2 Additional markets with frictions

The next step is to allow for additional markets: alternative forms of housing finance to mortgages, and a rental market for houses, but where these additional markets are subject to frictions. Trade in those additional markets may or may not take place depending on whether the gains from trade outweigh the frictions. Crucially, whether these markets are active or not is endogenous to the conduct of monetary policy. It is shown that the main conclusions are robust to endogenizing the incompleteness of markets.

Moving away from the abstract notion of complete contingent securities markets, now consider an extension of the model of section 2 where there are two specific additional markets subject to frictions: a rental market for houses, and a market for equity shares in houses.

The lifetime utility function (1) is replaced by:

$$U_t = \log C_{\mathrm{y},t} + \beta \mathbb{E}_t \left[\log C_{\mathrm{m},t+1} + \Theta(H_{\mathrm{m},t+1} + H_{\mathrm{r},t+1} - \underline{H})\right] + \beta^2 \mathbb{E}_t \log C_{\mathrm{o},t+2}, \tag{38}$$

where the new variable $H_{\mathrm{r},t}$ denotes housing services acquired through the rental market rather than homeownership. Here, rented housing is fundamentally the same as owner-occupied housing, so utility depends on the sum $H_{\mathrm{m},t} + H_{\mathrm{r},t}$. The nominal rent is Z_t, which is paid at date t for renting one unit of housing during time period t. The supply of houses for rent at time t is equal to $H_{\mathrm{o},t} - \underline{H}$, the number of housing units held by the old in excess of their own housing demand.

The friction in the rental market is contract enforceability combined with asymmetric information. A fraction χ_{r} of rental payments will be unenforceable. Renters are of two types: those that always pay (fraction $1 - \chi_{\mathrm{r}}$) and those that always default (fraction χ_{r}). Individual renters know their type, but landlords cannot distinguish the two types in advance (the defaulting type will mimic the repaying type when contracts are written). For simplicity, both types pool consumption risk.

There is also a market for housing equity shares, which is a form of financing for borrowers where payments are proportional to the value of a house. Formally, the seller of one unit of housing equity at date t makes a nominal payment to the buyer equal to the value of a house V_{t+1} at date $t+1$. The net housing equity share positions of the young and the middle aged at the end of period t are denoted by $e_{\mathrm{y},t}$ and $e_{\mathrm{m},t}$ respectively, where a positive value indicates a purchase and a negative value a sale. The nominal price of a unit of housing equity is S_t when it is sold.

The friction in the market for housing equity shares is also contract enforceability with asymmetric information. A fraction χ_{e} of payments to holders of equity shares will be unenforceable. Analogous to the assumptions for renters, there are two types of borrowers (fractions $1 - \chi_{\mathrm{e}}$ and χ_{e} of all borrowers) who know their type, but investors cannot distinguish them ex ante.

The budget identities (4) of the young, middle aged, and old are replaced by:

$$C_{\mathrm{y},t} + \frac{V_t H_{\mathrm{m},t+1}}{P_t} + \frac{Q_t B_{\mathrm{y},t}}{P_t} + \frac{S_t e_{\mathrm{y},t}}{P_t} = \frac{V_t H_{\mathrm{y},t}}{P_t}; \tag{39a}$$

$$C_{\mathrm{m},t} + \frac{Z_t H_{\mathrm{r},t}}{P_t} + \frac{Q_t B_{\mathrm{m},t}}{P_t} + \frac{V_t H_{\mathrm{o},t+1}}{P_t} + \frac{S_t e_{\mathrm{m},t}}{P_t} = y_{\mathrm{m},t} + \frac{V_t H_{\mathrm{m},t}}{P_t} + \frac{B_{\mathrm{y},t-1}}{P_t} + \frac{V_t e_{\mathrm{y},t-1}}{P_t} + \frac{X_t}{P_t}; \tag{39b}$$

$$\text{and } C_{\mathrm{o},t} = \frac{B_{\mathrm{m},t-1}}{P_t} + \frac{Z_t H_{\mathrm{o},t}}{P_t} + \frac{V_t H_{\mathrm{o},t}}{P_t} + \frac{V_t e_{\mathrm{m},t-1}}{P_t} - \frac{X_t}{P_t}, \tag{39c}$$

where X_t denotes the sum of defaults on rental contracts and housing equity shares, which is subtracted from the budget identity of the old. Defaults are added to a single budget identity for the middle aged because the repaying and defaulting types pool consumption risk. The first-order condition for maximizing utility (38) with respect to house purchases $H_{\mathrm{m},t+1}$ subject to (39) is:

$$\frac{V_t}{P_t C_{\mathrm{y},t}} = \beta \mathbb{E}_t \left[\Theta'(H_{\mathrm{m},t+1} + H_{\mathrm{r},t+1}) + \frac{V_{t+1}}{P_{t+1} C_{\mathrm{m},t+1}} \right], \tag{40}$$

which replaces (7a). The first-order conditions with respect to bond holdings $B_{a,t}$ remain (7b).

Considering respectively an individual who is a repaying type of renter and a repaying type of borrower, the first-order conditions for maximizing lifetime expected utility (38) with respect to $H_{\mathrm{r},t}$ and $e_{\mathrm{y},t}$ subject to the budget identities (39) are:

$$\frac{Z_t}{P_t C_{\mathrm{m},t}} = \Theta'(H_{\mathrm{m},t} + H_{\mathrm{r},t}), \quad \text{and} \quad \frac{S_t}{P_t C_{\mathrm{y},t}} = \beta \mathbb{E}_t \left[\frac{V_{t+1}}{P_{t+1} C_{\mathrm{m},t+1}} \right], \tag{41}$$

which are derived taking defaults X_t as given because the individual knows he is a repaying type. The defaulting types mimic the choices of $H_{\mathrm{r},t}$ and $e_{\mathrm{y},t}$ implied by (41), so all young individuals choose the same $e_{\mathrm{y},t}$ and all middle-aged individuals choose the same $H_{\mathrm{r},t}$.

Given the default and asymmetric information frictions, investors' losses X_t at date t are:

$$X_t = \chi_{\mathrm{r}} Z_t H_{\mathrm{o},t} + \chi_{\mathrm{e}} V_t e_{\mathrm{m},t-1}. \tag{42}$$

While investors cannot distinguish individuals' types, they know the fractions of repaying and defaulting types in the population, and they therefore take account of (42) when choosing $H_{\mathrm{o},t}$ and $e_{\mathrm{m},t}$. Given non-negativity constraints on $H_{\mathrm{o},t}$ and $e_{\mathrm{m},t}$, the conditions for maximizing lifetime utility (38) with respect to $H_{\mathrm{o},t+1}$ and $e_{\mathrm{m},t}$ subject to (39) and (42) are as follows:

$$H_{\mathrm{o},t+1} \geq \underline{H}, \quad \text{and} \quad \frac{V_t}{P_t C_{\mathrm{m},t}} \geq \beta \mathbb{E}_t \left[\frac{(1-\chi_{\mathrm{r}}) Z_{t+1} + V_{t+1}}{P_{t+1} C_{\mathrm{o},t+1}} \right] \text{ with equality if } H_{\mathrm{o},t+1} > \underline{H}; \tag{43}$$

$$e_{\mathrm{m},t} \geq 0, \quad \text{and} \quad \frac{S_t}{P_t C_{\mathrm{m},t}} \geq \beta (1-\chi_{\mathrm{e}}) \mathbb{E}_t \left[\frac{V_{t+1}}{P_{t+1} C_{\mathrm{o},t+1}} \right] \text{ with equality if } e_{\mathrm{m},t} > 0. \tag{44}$$

Two new market-clearing conditions are added for the rental and equity share markets:

$$H_{\mathrm{r},t} = H_{\mathrm{o},t} - \underline{H} \tag{45}$$

$$e_{\mathrm{y},t} + e_{\mathrm{m},t} = 0, \tag{46}$$

where $H_{\mathrm{o},t} - \underline{H}$ is the supply of investment properties. All other aspects of the model from section 2 are unchanged.

For simplicity, the following restriction is imposed so that the strengths χ_{r} and χ_{e} of the rental and equity share frictions are both indexed by a common parameter χ (with $0 \leq \chi \leq 1$):

$$\chi_{\mathrm{r}} = \chi, \quad \text{and} \quad \chi_{\mathrm{e}} = (1 - \beta\delta)\chi, \tag{47}$$

where β is as defined in (18). Define a variable σ_t as follows:

$$\sigma_t = \frac{V_t(H_{\mathrm{o},t+1} - \underline{H}) + S_t e_{\mathrm{m},t}}{Q_t B_{\mathrm{m},t} + V_t(H_{\mathrm{o},t+1} - \underline{H}) + S_t e_{\mathrm{m},t}}, \tag{48}$$

which denotes the fraction of total savings held as rental housing and housing equity shares.

Result 7 *Conditional on a monetary policy regime, there exists a unique equilibrium σ_t with ϕ_t replaced by $\eta_t = \sigma_{t-1}(1 - \beta\delta)\chi + (1 - \sigma_{t-1})\phi_t$ and $d_t = \alpha\beta\delta(1 - \sigma_t)/(1 - \alpha\phi_t)$. With no frictions ($\chi = 0$), the equilibrium is the same as with complete markets in Step 9. In the presence of frictions $\chi > 0$, there is no trade in the additional markets ($H_{\mathrm{o},t+1} = 0$ and $e_{\mathrm{m},t} = 0$, and thus $\sigma_t = 0$) if and only if $\xi_t \leq \bar{\xi}$ where $\bar{\xi} = (1 - \beta\delta)\chi/(1 - (1 - \beta\delta)\chi)$. When $\xi_t > \bar{\xi}$, there is a positive equilibrium value of σ_t that is decreasing in χ and increasing in spreads of financial conditions.*

PROOF See appendix A.16. ■

6 Conclusions

This paper revisits an old debate about whether monetary policy should be influenced by financial stability concerns. The current consensus view is that monetary policy should focus on stabilizing inflation rates for goods and services, but should not concern itself with asset-price 'bubbles' or 'excessive' levels of debt. To the extent that these are seen as problems for policymakers to address, the solutions are believed to lie in the domain of regulation or macroprudential policy. This paper challenges the consensus view by presenting a simple theoretical mechanism based on incomplete markets through which the monetary policy actions of the central bank can have long-lasting effects on real interest rates and the level and volatility of asset prices.

In the theory presented here, loose monetary policy creates the conditions for excessive growth in lending, booms in asset prices, and ultimately financial crises: rare events where asset prices drop dramatically and result in painful deleveraging. The central bank is responsible for these problems in the sense that tighter monetary policy could have prevented them. However, the theory also explains why, for political economy reasons, the central bank will find tough action difficult: taking away the punch bowl will not be popular.

How should the challenges for monetary policy highlighted by this paper be addressed? One analogy is with the old problem of the 'inflation bias'. There, price stability helps an economy operate more efficiently, but policymakers face pressure to reduce unemployment by trying to exploit a Phillips curve, which results in excessively high inflation. The inflation bias has been conquered through a combination of conservative central bankers, central bank independence, and the institutional framework provided by inflation targeting. Here, policymakers face pressure for low interest rates even though this results in inefficient fluctuations in asset prices and lending: a 'financial instability' bias. Overcoming this problem reaffirms the need for central bank independence and conservative central bankers: but now 'conservative' in the sense of standing up for the interests of savers rather than ignoring unemployment. More systematically, the results of the paper suggest a case for embedding financial stability concerns into the monetary policy framework.

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A Appendices

A.1 Proof of Step 1

Take the second equation in (14d) and substitute for $(1+r_{t+1})/(1+g_{t+1})$ using the first equation from (14a), and for $c_{\text{o},t+1}$ using the third equation in (14b):

$$\frac{1}{c_{\text{m},t}} = \beta \mathbb{E}_t \left[\frac{b_{t+1}h_{t+1}}{d_t} \frac{1}{b_{t+1}h_{t+1}} \right].$$

Cancelling terms immediately shows that $d_t = \beta c_{\text{m},t}$.

A.2 Proof of Step 2

Comparing the definition of the leverage ratio $\lambda_t = -Q_t B_{\text{y},t}/V_t(H_{\text{m},t+1} - \underline{H})$ to the definitions of h_t and d_t from (12) and using (2) and $H_{\text{m},t} = L - 2\underline{H}$ from housing-market clearing (6b), it follows that $\lambda_t = d_t/h_t$, or $h_t = d_t/\lambda_t$. The first equation from (14b) is $c_{\text{y},t} + h_t = d_t$, from which it follows that $c_{\text{y},t} = (1-1/\lambda_t)d_t$, and hence $h_t/c_{\text{y},t} = 1/(\lambda_t - 1)$. As h_t and $c_{\text{y},t}$ must be non-negative, so is d_t, and thus $\lambda_t \geq 1$ is required at all dates. The result $c_{\text{m},t+1} = d_{t+1}/\beta$ from Step 1 implies $h_{t+1}/c_{\text{m},t+1} = \beta/\lambda_{t+1}$.

Now define $\delta_t = \beta(1-1/\lambda_t)$, noting the admissible range is $0 \leq \delta_t \leq \beta$. Rearranging the definition shows that $\lambda_t = \beta/(\beta - \delta_t)$ and thus $1/(\lambda_t - 1) = \beta/\delta_t - 1$ and $\beta/\lambda_{t+1} = \beta - \delta_{t+1}$. Together with the results above, these formulas can be substituted for $h_t/c_{\text{y},t}$ and $h_{t+1}/c_{\text{m},t+1}$ in (14c) to obtain an equivalent equation:

$$\mathbb{E}_t \delta_{t+1} = \beta^{-1} + \beta + \theta - \frac{1}{\delta_t}.$$

This defines an expectational difference equation $\mathbb{E}_t \delta_{t+1} = F(\delta_t)$, where $F(\delta) = \beta^{-1} + \beta + \theta - \delta^{-1}$. A steady-state solution $\delta = F(\delta)$ is a root of the quadratic equation $G(\delta) = 0$, where $G(\delta) = \delta^2 - (\beta^{-1} + \beta + \theta)\delta + 1$. Since $G(0) = 1$ and $G(\beta) = -\beta\theta$, there exists a unique economically meaningful steady state δ between 0 and β. The second root is a larger positive number because $G(\delta)$ is positive for large δ. As the product of the roots of $G(\delta) = 0$ is 1, the steady-state δ is less than 1 and can be found by inverting the formula for the larger root and noting that $(\beta^{-1} + \beta + \theta)^2 - 4 = (\beta^{-1}(1-\beta)^2 + \theta)(\beta^{-1}(1+\beta)^2 + \theta)$:

$$\delta = \frac{2}{\beta^{-1} + \beta + \theta + \sqrt{(\beta^{-1}(1-\beta)^2 + \theta)(\beta^{-1}(1+\beta)^2 + \theta)}}, \tag{A.1}$$

The equation for δ_t is equivalent to the stochastic difference equation $\delta_t = F(\delta_{t-1}) + \upsilon_t$, where $\upsilon_t = \delta_t - \mathbb{E}_{t-1}\delta_t$ must be a martingale difference sequence ($\mathbb{E}_{t-1}\upsilon_t = 0$). The function $F(\delta)$ has properties $F(0) = -\infty$, $F(\beta) = \beta + \theta$, and $F'(\delta) = \delta^{-2}$. It is monotonic with a gradient greater than 1 at the steady state ($\delta < 1$). Since there must be positive-probability realizations of both $\upsilon_t \geq 0$ and $\upsilon_t \leq 0$ for all t and for any past history, it follows that if $\delta_t \neq \delta$ then there exists a positive probability path such that $\delta_{t+\ell} < 0$ or $\delta_{t+\ell} > \beta$ for some finite ℓ. Such a value of $\delta_{t+\ell}$ is impossible, so the unique equilibrium is $\delta_t = \delta$ for all t. This implies $\lambda_t = (1-\delta/\beta)^{-1}$, where the value of δ from (A.1) depends only on β and θ.

It can be seen immediately from (A.1) that δ is decreasing in θ, which implies λ is also decreasing in θ because λ is increasing in δ. The effect of β on λ is found by first considering the effect of changing this parameter on δ. Since the equation $\delta^2 - (\beta^{-1} + \beta + \theta)\delta + 1 = 0$ implicitly defines δ as a function of β and θ, the implicit function theorem implies:

$$\frac{\partial \delta}{\partial \beta} = -\frac{-(1-\beta^{-2})\delta}{2\delta - (\beta^{-1} + \beta + \theta)} = \frac{\delta(1-\beta^{-2})}{2\delta - (\delta + \delta^{-1})} = \frac{\delta^2(1-\beta^2)}{\beta^2(1-\delta^2)}, \tag{A.2}$$

where the second equality uses $\beta^{-1} + \beta + \theta = \delta + \delta^{-1}$. This result can be used to differentiate the ratio δ/β:

$$\frac{\partial(\delta/\beta)}{\partial \beta} = \frac{1}{\beta}\frac{\partial \delta}{\partial \beta} - \frac{\delta}{\beta^2} = \frac{\delta^2(1-\beta^2)}{\beta^3(1-\delta^2)} - \frac{\delta}{\beta^2} = -\frac{\delta(1+\beta\delta)(\beta-\delta)}{\beta^3(1-\delta^2)},$$

which is negative because $0 < \delta < \beta$ and $\delta < 1$. Since λ is increasing in the ratio δ/β it follows that λ is decreasing in β.

A.3 Proof of Step 3

Using (16) it follows that $d_t = \lambda h_t$ and hence $c_{\mathrm{y},t} = (\lambda - 1)h_t$. Combining this with $c_{\mathrm{m},t+1} = d_{t+1}/\beta$ from Step 1 leads to $c_{\mathrm{m},t+1} = (\lambda/\beta)h_{t+1}$. Substituting these results for $c_{\mathrm{y},t}$ and $c_{\mathrm{m},t+1}$ in the first equation from (14d), and using (14e) to substitute for $1 + r_{t+1}$:

$$\frac{1}{(\lambda-1)h_t} = \beta\mathbb{E}_t\left[\frac{(1+i_t)}{(1+\gamma_{t+1})(1+g_{t+1})}\frac{\beta}{\lambda h_{t+1}}\right].$$

Substituting for $(1+\gamma_{t+1})(1+g_{t+1})h_{t+1}/h_t$ using the second equation in (14a):

$$1 = \beta\mathbb{E}_t\left[\beta\left(1-\frac{1}{\lambda}\right)\frac{1+i_t}{1+\pi_{t+1}}\right],$$

and since the constant λ from Step 2 is $\lambda = (1-\delta/\beta)^{-1}$, a rearrangement of the equation shows that $\delta = \beta(1-1/\lambda)$. It follows that the equilibrium condition becomes:

$$1 = \beta\delta\mathbb{E}_t\left[\frac{1+i_t}{1+\pi_{t+1}}\right],$$

and solving for i_t leads to the formula in (18). Since $\mathbb{E}_t[(1+i_t)/(1+\pi_{t+1})] = 1/\beta\delta$, taking the conditional expectation of equation (17) immediately yields the result for $\mathbb{E}_t b_{t+1}$ in (18). Using the findings from the proof of Step 2, the coefficient δ is the smaller positive root of the quadratic equation $G(\delta) = \delta^2 - (\beta^{-1} + \beta + \theta)\delta + 1 = 0$. Noting that $G(\beta^{-1}) = -\theta\beta^{-1} < 0$, it follows that $\delta < \beta^{-1}$ and hence $0 < \beta\delta < 1$. It can be seen from (A.1) that δ is decreasing in θ, so the combined coefficient $\beta\delta$ is also decreasing in θ. The formula for the derivative of δ in (A.2) can be used to show:

$$\frac{\partial(\beta\delta)}{\partial\beta} = \delta + \beta\frac{\partial\delta}{\partial\beta} = \delta + \frac{\delta^2(1-\beta^2)}{\beta(1-\delta^2)} = \frac{\delta(\beta+\delta)(1-\beta\delta)}{\beta(1-\delta^2)},$$

which is positive because $\beta\delta < 1$. Thus, the combined coefficient $\beta\delta$ is increasing in β.

The formula in (19) for R_t follows immediately by combining (8) and (9). The housing return $\hat{R}_t$ in (9) is the sum of capital gains π_t and an imputed rental yield Z_t/V_{t-1}. The imputed rental yield is:

$$\frac{Z_t}{V_{t-1}} = (1+\pi_t)\frac{\Theta'(H_{\mathrm{m},t}-\underline{H})P_t C_{\mathrm{m},t}}{V_t} = (1+\pi_t)\frac{(H_{\mathrm{m},t}-\underline{H})\Theta'(H_{\mathrm{m},t}-\underline{H})\frac{C_{\mathrm{m},t}}{Y_t}}{\frac{V_t(H_{\mathrm{m},t}-\underline{H})}{P_t y_t}} = (1+\pi_t)\frac{\theta c_{\mathrm{m},t}}{h_t},$$

where the first equality uses the definition of house-price inflation π_t and the imputed rent Z_t from (9), the second equality uses (3), and the third equality uses housing-market equilibrium (6b) and (7a) to deduce $H_{\mathrm{m},t} - \underline{H} = L - 3\underline{H}$, together with the definitions of $c_{\mathrm{m},t}$, h_t, and θ from (12) and (14c). Using (9), $c_{\mathrm{m},t} = d_t/\beta$ from Step 1, and $d_t = \lambda h_t$ from (16), the return on housing satisfies:

$$1+\hat{R}_t = (1+\pi_t)\left(1+\frac{\theta c_{\mathrm{m},t}}{h_t}\right) = \left(1+\frac{\theta\lambda}{\beta}\right)(1+\pi_t) = \left(1+\frac{\theta}{\beta-\delta}\right)(1+\pi_t),$$

where the final equality follows from the definition of $\lambda = (1-\delta/\beta)^{-1}$. The quadratic equation for δ implies $\theta = \delta^{-1} - \beta^{-1} - \beta + \delta$, which can be written equivalently as $\theta = (\beta-\delta)/(\beta^{-1}\delta^{-1} - 1)$. Substituting this into the equation above implies $1 + \hat{R}_t = (1+\pi_t)/\beta\delta$, confirming the claim in (19). The expected excess return on housing from (10) can then be calculated using the expressions for R_t and $\hat{R}_t$ in (19):

$$\xi_t = \frac{\mathbb{E}_t[1+\hat{R}_{t+1}] - (1+i_t)}{(1+i_t)} = \frac{\mathbb{E}_t[1+\pi_{t+1}]}{\beta\delta(1+i_t)} - 1.$$

Since (18) implies $\beta\delta(1+i_t) = 1/\mathbb{E}_t[(1+\pi_{t+1})^{-1}]$ in equilibrium, the formula for ξ_t in (19) is obtained. If π_{t+1} is perfectly predictable using information available at date t then $\mathbb{E}_t[(1+\pi_{t+1})^{-1}] = (\mathbb{E}_t[1+\pi_{t+1}])^{-1}$, in which case $\xi_t = 0$. Since $(1+\pi_{t+1})^{-1}$ is a strictly convex function of π_{t+1}, whenever π_{t+1} has a non-degenerate probability distribution, Jensen's inequality implies $\mathbb{E}_t[(1+\pi_{t+1})^{-1}] > (1+\mathbb{E}_t\pi_{t+1})^{-1}$ and thus $\xi_t > 0$. Owing to the strict convexity of $(1+\pi_{t+1})^{-1}$, $\mathbb{E}_t[(1+\pi_{t+1})^{-1}]$ increases with any mean-preserving spread of π_{t+1}, which therefore increases ξ_t because it does not affect $\mathbb{E}_t[1+\pi_{t+1}]$.

A.4 Proof of Step 4

Taking equation (24) and collecting terms in h_t on one side:

$$\left(1 - \frac{\beta}{\lambda(1+\beta)} + \frac{\beta}{\lambda(1+\beta)}\frac{\lambda}{\beta\delta}(1-\phi_t)\right)h_t = \frac{\beta}{\lambda(1+\beta)}.$$

The value of ϕ_t must satisfy $-\infty \leq \phi_t \leq 1$ in any equilibrium. Together with $\lambda = (1-\delta/\beta)^{-1} > 1$ as $\delta < \beta$, this implies the coefficient of h_t above is strictly positive and thus there exists a unique solution of the equation for h_t given a value of financial conditions ϕ_t. Dividing both sides of the equation by the positive term $1-(\beta/\lambda(1+\beta))(1-\lambda/\beta\delta)$ leads to:

$$(1-\alpha\phi_t)h_t = \frac{\alpha\beta\delta}{\lambda}, \quad \text{with } \alpha = \frac{\frac{\beta}{\lambda(1+\beta)}\frac{\lambda}{\beta\delta}}{1-\frac{\beta}{\lambda(1+\beta)}\left(1-\frac{\lambda}{\beta\delta}\right)}, \tag{A.3}$$

where the coefficient α is as defined in the main text. This confirms the solution $h_t = \alpha\beta\delta/\lambda(1-\alpha\phi_t)$ for h_t. The solution for d_t follows immediately by substituting this result into $d_t = \lambda h_t$ from (16), and the solution for n_t is then obtained by using $n_t = ((1+\beta)/\beta)d_t$ from (23).

Simplifying the formula for α from (A.3) allows it to written as follows:

$$\alpha = \frac{\frac{1}{\delta(1+\beta)}}{1-\frac{\beta}{\lambda(1+\beta)}+\frac{1}{\delta(1+\beta)}} = \frac{1}{1+\delta(1+\beta)-\frac{\beta\delta}{\lambda}} = \frac{1}{1+\delta+\beta\delta\left(1-\frac{1}{\lambda}\right)}.$$

Rearranging the definition $\lambda = (1-\delta/\beta)^{-1}$ leads to $1-1/\lambda = \delta/\beta$, and substituting into the expression for α confirms that $\alpha = 1/(1+\delta+\delta^2)$. The proof of Step 2 shows that $0<\delta<1$, from which it follows that $1/3 < \alpha < 1$. It is also shown there that δ is decreasing in θ, which implies α is increasing in θ. Using equation (A.2) from the proof of Step 2 together with $0<\delta<1$ demonstrates that δ is increasing in β when $\beta < 1$, and decreasing in β when $\beta > 1$. These findings immediately imply that α is increasing in the distance of the discount factor β from 1.

A.5 Proof of Result 1

Adding α multiplied by (26a) to $1-\alpha$ multiplied by (26b):

$$\alpha\frac{1+r_{t+1}}{1+g_{t+1}} + (1-\alpha)\frac{1+\hat{r}_{t+1}}{1+g_{t+1}} = \frac{\alpha(1-\alpha\phi_t)(1-\phi_{t+1})}{\beta\delta(1-\alpha\phi_{t+1})} + \frac{(1-\alpha)(1-\alpha\phi_t)}{\beta\delta(1-\alpha\phi_{t+1})} = \frac{1-\alpha\phi_t}{\beta\delta}. \tag{A.4}$$

Taking unconditional expectations of both sides and using $\mathbb{E}\phi_t = 0$ implied by (22):

$$\alpha\mathbb{E}\left[\frac{1+r_{t+1}}{1+g_{t+1}}\right] + (1-\alpha)\mathbb{E}\left[\frac{1+\hat{r}_{t+1}}{1+g_{t+1}}\right] = \frac{1-\alpha\mathbb{E}\phi_t}{\beta\delta} = \frac{1}{\beta\delta},$$

which shows that an average of $\mathbb{E}[(1+r_{t+1})/(1+g_{t+1})]$ and $\mathbb{E}[(1+\hat{\rho}_{t+1})/(1+g_{t+1})]$ with weights α and $1-\alpha$ is always equal to $1/\beta\delta$. Multiplying both sides of (A.4) by $1+g_{t+1}$ and taking expectations conditional on date-t information implies $\alpha(1+\rho_t)+(1-\alpha)(1+\hat{\rho}_t) = ((1-\alpha\phi_t)/\beta\delta)(1+\mathbb{E}_t g_{t+1})$, recalling the definitions of ρ_t and $\hat{\rho}_t$ in (11). Dividing by $1+\mathbb{E}_t g_{t+1}$, taking unconditional expectations, and using $\mathbb{E}\phi_t = 0$ leads to the result that the weighted average of $\mathbb{E}[(1+\rho_t)/(1+\mathbb{E}_t g_{t+1})]$ and $\mathbb{E}[(1+\hat{\rho}_t)/(1+\mathbb{E}_t g_{t+1})]$ is always equal to $1/\beta\delta$.

Taking expectations of (26a) and (26b) conditional on date-t information:

$$\mathbb{E}_t\left[\frac{1+r_{t+1}}{1+g_{t+1}}\right]=\frac{1-\alpha\phi_t}{\beta\delta}\mathbb{E}_t\left[\frac{1-\phi_{t+1}}{1-\alpha\phi_{t+1}}\right],\quad\text{and}\ \mathbb{E}_t\left[\frac{1+\hat{r}_{t+1}}{1+g_{t+1}}\right]=\frac{1-\alpha\phi_t}{\beta\delta}\mathbb{E}_t\left[\frac{1}{1-\alpha\phi_{t+1}}\right].\qquad\text{(A.5)}$$

Let $F_S(\phi)=(1-\phi)/(1-\alpha\phi)$, which has first derivative $F_S'(\phi)=-(1-\alpha)/(1-\alpha\phi)^2$. The second derivative is $F_S'(\phi)=-2\alpha(1-\alpha)/(1-\alpha\phi)^3$ which is negative for all $\phi\in[-\infty,1]$ because $\alpha<1$, so $F_S(\phi)$ is a strictly concave function of ϕ. Likewise, let $F_B(\phi)=1/(1-\alpha\phi)$, which has first derivative $F_B'(\phi)=\alpha/(1-\alpha\phi)^2$. The second derivative is $F_B''(\phi)=2\alpha^2/(1-\alpha\phi)^3$, which is positive over the range of ϕ, establishing that $F_B(\phi)$ is a strictly convex function.

Since $F_S(\phi)$ is a strictly concave function and $F_B(\phi)$ is a strictly convex function, Jensen's inequality implies $\mathbb{E}_tF_S(\phi_{t+1})<F_S(\mathbb{E}_t\phi_{t+1})$ and $\mathbb{E}_tF_B(\phi_{t+1})>F_B(\mathbb{E}_t\phi_{t+1})$ for any non-degenerate distribution of ϕ_{t+1}. As (22) requires $\mathbb{E}_t\phi_{t+1}=0$, it follows that $\mathbb{E}_t[(1-\phi_{t+1})/(1-\alpha\phi_{t+1})]<1<\mathbb{E}_t[1/(1-\alpha\phi_{t+1})]$. Moreover, $\mathbb{E}_t[(1-\phi_{t+1})/(1-\alpha\phi_{t+1})]$ and $\mathbb{E}_t[1/(1-\alpha\phi_{t+1})]$ are respectively decreasing and increasing with any spread of ϕ_{t+1} (which given 22 must necessarily remain with mean zero). Since the coefficients $(1-\alpha\phi_t)/\beta\delta$ in (A.5) are positive for all valid $\phi_t\in[-\infty,1]$, $\mathbb{E}_t[(1+r_{t+1})/(1+g_{t+1})]$ and $\mathbb{E}_t[(1+\hat{r}_{t+1})/(1+g_{t+1})]$ are respectively decreasing and increasing with spreads of ϕ_{t+1}. Furthermore, $\mathbb{E}_t[(1+r_{t+1})/(1+g_{t+1})]<(1-\alpha\phi_t)/\beta\delta$ and $\mathbb{E}_t[(1+\hat{r}_{t+1})/(1+g_{t+1})]>(1-\alpha\phi_t)/\beta\delta$ for any non-degenerate distribution of ϕ_{t+1}. Taking unconditional expectations and noting that $\mathbb{E}[(1-\alpha\phi_t)/\beta\delta]=1/\beta\delta$, it follows that $\mathbb{E}[(1+r_{t+1})/(1+g_{t+1})]$ is always less than $1/\beta\delta$ and $\mathbb{E}[(1+\hat{r}_{t+1})/(1+g_{t+1})]$ is always greater than $1/\beta\delta$. If ϕ_{t+1} has a degenerate distribution then it must equal 0 with probability one, in which case both $\mathbb{E}[(1+r_{t+1})/(1+g_{t+1})]$ and $\mathbb{E}[(1+\hat{r}_{t+1})/(1+g_{t+1})]$ are equal to $1/\beta\delta$ using (26a) and (26b).

A.6 Proof of Result 2

Substituting housing demand $H_{o,t}=\underline{H}$ from (7a), $H_{y,t}=\underline{H}$ from (3), and housing supply (2) into the housing market clearing condition (6b) implies $H_{m,t}=L-2\underline{H}$. It follows that $\Theta(H_{m,t}-\underline{H})=\Theta(L-3\underline{H})=$ t.i.p., where t.i.p. denotes terms independent of monetary policy. Note that $\log C_{a,t}=\log c_{a,t}+\log Y_t$ using the definition of the consumption ratio $c_{a,t}$ from (12), and since real GDP Y_t from (2) is exogenous, this means that $\log C_{a,t}=\log c_{a,t}+$t.i.p.. Taking logarithms of both sides of the expressions for the consumption ratios $c_{a,t}$ in (27) implies $\log c_{y,t}=-\log(1-\alpha\phi_t)+$t.i.p., $c_{m,t}=-\log(1-\alpha\phi_t)+$t.i.p., and $c_{o,t}=\log(1-\phi_t)-\log(1-\alpha\phi_t)+$t.i.p., which when combined with the earlier results allows the continuation utilities (28) to be written as:

$$U_{y,t}=-\log(1-\alpha\phi_t)-\beta\mathbb{E}_t\log(1-\alpha\phi_{t+1})+\beta^2\mathbb{E}_t[\log(1-\phi_{t+2})-\log(1-\alpha\phi_{t+2})]+\text{t.i.p.};$$
$$U_{m,t}=-\log(1-\alpha\phi_t)+\beta\mathbb{E}_t[\log(1-\phi_{t+1})-\log(1-\alpha\phi_{t+1})]+\text{t.i.p.};$$
$$U_{o,t}=\log(1-\phi_t)-\log(1-\alpha\phi_t)+\text{t.i.p.}.$$

By defining $w_{S,t}=\mathbb{E}_{t-1}[\log(1-\phi_t)-\log(1-\alpha\phi_t)]$ and $w_{B,t}=-\mathbb{E}_{t-1}[\log(1-\alpha\phi_t)]$ and taking conditional expectations of the continuation utilities above using date $t-1$ information leads to the expressions given for $\mathbb{E}_{t-1}U_{y,t}$, $\mathbb{E}_{t-1}U_{m,t}$, and $\mathbb{E}_{t-1}U_{o,t}$. Thus, in respect of the welfare effects of the monetary policy regime, all individuals at a point in time fall into a group of 'borrowers' (B) (the young and middle aged) or a group of 'savers' (S) (the old).

Consider the functions $f_S(\phi)=\log(1-\phi)-\log(1-\alpha\phi)$ and $f_B(\phi)=-\log(1-\alpha\phi)$, which have first derivatives $f_S'(\phi)=\alpha/(1-\alpha\phi)-1/(1-\phi)$ and $f_B'(\phi)=\alpha/(1-\alpha\phi)$. The second derivatives are:

$$f_S''(\phi)=\frac{\alpha^2}{(1-\alpha\phi)^2}-\frac{1}{(1-\phi)^2}=-\frac{(1-\alpha)\left((1-\alpha)+2\alpha(1-\phi)\right)}{((1-\alpha\phi)(1-\phi))^2},\quad\text{and}\ f_B''(\phi)=\frac{\alpha^2}{(1-\alpha\phi)^2},$$

implying $f_S''(\phi)<0$ and $f_B''(\phi)>0$ for all $\phi\in[-\infty,1]$ because $\alpha<1$. Hence, $w_{S,t}=\mathbb{E}_{t-1}f_S(\phi_t)$ and $w_{B,t}=\mathbb{E}_{t-1}f_B(\phi_t)$, where $f_S(\phi)$ and $f_B(\phi)$ are respectively strictly concave and strictly convex functions of ϕ. As shown in Result 1, the real return on housing increases and the real return on bonds decreases with spreads of financial conditions ϕ_t around its conditional mean $\mathbb{E}_{t-1}\phi_t=0$. Owing to the concavity and convexity of $f_S(\phi)$ and $f_B(\phi)$ respectively, these spreads of ϕ_t decrease the welfare of savers and increase the welfare of borrowers.

A.7 Proof of Step 5

Substituting the continuation utilities (28) into the social welfare function (29):

$$W_t = \mathbb{E}_{t-1}\left[\Omega_{t-2|t}\left\{\log C_{\text{o},t}\right\} + \Omega_{t-1|t}\left\{\log C_{\text{m},t} + \Theta(H_{\text{m},t} - \underline{H}) + \beta \mathbb{E}_t \log C_{\text{o},t+1}\right\}\right]$$
$$+ \mathbb{E}_{t-1}\left[\sum_{\ell=0}^{\infty} \Omega_{t+\ell|t}\left\{\log C_{\text{y},t+\ell} + \beta \mathbb{E}_{t+\ell} \log C_{\text{m},t+\ell+1} + \Theta(H_{\text{m},t+\ell+1} - \underline{H}) + \beta^2 \mathbb{E}_{t+\ell} \log C_{\text{o},t+\ell+2}\right\}\right].$$

Applying the law of iterated expectations and grouping terms corresponding to the same time period:

$$W_t = \mathbb{E}_{t-1}\left[\sum_{\ell=0}^{\infty}\left\{\Omega_{t+\ell|t}\log C_{\text{y},t+\ell} + \beta^{\min\{1,\ell\}}\Omega_{t+\ell-1|t}\log C_{\text{m},t+\ell} + \beta^{\min\{2,\ell\}}\Omega_{t+\ell-2|t}\log C_{\text{o},t+\ell}\right\}\right]$$
$$+ \mathbb{E}_{t-1}\left[\sum_{\ell=0}^{\infty}\beta^{\min\{1,\ell\}}\Omega_{t+\ell-1|t}\Theta(H_{\text{m},t+\ell} - \underline{H})\right]. \quad \text{(A.6)}$$

In any competitive equilibrium, $H_{\text{y},t} = H_{\text{o},t} = \underline{H}$, hence $H_{\text{m},t} = L - 2\underline{H}$ using (6b), from which it follows that

$$\mathbb{E}_{t-1}\left[\sum_{\ell=0}^{\infty}\beta^{\min\{1,\ell\}}\Omega_{t+\ell-1|t}\Theta(H_{\text{m},t+\ell} - \underline{H})\right] = \Theta(L - 3\underline{H})\left(\Omega_{t-1|t} + \beta\sum_{\ell=0}^{\infty}\Omega_{t+\ell|t}\right).$$

Substituting this and $\log C_{a,t} = \log c_{a,t} + \log Y_t$ from the definition (12) into (A.6):

$$W_s = \mathbb{E}_{s-1}\left[\sum_{t=s}^{\infty}\left\{\Omega_{t|s}\log c_{\text{y},t} + \beta^{\min\{1,t-s\}}\Omega_{t-1|s}\log c_{\text{m},t} + \beta^{\min\{2,t-s\}}\Omega_{t-2|s}\log c_{\text{o},t}\right\}\right]$$
$$+ \mathbb{E}_{s-1}\left[\sum_{t=s}^{\infty}\Delta_{t|s}\log Y_t\right] + \Theta(L - 3\underline{H})\left(\Omega_{s-1|s} + \beta\sum_{t=s}^{\infty}\Omega_{t|s}\right), \quad \text{(A.7)}$$

where s denotes the starting time period for the welfare calculation, and where $\Delta_{t|s} = \Omega_{t|s} + \beta^{\min\{1,t-s\}}\Omega_{t-1|s} + \beta^{\min\{2,t-s\}}\Omega_{t-2|s}$ is defined. Having the positive sum $\sum_{t=s}^{\infty}\Omega_{t|s} < \infty$ ensures the final term in (A.7) is finite. Using the definition of real GDP growth g_t from (2), $\log Y_t = \log Y_{s-1} + \sum_{\ell=0}^{t-s}\log(1 + g_{s+\ell})$, and the bounds on growth in (2) imply $(t-s)\log(1+\underline{g}) \leq \sum_{\ell=0}^{t-s}\log(1+g_{s+\ell}) \leq (t-s)\log(1+\bar{g})$. It follows that:

$$(\log Y_{s-1} - s\log(1+\underline{g}))\sum_{t=s}^{\infty}\Delta_{t|s} - (\log(1+\underline{g}))\sum_{t=s}^{\infty}t\Delta_{t|s} \leq \mathbb{E}_{s-1}\left[\sum_{t=s}^{\infty}\Delta_{t|s}\log Y_t\right] \leq (\log(1+\bar{g}))\sum_{t=s}^{\infty}t\Delta_{t|s}$$
$$+ (\log Y_{s-1} - s\log(1+\bar{g}))\sum_{t=s}^{\infty}\Delta_{t|s},$$

and therefore $\sum_{t=s}^{\infty}t\Delta_{t|s} < \infty$ (which implies $\sum_{t=s}^{\infty}\Delta_{t|s} < \infty$) is sufficient for the final two terms in (A.7) to be finite as $-1 < \underline{g}$ and $\bar{g} < \infty$. Those terms are independent of monetary policy, denoted t.i.p. in what follows. It can be seen from the definition of $\Delta_{t|s}$ that $\sum_{t=s}^{\infty}t\Omega_{t|s} < \infty$ implies $\sum_{t=s}^{\infty}t\Delta_{t|s} < \infty$, guaranteeing the terms independent of monetary policy in (A.7) are finite, hence

$$W_s = \mathbb{E}_{s-1}\left[\sum_{t=s}^{\infty}\left\{\Omega_{t|s}\log c_{\text{y},t} + \beta^{\min\{1,t-s\}}\Omega_{t-1|s}\log c_{\text{m},t} + \beta^{\min\{2,t-s\}}\Omega_{t-2|s}\log c_{\text{o},t}\right\}\right] + \text{t.i.p.} \quad \text{(A.8)}$$

is a well-defined expression for social welfare (A.7). Using the expressions for the consumption ratios in (27) and borrower and saver welfare $w_{\text{B},t}$ and $w_{\text{S},t}$ from Result 2, it follows that $\mathbb{E}_{s-1}c_{\text{y},t} = \log(\alpha\delta^2) + \mathbb{E}_{s-1}w_{\text{B},t}$, $\mathbb{E}_{s-1}c_{\text{m},t} = \log(\alpha\delta) + \mathbb{E}_{s-1}w_{\text{B},t}$, and $\mathbb{E}_{s-1}c_{\text{o},t} = \log\alpha + \mathbb{E}_{s-1}w_{\text{S},t}$ for all $t \geq s$. Together with the definition of $\omega_{t|s} = \beta^{\min\{2,t-s\}}\Omega_{t-2|s}/(\Omega_{t|s} + \beta^{\min\{1,t-s\}}\Omega_{t-1|s} + \beta^{\min\{2,t-s\}}\Omega_{t-2|s})$, (A.8) can be written as

$$W_s = \mathbb{E}_{s-1}\left[\sum_{t=s}^{\infty}\Delta_{t|s}\left\{(1-\omega_{t|s})w_{\text{B},t} + \omega_{t|s}w_{\text{S},t}\right\}\right] + \text{t.i.p.} = \sum_{t=s}^{\infty}\Delta_{t|s}\mathbb{E}_{s-1}w_t + \text{t.i.p.},$$

where $w_t = (1-\omega_{t|s})w_{\mathrm{B},t} + \omega_{t|s}w_{\mathrm{S},t}$. This follows by using the earlier definition of $\Delta_{t|s}$ and noting that $\Delta_{t|s}$ and $\omega_{t|s}$ must be $s-1$-measurable given the same property of $\Omega_{t|s}$. Since $\Omega_{t|s}$ is positive, $\omega_{t|s}$ must lie strictly between 0 and 1. Taking $w_{\mathrm{B},t} = -\mathbb{E}_{t-1}[\log(1-\alpha\phi_t)]$ and $w_{\mathrm{S},t} = \mathbb{E}_{t-1}[\log(1-\phi_t) - \log(1-\alpha\phi_t)]$ from Result 2, this yields $w_t = \mathbb{E}_{t-1}[\omega_{t|s}\log(1-\phi_t) - \log(1-\alpha\phi_t)]$. Given $\omega_{t|s}$, the variable w_t depends only on the probability distribution of ϕ_t and parameters.

The Ramsey problem (30) finds the supremum of W_s over the state-contingent path of nominal house-price inflation $\{\pi_t\}_{t=s}^{s+k-1}$ subject to the equilibrium conditions in (14) for all t, and taking $\{\pi_t\}_{t=s+k}^{\infty}$ as given. The equilibrium conditions imply (13), which means that $c_{a,t} \leq 1$, showing the expression for social welfare in (A.8) must be bounded above (terms independent of monetary policy are finite). It follows that there exists a supremum of social welfare subject to the equilibrium conditions.

Using Step 3, the equilibrium conditions imply (22), so ϕ_t must satisfy $\mathbb{E}_{t-1}\phi_t = 0$ and $\phi_t \in [-\infty, 1]$ for all t. Noting that $w_t = 0$ when $\phi_t = 0$ in all states of the world, which satisfies the constraints on ϕ_t, it follows that $w_t = 0$ is attainable, so the supremum of social welfare is finite. Since (22) shows that ϕ_t is entirely determined by the probability distribution of π_t, the Ramsey policymaker can choose $\{\phi_t\}_{t=s}^{s+k-1}$ subject to $\phi_t \in [-\infty, 1]$ and $\mathbb{E}_{t-1}\phi_t = 0$, taking as given $\{\phi_t\}_{t=s+k}^{\infty}$. Thus, the policymaker's choices for $t = s, \ldots, s+k-1$ have no effect on $\sum_{t=s+k}^{\infty}\Delta_{t|s}\mathbb{E}_{s-1}w_t$ and hence $W_s = \sum_{t=s}^{s+k-1}\Delta_{t|s}\mathbb{E}_{s-1}w_t + \text{t.i.p.}$ Since each term $\mathbb{E}_{s-1}w_t$ enters W_s additively with a strictly positive coefficient and depends only on ϕ_t, and the constraints on ϕ_t are independent for each date t, the supremum of W_s can be obtained by finding the supremums of w_t over the probability distribution of ϕ_t subject to $\mathbb{E}_{t-1}\phi_t = 0$ and $\phi_t \in [-\infty, 1]$ for all $t = s, \ldots, s+k-1$, as stated in (31).

A.8 Proof of Result 3

According to Step 5, the Ramsey solution for financial conditions ϕ_t is determined by (31). Given a value of $\omega_{t|s} = \omega$ (satisfying $0 \leq \omega \leq 1$), the problem in (31) has a time-invariant form:

$$\sup_{\phi}\mathbb{E}\left[\omega\log(1-\phi) - \log(1-\alpha\phi)\right] \text{ subject to } \mathbb{E}\phi = 0 \text{ and } \phi \in [-\infty, 1], \tag{A.9}$$

where α is the parameter from Step 4 (satisfying $1/3 < \alpha < 1$). The objective function above comes from $(1-\omega_{t|s})w_{\mathrm{B},t} + \omega_{t|s}w_{\mathrm{S},t} = \mathbb{E}_{t-1}[\omega_{t|s}\log(1-\phi_t) - \log(1-\alpha\phi_t)]$ in Step 5, and the time subscripts can be dropped because of the time-invariant form of the constraints $\mathbb{E}_{t-1}\phi_t = 0$ and $\phi_t \in [-\infty, 1]$, and because the relative weight on savers $\omega_{t|s}$ is known given date $t-1$ information. It is known from Step 5 that the supremum in (A.9) exists, so the problem has a solution.

Since neither of the exogenous variables g_t and ψ_t appear in the problem (31), this means (A.9) can be stated equivalently as the choice of a distribution function $F(\cdot)$ for ϕ on support $[-\infty, 1]$:

$$\sup_{F(\cdot)}\int_{-\infty}^{1}\upsilon(\phi)\mathrm{d}F(\phi) \text{ s.t. } \int_{-\infty}^{1}\phi\mathrm{d}F(\phi) = 0, \quad \text{where } \upsilon(\phi) = \omega\log(1-\phi) - \log(1-\alpha\phi), \tag{A.10}$$

with $\upsilon(\phi)$ denoting the realized value of social welfare if financial conditions take value ϕ. The probability distribution $F(\phi)$ is generated by the mapping from the exogenous state of the world to the realization of ϕ together with the stochastic process for the exogenous state, but the exact form of this mapping from exogenous state to ϕ has no bearing on the value of the objective function. Any mapping that gives rise to the same probability distribution $F(\phi)$ leads to the same value of the objective function. Since the exogenous variables g_t and ψ_t have continuous distributions, it is possible to generate any valid probability distribution $F(\phi)$ using some mapping from the exogenous state to the realization of ϕ_t.

With democracy (32), the weight on savers is $\omega = 1/3$, but the more general case of $\omega < \alpha$ is considered here (recalling $\alpha > 1/3$). The solution of the problem (A.10) can be analysed using the Lagrangian:

$$\Lambda = \int_{-\infty}^{1}\upsilon(\phi)\mathrm{d}F(\phi) - \mu\int_{-\infty}^{1}\phi\mathrm{d}F(\phi), \tag{A.11}$$

where μ is the multiplier on the constraint $\mathbb{E}\phi = 0$. Taking the first-order conditions of the Lagrangian, if $\upsilon'(\phi) = \mu$ does not hold for every value of $\phi \in (-\infty, 1]$ receiving positive density or mass in the probability distribution $F(\phi)$ then $\mathbb{E}\upsilon(\phi)$ can be increased by a strictly positive amount by choosing a different probability distribution. The first derivative of realized social welfare is $\upsilon'(\phi) = \alpha/(1-\alpha\phi) - \omega/(1-\phi) = (\alpha - \omega - \alpha(1-\omega)\phi)/(1-\alpha\phi)(1-\phi)$. It can be seen that in the range $\phi \in [-\infty, 1]$, since the terms in the denominator are non-negative, $\upsilon'(\phi)$ is positive if $\phi < \hat{\phi}$ and negative if $\phi > \hat{\phi}$ where $\hat{\phi} = (\alpha-\omega)/\alpha(1-\omega)$. The second derivative of $\upsilon(\phi)$ is:

$$\upsilon''(\phi) = \frac{\alpha^2}{(1-\alpha\phi)^2} - \frac{\omega}{(1-\phi)^2} = \frac{\left(\alpha - \sqrt{\omega} - \alpha(1-\sqrt{\omega})\phi\right)\left((1-\alpha)\sqrt{\omega} + \alpha(1+\sqrt{\omega})(1-\phi)\right)}{(1-\alpha\phi)^2(1-\phi)^2},$$

where the second expression follows by taking a common denominator and using the formula for the difference of two squares. The terms in the denominator are non-negative, and for $\phi \in [-\infty, 1]$, the second term in the numerator is strictly positive. It follows that in the range $\phi \in [-\infty, 1]$, $\upsilon''(\phi)$ is positive if $\phi < (\alpha - \sqrt{\omega})/\alpha(1-\sqrt{\omega})$ and negative if $\phi > (\alpha - \sqrt{\omega})/\alpha(1-\sqrt{\omega})$. Hence $\upsilon'(\phi)$ can switch from increasing to decreasing in ϕ at most once, and thus there is a maximum of two values of ϕ where $\upsilon'(\phi) = \mu$ for any μ. Attention can therefore be restricted to probability distributions of ϕ with one or two point masses.

Consider the case of a distribution of ϕ with two mass points, $\phi = \underline{\phi}$ with probability ε, and $\phi = \bar{\phi}$ with probability $1-\varepsilon$. The Lagrangian (A.11) for this simpler problem reduces to:

$$\Lambda = \varepsilon\upsilon(\underline{\phi}) + (1-\varepsilon)\upsilon(\bar{\phi}) - \mu\left(\varepsilon\underline{\phi} + (1-\varepsilon)\bar{\phi}\right).$$

Suppose $\underline{\phi}$ and $\bar{\phi}$ lie in a bounded interval within $(-\infty, 1]$. Since $\varepsilon \in [0,1]$ and $\underline{\phi}$ and $\bar{\phi}$ are restricted to a compact interval on which $\upsilon(\phi)$ is continuous (if $\omega > 0$, a neighbourhood of $\phi = 1$ can be excluded because $\lim_{\phi\to 1}\upsilon(\phi) = -\infty$), there exists a maximum for $\mathbb{E}\upsilon(\phi)$ subject to the constraint $\mathbb{E}\phi = 0$. If this maximum features $\varepsilon \in (0,1)$ then the first-order necessary conditions of the Lagrangian with respect to $\underline{\phi}$, $\bar{\phi}$, and ε are:

$$\upsilon'(\underline{\phi}) = \frac{\upsilon(\bar{\phi}) - \upsilon(\underline{\phi})}{\bar{\phi} - \underline{\phi}} = \upsilon'(\bar{\phi}) = \mu, \quad \text{and} \quad \varepsilon\underline{\phi} + (1-\varepsilon)\bar{\phi} = 0. \tag{A.12}$$

Suppose there exists a solution of these equations for $\underline{\phi}$, $\bar{\phi}$, ε, and μ. Since $\upsilon(\phi)$ is a continuously differentiable function for $\phi \in (-\infty, 1)$, the mean value theorem together with the first two equations in (A.12) implies there exists a $\tilde{\phi}$ such that $\underline{\phi} < \tilde{\phi} < \bar{\phi}$ and $\upsilon'(\tilde{\phi}) = \mu$. But that would mean there are three values of ϕ satisfying the equation $\upsilon'(\phi) = \mu$, which contradicts the properties of $\upsilon(\phi)$ established earlier. It follows the solution of (A.10) is not a two-point distribution within some bounded interval. That leaves two possibilities, a degenerate probability distribution ($\varepsilon = 0$ or $\varepsilon = 1$), or the case where a higher value of $\mathbb{E}\upsilon(\phi)$ can always be attained by choosing $\underline{\phi}$ and $\bar{\phi}$ outside any bounded interval.

If the probability distribution of ϕ is degenerate then the constraint $\mathbb{E}\phi = 0$ requires $\phi = 0$ with probability one. In this case, the value of the objective function would be $\mathbb{E}\upsilon(\phi) = \upsilon(0)$. As $\alpha < 1$ and $\omega \geq 0$, it follows that $\hat{\phi} = (\alpha-\omega)/\alpha(1-\omega)$ does not exceed 1. Since $\upsilon(\phi)$ is increasing for $\phi < \hat{\phi}$ and decreasing for $\phi > \hat{\phi}$ as shown earlier, the unconstrained maximum value of $\upsilon(\phi)$ on the interval $[-\infty, 1]$ is $\upsilon(\hat{\phi})$. This means $\sup\mathbb{E}\upsilon(\phi) \leq \upsilon(\hat{\phi})$. With $\omega < \alpha$, it must be the case that $\hat{\phi} \in (0,1]$, and thus $\upsilon(0) < \upsilon(\hat{\phi})$.

Taking the two-point probability distribution $\underline{\phi}$ and $\bar{\phi}$ with probabilities ε and $1-\varepsilon$, fix the value of $\bar{\phi}$ to $x/(1+x)$ for some $x > 0$, so $0 < \bar{\phi} \leq 1$. Given the constraint $\mathbb{E}\phi = 0$, $\underline{\phi}$ must satisfy the equation $\varepsilon\underline{\phi} + (1-\varepsilon)\bar{\phi} = 0$ and hence $\underline{\phi} = -(1-\varepsilon)x/\varepsilon(1+x)$ conditional on the values of $x > 0$ and $0 < \varepsilon < 1$. This is the probability distribution of ϕ specified in (33). Note that $1 - \underline{\phi} = ((1-\varepsilon)\bar{\phi} + \varepsilon)/\varepsilon$ and $1 - \alpha\underline{\phi} = ((1-\varepsilon)\alpha\bar{\phi} + \varepsilon)/\varepsilon$ and thus $\upsilon(\underline{\phi}) = \omega\log((1-\varepsilon)\bar{\phi} + \varepsilon) - \log((1-\varepsilon)\alpha\bar{\phi} + \varepsilon) + (1-\omega)\log\varepsilon$. With $\lim_{\varepsilon\to 0}\varepsilon\log\varepsilon = 0$ and $\alpha > 0$, it follows that $\lim_{\varepsilon\to 0}\varepsilon\upsilon(\underline{\phi}) = 0$ for any $\bar{\phi} \in (0,1]$. With the two-point probability distribution, the value of the objective function is $\mathbb{E}\upsilon(\phi) = (1-\varepsilon)\upsilon(\bar{\phi}) + \varepsilon\upsilon(\underline{\phi})$, and therefore $\mathbb{E}\upsilon(\phi)$ approaches $\upsilon(\bar{\phi})$ as ε tends to zero. Since $\mathbb{E}\upsilon(\phi)$ can be made arbitrarily close to $\upsilon(\bar{\phi})$ for any $\bar{\phi} \in (0,1]$ while satisfying $\mathbb{E}\phi = 0$, it follows that $\sup\mathbb{E}\upsilon(\phi) \geq \upsilon(\bar{\phi})$ for all $\bar{\phi} \in (0,1]$. Since it is known that $\sup\mathbb{E}\upsilon(\phi) \leq \upsilon(\hat{\phi})$ and $0 < \hat{\phi} \leq 1$, this argument establishes that $\sup\mathbb{E}\upsilon(\phi) = \upsilon(\hat{\phi})$.

As $\sup\mathbb{E}\upsilon(\phi)$ is strictly greater than $\upsilon(0)$, the degenerate distribution of ϕ is not the solution of (A.9). Instead, it is (33) with $x_t = x$ satisfying $x/(1+x) = \hat{\phi} = (\alpha-\omega)/\alpha(1-\omega)$ and ε_t small but positive (where the supremum is approached as $\varepsilon_t \to 0$). Rearranging the equation for x yields $x = (\alpha-\omega)/(1-\alpha)\omega$, which is $(3\alpha-1)/(1-\alpha)$ when $\omega=1/3$ in the case of democracy (32).

A.9 Proof of Step 6

The interest-rate feedback rule for i_t implies

$$\frac{1+i_t}{1+\mathbb{E}_t\pi_{t+1}}=\frac{\zeta\left(\max\left\{\phi_t-\frac{x}{1+x},0\right\}\right)}{\beta\delta\left(1+\frac{(1-\varepsilon)x^2}{x+\varepsilon}\right)}, \tag{A.13}$$

where $x\geq 0$, $0<\varepsilon<1$, and $\zeta(\cdot)$ is a function satisfying $\zeta(0)=1$ and $\zeta'(\cdot)>0$. The requirement that financial conditions ϕ_t are a martingale difference sequence ($\mathbb{E}_{t-1}\phi_t=0$) is strengthened to ϕ_t being an i.i.d. stochastic process, which must have zero mean. Using (20) and the expression for the housing risk premium $\xi_t=\mathbb{E}_t[\phi_{t+1}/(1-\phi_{t+1})]$, it follows that equilibrium interest rates and house-price inflation must satisfy:

$$\frac{1+i_t}{1+\mathbb{E}_t\pi_{t+1}}=\frac{1}{\beta\delta(1+\xi)},\quad \text{with } \xi=\mathbb{E}\left[\frac{\phi_t}{1-\phi_t}\right],$$

where the time-invariance of the housing risk premium comes from the i.i.d. property of ϕ_t. Combining these equations with the interest-rate feedback rule (A.13) implies that the equilibrium probability distribution of financial conditions ϕ_t must satisfy the following in all states of the world:

$$\mathbb{E}\left[\frac{\phi_t}{1-\phi_t}\right]=\frac{1+\frac{(1-\varepsilon)x^2}{x+\varepsilon}}{\zeta\left(\max\left\{\phi_t-\frac{x}{1+x},0\right\}\right)}-1,\quad \text{and } \mathbb{E}[\phi_t]=0.$$

Using $\zeta'(\cdot)>0$, the right-hand side of the first equation is strictly decreasing in ϕ_t for $\phi_t>x/(1+x)$, while the left-hand side does not depend on the particular realization of ϕ_t, so it follows that if the distribution is to have support to the right of $x/(1+x)$ then it must be a degenerate distribution. However, since $x/(1+x)\geq 0$, such a degenerate distribution with $\phi_t>x/(1+x)$ is inconsistent with the equilibrium condition $\mathbb{E}[\phi_t]=0$. Hence, the equilibrium distribution of ϕ_t must have support within the interval $[-\infty,x/(1+x)]$. Using $\zeta(0)=1$, the probability distribution of ϕ_t must therefore satisfy all of:

$$\mathbb{E}\left[\frac{\phi_t}{1-\phi_t}\right]=\frac{(1-\varepsilon)x^2}{x+\varepsilon},\quad \mathbb{E}[\phi_t]=0,\quad \text{and } \phi_t\in\left[-\infty,\frac{x}{1+x}\right]. \tag{A.14}$$

Observe that the distribution of financial conditions in (33) with $x_t=x$ and $\varepsilon_t=\varepsilon$ satisfies $\mathbb{E}[\phi_t]=0$ and lies in the interval $[-\infty,x/(1+x)]$, and moreover:

$$\mathbb{E}\left[\frac{\phi_t}{1-\phi_t}\right]=(1-\varepsilon)\left(\frac{\frac{x}{1+x}}{1-\frac{x}{1+x}}\right)-\varepsilon\left(\frac{\frac{(1-\varepsilon)x}{\varepsilon(1+x)}}{1+\frac{(1-\varepsilon)x}{\varepsilon(1+x)}}\right)=(1-\varepsilon)x-\frac{\varepsilon(1-\varepsilon)x}{x+\varepsilon}=\frac{(1-\varepsilon)x^2}{x+\varepsilon}. \tag{A.15}$$

The probability distribution (33) is therefore consistent with the conditions (A.14). It is now shown that any other distribution satisfying (A.14) as ε approaches zero must approach (33).

Note that $\phi_t/(1-\phi_t)$ is a strictly increasing function of ϕ_t, so subject to $\phi_t\in[-\infty,x/(1+x)]$, the maximum value $\phi_t/(1-\phi_t)$ takes is x, and hence $\mathbb{E}[\phi_t/(1-\phi_t)]\leq x$ for any probability distribution of ϕ_t with support $[-\infty,x/(1+x)]$. Conditional on having to satisfy $\mathbb{E}[\phi_t]=0$, the supremum of $\mathbb{E}_t[\phi_t/(1-\phi_t)]$ exists and cannot exceed x. Taking the limit as $\varepsilon\to 0$ in (A.15), $\mathbb{E}[\phi_t/(1-\phi_t)]$ approaches x, hence $\sup\mathbb{E}[\phi_t/(1-\phi_t)]=x$ subject to $\mathbb{E}[\phi_t]=0$ and $\phi_t\in[-\infty,x/(1+x)]$. Since $\phi_t/(1-\phi_t)$ is strictly convex in ϕ_t, the expectation $\mathbb{E}[\phi_t/(1-\phi_t)]$ cannot be maximized subject to $\mathbb{E}[\phi_t]=0$ with positive mass or density in the interior of $[-\infty,x/(1+x)]$. Therefore, as ε tends to zero, a probability distribution of ϕ_t consistent with (A.14) must approach (33) for small but positive ε.

A.10 Proof of Step 7

Given the minimum housing need $\underline{H}$, but utility from housing in excess of $\underline{H}$ is received only by the middle-aged, it follows that subject to the housing resource constraint (6b), any Pareto efficient allocation must have $H_{y,t} = \underline{H}$, $H_{o,t} = \underline{H}$, and $H_{m,t} = L - 2\underline{H}$, the same allocation of housing as in any competitive equilibrium.

The allocation of consumption goods is subject to (6a) for all $t \geq s$. Considering the allocation at just one specific date t across the three generations alive at that date, Pareto efficiency requires that $C_{y,t}$, $C_{m,t}$, and $C_{o,t}$ maximize one of the expected utilities $\mathbb{E}_{s-1}U_{y,t}$, $\mathbb{E}_{s-1}U_{m,t}$, or $\mathbb{E}_{s-1}U_{o,t}$ from (28) subject to (6a) and to achieving given values of the other expected utilities. The first-order conditions for this problem imply $C_{a,t}$ is the ratio of a $s-1$-measurable term (specific to a) and the Lagrangian multiplier on the date-t resource constraint $C_{y,t} + C_{m,t} + C_{o,t} = Y_t$. This demonstrates $C_{a,t}/\mathbb{E}_{s-1}C_{a,t}$ is the equalized for all $a \in \{y, m, o\}$, that is, the unpredictable component of each person's consumption is proportional to their own expected consumption: a risk-sharing condition. Given that $c_{a,t} = C_{a,t}/Y_t$, this can be equivalently expressed as $c_{a,t}/\mathbb{E}_{s-1}c_{a,t}$ equalized across $a \in \{y, m, o\}$, which must hold for $t \geq s$.

Now consider the allocation of consumption for two adjacent dates t and $t+1$ and only for those individuals whose lives overlap on both dates, namely those who are young or middle-aged at date t, and who will be middle-aged or old at date $t+1$. Pareto efficiency requires maximizing either expected utility $\mathbb{E}_{s-1}U_{y,t}$ or $\mathbb{E}_{s-1}U_{m,t}$ subject to a given value of the other expected utility and the resource constraints (6a) at dates t and $t+1$. This implies both $C_{m,t+1}/C_{y,t}$ and $C_{o,t+1}/C_{m,t}$ are equal to a common term proportional to the ratio of the Lagrangian multipliers on the dates $t+1$ and t resource constraints. It follows that $C_{m,t+1}/C_{y,t} = C_{o,t+1}/C_{m,t}$, which means expected consumption growth must be equalized across generations whose lives overlap, that is, $\mathbb{E}_t C_{m,t+1}/C_{y,t} = \mathbb{E}_t C_{o,t+1}/C_{m,t}$: a consumption smoothing condition. Since $c_{a,t} = C_{a,t}/Y_t$, those conditions can be stated equivalently as $c_{m,t+1}/c_{y,t} = c_{o,t+1}/c_{m,t}$ and $\mathbb{E}_t c_{m,t+1}/c_{y,t} = \mathbb{E}_t c_{o,t+1}/c_{m,t}$.

The perpetual sequence of generations means that it is also necessary to consider whether adjustments to the consumption allocation across an infinite number of time periods can make some generation better off without any being worse off. Starting from an initial allocation that fails $\liminf_{t\to\infty} \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}] \leq 1/\beta$, suppose at each date t from s onwards, a positive fraction τ_t of the consumption of the middle-aged $C_{m,t}$ is transferred to the old at that date. Feasibility requires $\tau_t < 1$ for all t. This makes the initial generation of old at date s better off. For infinitesimal τ_t, the conditions for all other generations to be no worse off is $\tau_t C_{m,t}/C_{m,t} \leq \beta \mathbb{E}_t[\tau_{t+1} C_{m,t+1}/C_{o,t+1}]$. Using the definition of $c_{a,t} = C_{a,t}/Y_t$, this is equivalent to $\mathbb{E}_t[(\tau_{t+1}/\tau_t)(c_{m,t+1}/c_{o,t+1})] \geq 1/\beta$.

Start from a given τ_s. A sequence $\{\tau_t\}_{t=s}^{\infty}$ is constructed recursively using $\tau_{t+1} = (\beta \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}])^{-1}\tau_t$ for all $t \geq s$. Note that this implies $\mathbb{E}_t[(\tau_{t+1}/\tau_t)(c_{m,t+1}/c_{o,t+1})] = 1/\beta$. The recursion can be iterated to deduce $\tau_t = \tau_s \prod_{\ell=1}^{t-s}(\beta \mathbb{E}_{s+\ell-1}[c_{m,s+\ell}/c_{o,s+\ell}])^{-1}$. Since $\beta \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}]$ is non-negative for all t, it follows that $\limsup_{t\to\infty}(\beta \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}])^{-1} = (\liminf_{t\to\infty} \beta \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}])^{-1}$, and hence the property of the allocation $\liminf_{t\to\infty} \beta \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}] > 1$ implies $\limsup_{t\to\infty}(\beta \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}])^{-1} < 1$. Thus, the constructed sequence satisfies $\lim_{t\to\infty} \tau_t = 0$ for any τ_s, and hence the whole sequence $\{\tau_t\}$ can be made arbitrarily small for an appropriate choice of τ_s. Therefore, there exists a feasible sequence of transfers that makes one generation better off without making any other generation worse off. This means a necessary condition for Pareto efficient is $\liminf_{t\to\infty} \mathbb{E}_t[c_{m,t+1}/c_{o,t+1}] \leq 1/\beta$, which the nature of the transfers in this argument reveals to be a dynamic efficiency condition. Exactly analogous reasoning for transfers from the young to the middle-aged shows that $\liminf_{t\to\infty} \mathbb{E}_t[c_{y,t+1}/c_{m,t+1}] \leq 1/\beta$ is also a requirement for dynamic efficiency.

Now take the social welfare function (29). Since the welfare-maximizing allocation of housing must coincide with the competitive-equilibrium allocation, and as Y_t is taken as given by the planner, the same steps from the proof of Step 5 used to derive the expression for social welfare in (A.8) apply here, and hence:

$$W_s = \mathbb{E}_{s-1}\left[\sum_{t=s}^{\infty}\left\{\tilde{\Omega}_{t|s}\log c_{y,t} + \beta\tilde{\Omega}_{t-1|s}\log c_{m,t} + \beta^2\tilde{\Omega}_{t-2|s}\log c_{o,t}\right\}\right] + \text{t.i.p.}, \tag{A.16}$$

where $\tilde{\Omega}_{s-2|s} = \beta^{-2}\Omega_{s-2|s}$, $\tilde{\Omega}_{s-1|s} = \beta^{-1}\Omega_{s-1|s}$, and $\tilde{\Omega}_{t|s} = \Omega_{t|s}$ for all $t \geq s$ denote the scaled sequence of positive and $s-1$-measurable weights, and t.i.p. is now terms independent of the planner's choice of the consumption allocation $c_{a,t}$. Social welfare (A.16) depends only on $c_{a,t}$, and the resource constraint (6a) is equivalent to (13) in terms of $c_{a,t}$. Following the proof of Step 5, existence of a finite supremum of W_s subject to the resource constraints requires $\sum_{t=s}^{\infty}\tilde{\Omega}_{t|s} < \infty$ (equivalent to $\sum_{t=s}^{\infty}\Omega_{t|s} < \infty$), and is guaranteed by $\sum_{t=s}^{\infty} t\tilde{\Omega}_{t|s} < \infty$. The Lagrangian for maximizing (A.16) subject to (13) for all $t \geq s$ is:

$$\Lambda_s = \mathbb{E}_{s-1}\left[\sum_{t=s}^{\infty}\left\{\tilde{\Omega}_{t|s}\log c_{\mathrm{y},t} + \beta\tilde{\Omega}_{t-1|s}\log c_{\mathrm{m},t} + \beta^2\tilde{\Omega}_{t-2|s}\log c_{\mathrm{o},t} + \Psi_{t|s}\left(1 - c_{\mathrm{y},t} - c_{\mathrm{m},t} - c_{\mathrm{o},t}\right)\right\}\right],$$

where $\Psi_{t|s}$ denotes the Lagrangian multiplier on the date-t resource constraint for the allocation chosen from date s. The first-order conditions are:

$$\frac{\tilde{\Omega}_{t|s}}{c_{\mathrm{y},t}} = \frac{\beta\tilde{\Omega}_{t-1|s}}{c_{\mathrm{m},t}} = \frac{\beta^2\tilde{\Omega}_{t-2|s}}{c_{\mathrm{o},t}} = \Psi_{t|s}, \tag{A.17}$$

and these first-order conditions can be rearranged to deduce $c_{\mathrm{y},t} = \tilde{\Omega}_{t|s}/\Psi_{t|s}$, $c_{\mathrm{m},t} = \beta\tilde{\Omega}_{t-1|s}/\Psi_{t|s}$, and $c_{\mathrm{o},t} = \beta^2\tilde{\Omega}_{t-2|s}$. It follows immediately that $c_{\mathrm{y},t}/\mathbb{E}_{s-1}c_{\mathrm{y},t} = c_{\mathrm{m},t}/\mathbb{E}_{s-1}c_{\mathrm{m},t} = c_{\mathrm{o},t}/\mathbb{E}_{s-1}c_{\mathrm{o},t}$ and $\mathbb{E}_t c_{\mathrm{m},t+1}/c_{\mathrm{y},t} = \mathbb{E}_t c_{\mathrm{o},t+1}/c_{\mathrm{m},t}$. Observing $\mathbb{E}_t[c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}] = (\tilde{\Omega}_{t|s}/\tilde{\Omega}_{t-1|s})/\beta$ and $\mathbb{E}_t[c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1}] = (\tilde{\Omega}_{t+1|s}/\tilde{\Omega}_{t|s})/\beta$ and noting $\liminf_{t\to\infty}\tilde{\Omega}_{t+1|s}/\tilde{\Omega}_{t|s} \leq 1$ is necessary for $\sum_{t=s}^{\infty}\tilde{\Omega}_{t|s} < \infty$ implies that $\liminf_{t\to\infty}\mathbb{E}_t[c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1}] \leq 1/\beta$ and $\liminf_{t\to\infty}\mathbb{E}_t[c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}] \leq 1/\beta$ must hold.

Conversely, consider a consumption allocation $\{c_{a,t}\}_{t=s}^{\infty}$ that satisfies (13), $c_{\mathrm{y},t}/\mathbb{E}_{s-1}c_{\mathrm{y},t} = c_{\mathrm{m},t}/\mathbb{E}_{s-1}c_{\mathrm{m},t} = c_{\mathrm{o},t}/\mathbb{E}_{s-1}c_{\mathrm{o},t}$, and $\mathbb{E}_t c_{\mathrm{m},t+1}/c_{\mathrm{y},t} = \mathbb{E}_t c_{\mathrm{o},t+1}/c_{\mathrm{m},t}$ for all $t \geq s$, and $\limsup_{t\to\infty}\mathbb{E}_t[c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1}] < 1/\beta$ and $\limsup_{t\to\infty}\mathbb{E}_t[c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}] < 1/\beta$. It will be shown there exists a well-defined sequence of weights $\{\tilde{\Omega}_t\}_{t=s-2}^{\infty}$ and Lagrangian multipliers $\{\Psi_{t|s}\}_{t=s}^{\infty}$ such that the allocation satisfies the first-order conditions (A.17) and the social welfare function (A.16) converges. Note that the first-order conditions are homogeneous of degree zero in the weights and the multipliers, so one normalization (a scaling by a $s-1$-measurable variable) can be imposed without loss of generality. The normalization adopted is $\tilde{\Omega}_{s-2|s} = 1$.

The sequence of weights is constructed recursively for all $t \geq s$ using $\tilde{\Omega}_{t|s} = \beta(c_{\mathrm{y},t}/c_{\mathrm{m},t})\tilde{\Omega}_{t-1|s}$ starting from $\tilde{\Omega}_{s-2|s} = 1$ and $\tilde{\Omega}_{s-1|s} = \beta c_{\mathrm{m},s}/c_{\mathrm{o},s}$. The risk-sharing condition implies $c_{\mathrm{y},t}/c_{\mathrm{m},t} = \mathbb{E}_{s-1}c_{\mathrm{y},t}/\mathbb{E}_{s-1}c_{\mathrm{m},t}$ and $c_{\mathrm{m},t}/c_{\mathrm{o},t} = \mathbb{E}_{s-1}c_{\mathrm{m},t}/\mathbb{E}_{s-1}c_{\mathrm{o},t}$, hence the ratios $c_{\mathrm{y},t}/c_{\mathrm{m},t}$ and $c_{\mathrm{m},t}/c_{\mathrm{o},t}$ are $s-1$-measurable for all $t \geq s$. The definition of the weights $\{\tilde{\Omega}_{t|s}\}_{t=s-2}^{\infty}$ therefore results in a strictly positive and $s-1$-measurable sequence.

Now consider any $t > s$. The risk-sharing condition requires $c_{\mathrm{m},t}/\mathbb{E}_{s-1}c_{\mathrm{m},t} = c_{\mathrm{o},t}/\mathbb{E}_{s-1}c_{\mathrm{o},t}$ and taking expectations conditional on date $t-1$ information and rearranging yields $\mathbb{E}_{s-1}c_{\mathrm{m},t}/\mathbb{E}_{s-1}c_{\mathrm{o},t} = \mathbb{E}_{t-1}c_{\mathrm{m},t}/\mathbb{E}_{t-1}c_{\mathrm{o},t}$. Combining this with the consumption smoothing condition $\mathbb{E}_{t-1}c_{\mathrm{m},t}/c_{\mathrm{y},t-1} = \mathbb{E}_{t-1}c_{\mathrm{o},t}/c_{\mathrm{m},t-1}$ yields $c_{\mathrm{m},t}/c_{\mathrm{o},t} = c_{\mathrm{y},t-1}/c_{\mathrm{m},t-1}$. Hence, using the recursive definition of $\tilde{\Omega}_{t|s}$ for any $t > s$, $\tilde{\Omega}_{t-1|s} = \beta(c_{\mathrm{y},t-1}/c_{\mathrm{m},t-1})\tilde{\Omega}_{t-2|s} = \beta(c_{\mathrm{m},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s}$ and $\tilde{\Omega}_{t|s} = \beta(c_{\mathrm{y},t}/c_{\mathrm{m},t})\tilde{\Omega}_{t-1|s} = \beta(c_{\mathrm{y},t}/c_{\mathrm{m},t})\beta(c_{\mathrm{m},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s} = \beta^2(c_{\mathrm{y},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s}$. Note that since $\tilde{\Omega}_{s-2|s} = 1$, the definition of $\tilde{\Omega}_{s-1|s}$ directly implies $\tilde{\Omega}_{s-1} = \beta(c_{\mathrm{m},s}/c_{\mathrm{o},s})\tilde{\Omega}_{s-2|s}$, and substituting the definition of $\tilde{\Omega}_{s-1|s}$ into $\tilde{\Omega}_{s|s}$ implies $\tilde{\Omega}_{s|s} = \beta^2(c_{\mathrm{y},s}/c_{\mathrm{o},s})\tilde{\Omega}_{s-2|s}$. Putting these results together, it has been shown that $\tilde{\Omega}_{t-1|s} = \beta(c_{\mathrm{m},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s}$ and $\tilde{\Omega}_{t|s} = \beta^2(c_{\mathrm{y},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s}$ for all $t \geq s$.

The recursion for $\tilde{\Omega}_{t|s}$ gives $\mathbb{E}_t[c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1}] = (\tilde{\Omega}_{t+1|s}/\tilde{\Omega}_{t|s})/\beta$ and $\mathbb{E}_t[c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}] = (\tilde{\Omega}_{t|s}/\tilde{\Omega}_{t-1|s})/\beta$, so the dynamic efficiency conditions $\limsup_{t\to\infty}\mathbb{E}_t[c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1}] < 1/\beta$ and $\limsup_{t\to\infty}\mathbb{E}_t[c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}] < 1/\beta$ imply that the sequence $\tilde{\Omega}_{t|s}$ must satisfy $\limsup_{t\to\infty}\tilde{\Omega}_{t|s}/\tilde{\Omega}_{t-1|s} < 1$. Since the limit of $t/(t-1)$ is 1 as t becomes large, this also shows $\limsup_{t\to\infty}(t\tilde{\Omega}_{t|s})/((t-1)\tilde{\Omega}_{t-1|s}) < 1$, which guarantees $\sum_{t=s}^{\infty} t\tilde{\Omega}_{t|s} < \infty$. The argument from the proof of Step 5 then demonstrates that the social welfare function is finite and has a well-defined constrained maximum subject to (13). That constrained maximum is characterized by the first-order conditions (A.17).

Define $\Psi_{t|s} = \tilde{\Omega}_{t|s} + \beta\tilde{\Omega}_{t-1|s} + \beta^2\tilde{\Omega}_{t-2|s}$ for all $t \geq s$. Using the earlier results for $\tilde{\Omega}_{t|s}$ it follows that $\Psi_{t|s} = \beta^2(c_{\mathrm{y},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s} + \beta^2(c_{\mathrm{m},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s} + \beta^2(c_{\mathrm{o},t}/c_{\mathrm{o},t})\tilde{\Omega}_{t-2|s}$. Since the resource constraint (13) requires $c_{\mathrm{y},t} + c_{\mathrm{m},t} + c_{\mathrm{o},t} = 1$, this means that $\Psi_{t|s} = \beta^2\tilde{\Omega}_{t-2|s}/c_{\mathrm{o},t}$. Using $\beta^2\tilde{\Omega}_{t-2|s} = \beta\tilde{\Omega}_{t-1|s}c_{\mathrm{o},t}/c_{\mathrm{m},t}$ and $\beta^2\tilde{\Omega}_{t-2|s} = \tilde{\Omega}_{t-1|s}c_{\mathrm{o},t}/c_{\mathrm{y},t}$, it follows that the weights and Lagrangian multipliers satisfy all the first-order conditions (A.17) for the consumption allocation $\{c_{a,t}\}_{t=s}^{\infty}$. The original sequence of weights $\{\Omega_{t|s}\}_{t=s-2}^{\infty}$ can be recovered from $\{\tilde{\Omega}_{t|s}\}_{t=s-2}^{\infty}$ using $\Omega_{s-2|s} = \beta^2\tilde{\Omega}_{s-2|s}$, $\Omega_{s-1|s} = \beta\tilde{\Omega}_{s-1|s}$, and $\Omega_{t|s} = \tilde{\Omega}_{t|s}$ for all $t \geq s$. The allocation is the solution of the planner's problem for some set of weights, so it is Pareto efficient.

A.11 Proof of Step 8

In respect of risk sharing, note that (27) implies $c_{y,t} = \delta c_{m,t}$, so $c_{y,t}/\mathbb{E}_{s-1}c_{y,t} = c_{m,t}/\mathbb{E}_{s-1}c_{m,t}$ holds automatically for all t even though markets are incomplete. Suppose $c_{m,t}/\mathbb{E}_{t-1}c_{m,t} = c_{o,t}/\mathbb{E}_{t-1}c_{o,t}$ holds. Using (27) to deduce $c_{o,t}/c_{m,t} = (1-\phi_t)/\delta$, it follows that $\phi_t = 1 - \delta\mathbb{E}_{t-1}c_{o,t}/\mathbb{E}_{t-1}c_{m,t}$. This means ϕ_t is $t-1$-measurable, and hence $\phi_t = 0$ with probability one because of (22). Conversely, if $\phi_t = 0$ with probability one then (27) implies $c_{m,t} = \alpha\delta$ and $c_{o,t} = \alpha$. Therefore, $\phi_t = 0$ with probability one is equivalent to $c_{m,t}/\mathbb{E}_{t-1}c_{m,t} = c_{o,t}/\mathbb{E}_{t-1}c_{o,t}$.

The remaining requirement for risk sharing is $c_{m,t}/\mathbb{E}_{s-1}c_{m,t} = c_{o,t}/\mathbb{E}_{s-1}c_{o,t}$ for all $t \geq s$. If $\phi_t = 0$ with probability one for all $t \geq s$ then (27) immediately implies that risk sharing condition holds. Conversely, $c_{m,t}/\mathbb{E}_{s-1}c_{m,t} = c_{o,t}/\mathbb{E}_{s-1}c_{o,t}$ implies $\mathbb{E}_{t-1}c_{m,t}/\mathbb{E}_{s-1}c_{m,t} = \mathbb{E}_{t-1}c_{o,t}/\mathbb{E}_{s-1}c_{o,t}$ for any $t \geq s$ and thus $c_{m,t}/\mathbb{E}_{t-1}c_{m,t} = c_{o,t}/\mathbb{E}_{t-1}c_{o,t}$, which is known to imply $\phi_t = 0$ with probability one. The risk sharing conditions are therefore equivalent in the incomplete-markets economy either to $c_{m,t}/\mathbb{E}_{t-1}c_{m,t} = c_{o,t}/\mathbb{E}_{t-1}c_{o,t}$ or $\phi_t = 0$ with probability one. Finally, using (22), $\phi_t = 0$ with probability one is equivalent to nominal house-price inflation π_t being $t-1$-measurable, that is, $\pi_t = \mathbb{E}_{t-1}\pi_t$ with probability one.

In respect of consumption smoothing, consider the inequality $\mathbb{E}_t[(c_{m,t+1}/c_{y,t}) - (c_{o,t+1}/c_{m,t})] \geq 0$. Since (27) implies $c_{y,t} = \delta c_{m,t}$, which is non-negative, the inequality is equivalent to $\mathbb{E}_t[(c_{m,t+1}/\delta) - c_{o,t+1}] \geq 0$. Substituting from (27), $(c_{m,t+1}/\delta) - c_{o,t+1} = \alpha/(1-\alpha\phi_{t+1}) - \alpha(1-\phi_{t+1})/(1-\alpha\phi_{t+1}) = \alpha\phi_{t+1}/(1-\alpha\phi_{t+1})$, and therefore the inequality is equivalent to $\mathbb{E}_t[\phi_{t+1}/(1-\alpha\phi_{t+1})] \geq 0$, cancelling the positive term α. Since $\phi_{t+1}/(1-\alpha\phi_{t+1})$ is a strictly convex function of ϕ_{t+1}, and $\mathbb{E}_t\phi_{t+1} = 0$ given (22), Jensen's inequality implies the inequality always holds, and holds with equality if and only if $\phi_{t+1} = 0$ with probability one.

Using $\xi_t = \mathbb{E}_t[\phi_{t+1}/(1-\phi_{t+1})] \geq 0$, having $\phi_{t+1} = 0$ with probability one is equivalent to $\xi_t = 0$, which is in turn equivalent to $1 + i_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta$ given (20). Therefore, it has been shown that $\mathbb{E}_t[(c_{m,t+1}/c_{y,t}) - (c_{o,t+1}/c_{m,t})] = 0$, which is the consumption smoothing condition $\mathbb{E}_t c_{m,t+1}/c_{y,t} = \mathbb{E}_t c_{o,t+1}/c_{m,t}$, holds if and only if $i_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta - 1$. Consumption smoothing is implied by risk sharing because that gives $\phi_{t+1} = 0$ with probability one. Since the earlier inequality always holds, a failure of consumption smoothing must mean $\mathbb{E}_t c_{m,t+1}/c_{y,t} > \mathbb{E}_t c_{o,t+1}/c_{m,t}$, which is associated with $i_t < (1+\mathbb{E}_t\pi_{t+1})/\beta\delta - 1$ because $\xi_t > 0$.

Finally, in respect of dynamic efficiency, observe that (27) implies $c_{y,t+1}/c_{m,t+1} = \delta$, so it follows that $\limsup_{t\to\infty}\mathbb{E}_t[c_{y,t+1}/c_{m,t+1}] = \delta$. Since Step 3 implies $\beta\delta < 1$, this means that $\liminf_{t\to\infty}\mathbb{E}_t[c_{y,t+1}/c_{m,t+1}] < 1/\beta$. Equation (27) implies $\mathbb{E}_t[c_{m,t+1}/c_{o,t+1}] = \delta\mathbb{E}_t[1/(1-\phi_{t+1})] = \delta(1+\xi_t)$ noting that $1+\xi_t = \mathbb{E}_t[1/(1-\phi_{t+1})]$. Using (20) implies $1+\xi_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta(1+i_t)$, and hence $\mathbb{E}_t[c_{m,t+1}/c_{o,t+1}] = (1+\mathbb{E}_t\pi_{t+1})/\beta(1+i_t)$. Since β is positive and finite, the requirement $\liminf_{t\to\infty}\mathbb{E}_t[c_{m,t+1}/c_{o,t+1}] \leq 1/\beta$ is thus equivalent to $\liminf_{t\to\infty}(\mathbb{E}_t\pi_{t+1} - i_t)/(1+i_t) \leq 0$. As long as i_t is bounded away from -1 and ∞, this condition becomes $\liminf_{t\to\infty}(\mathbb{E}_t\pi_{t+1} - i_t) \leq 0$, or equivalently $\limsup_{t\to\infty}(i_t - \mathbb{E}_t\pi_{t+1}) \geq 0$. Note that risk sharing or consumption smoothing entail $i_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta - 1 > \mathbb{E}_t\pi_{t+1}$ because $0 < \beta\delta < 1$, so they imply that the dynamic efficiency conditions hold.

A.12 Proof of Result 4

There exist monetary policy regimes where the path of nominal house-price inflation π_t is predictable one time period in advance, that is, $\pi_t = \mathbb{E}_{t-1}\pi_t$ with probability one. Using (22), this implies financial conditions will satisfy $\phi_t = 0$ with probability one. Since $\xi_t = \mathbb{E}_t[\phi_{t+1}/(1-\phi_{t+1})]$, this in turn implies $\xi_t = 0$ for all t. Equation (20) then implies interest rate $i_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta - 1$, so the required nominal interest rate is higher than that in a equilibrium with financial crises (see 34 with $x_{t+1} > 0$).

Using Step 4 and $\phi_t = 0$, the house-price income and debt-to-GDP ratios are constant at $h_t = h^*$ and $d_t = d^*$, where $h^* = \alpha\beta\delta/\lambda$ and $d^* = \alpha\beta\delta$. In an equilibrium with financial crises and credit booms of size x, the house-price income ratio in the credit boom is $\bar{h} = \alpha\beta\delta(1+x)/\lambda(1+(1-\alpha)x)$, which is higher than h^* for any $x>0$ because $\alpha>0$. Likewise, the debt-to-GDP ratio in a credit boom is $\bar{d}=\alpha\beta\delta(1+x)/(1+(1-\alpha)x)$, which is always higher than d^*. The formulas for h_t and d_t given in Step 4 are strictly convex functions of ϕ_t, and as $\mathbb{E}_{t-1}\phi_t=0$ must hold, it follows that $\mathbb{E}h_t>h^*$ and $\mathbb{E}d_t>d^*$ in an equilibrium with financial crises.

Finally, $\pi_t=\mathbb{E}_{t-1}\pi_t$ is equivalent to the risk-sharing condition for efficiency according to Step 8, and $i_t=(1+\mathbb{E}_t\pi_{t+1})/\beta\delta-1$ is equivalent to the consumption smoothing condition. These imply the dynamic efficiency condition using Step 8, so the equilibrium with financial stability is a first-best allocation.

A.13 Proof of Result 5

An equilibrium with financial crises has financial conditions ϕ_t given by (33) for $x_t>0$ and small $\varepsilon_t>0$. Using (22), the non-degenerate distribution of ϕ_t implies $\pi_t\neq\mathbb{E}_{t-1}\pi_t$ with probability one, so given Step 8 the risk sharing requirement for efficiency is violated. The interest rate satisfies (34) implying $i_t>(1+\mathbb{E}_t\pi_{t+1})/\beta\delta-1$, hence the consumption smoothing requirement for efficiency is violated according to Step 8.

Considering a case with $x_t=x$ and small ε_t, it is known that $\xi_t=x$, and hence $1+i_t=(1+\mathbb{E}_t\pi_{t+1})/\beta\delta(1+x)$ using (20). Rearranging this equation leads to $\beta\delta(1+x)-1=(\mathbb{E}_t\pi_{t+1}-i_t)/(1+i_t)$, and thus there is dynamic inefficiency if $\beta\delta(1+x)-1>0$ according to Step 8, that is, $x>1/\beta\delta-1$. Using $x=(3\alpha-1)/(1-\alpha)$ from Result 3 and $\alpha=1/(1+\delta+\delta^2)$ from Step 4, it follows that $x=2/\delta(1+\delta)-1$, and hence dynamic inefficiency occurs if $2/\delta(1+\delta)>1/\beta\delta$, which is equivalent to $\delta<2\beta-1$. This inequality may or may not hold depending on parameters. If $\beta<1/2$ then $2\beta-1$ is negative, and since $\delta>0$ it follows that the equilibrium is dynamically efficient because the inequality is not true. But if $\beta>1$ then $2\beta-1>1$, implying the inequality must be true because $\delta<1$. Therefore, an equilibrium with financial crises is never a first-best allocation as risk sharing and consumption smoothing fail, but it may or may not be dynamically efficient.

Suppose ε_t is small for all t. Now consider an alternative allocation of consumption from some start date s for all $t\geq s$. Taking the values of x_t from the probability distributions of financial conditions in (33), define:

$$\tilde{c}_{\mathrm{y},t}=\frac{\alpha\delta^2}{1-\alpha\frac{x_t}{1+x_t}},\quad \tilde{c}_{\mathrm{m},t}=\frac{\alpha\delta}{1-\alpha\frac{x_t}{1+x_t}},\quad \text{and } \tilde{c}_{\mathrm{o},t}=\frac{\alpha\left(1-\frac{x_t}{1+x_t}\right)}{1-\alpha\frac{x_t}{1+x_t}}. \tag{A.18}$$

Noting that $\alpha(1+\delta+\delta^2)=1$ from Step 4, this allocation is feasible because it satisfies (13). The new consumption allocation (A.18) implies $\tilde{c}_{a,t}=\mathbb{E}_{t-1}\tilde{c}_{a,t}$ with probability one, so it features risk sharing (see Step 7), unlike the original equilibrium $c_{a,t}$.

The consumption smoothing requirement for efficiency from Step 7 is $(\mathbb{E}_t c_{\mathrm{m},t+1}/c_{\mathrm{y},t})/(\mathbb{E}_t c_{\mathrm{o},t+1}/c_{\mathrm{m},t})=1$. For the new allocation (A.18), it follows that $(\mathbb{E}_t\tilde{c}_{\mathrm{m},t+1}/\tilde{c}_{\mathrm{y},t})/(\mathbb{E}_t\tilde{c}_{\mathrm{o},t+1}/\tilde{c}_{\mathrm{m},t})=(\tilde{c}_{\mathrm{m},t+1}/\tilde{c}_{\mathrm{o},t+1})/\delta=1+x_{t+1}$, noting $\tilde{c}_{\mathrm{y},t}=\delta\tilde{c}_{\mathrm{m},t}$. This means consumption smoothing fails to the extent that $x_{t+1}>0$. With the probability distribution of ϕ_t taking the form (33), observe that:

$$\lim_{\varepsilon_t\to 0}\mathbb{E}_{t-1}\left[\frac{1}{1-\alpha\phi_t}\right]=\lim_{\varepsilon_t\to 0}\left((1-\varepsilon_t)\frac{1}{1-\alpha\frac{x_t}{1+x_t}}+\varepsilon_t\frac{\varepsilon_t(1+x_t)}{\alpha(1-\varepsilon_t)x_t+\varepsilon_t(1+x_t)}\right)=\frac{1}{1-\alpha\frac{x_t}{1+x_t}}.$$

Similar reasoning shows that $\lim_{\varepsilon_t\to 0}\mathbb{E}_{t-1}[(1-\phi_t)^{-1}]=1+x_t$ and $\lim_{\varepsilon_t\to 0}\mathbb{E}_{t-1}[(1-\phi_t)/(1-\alpha\phi_t)]=(1-x_t/(1+x_t))/(1-\alpha x_t/(1+x_t))$. Using (27) and comparing to (A.18), these observations imply the consumption allocation in an equilibrium with financial crises satisfies $\lim_{\varepsilon_t\to 0}\mathbb{E}_{t-1}c_{\mathrm{a},t}=\tilde{c}_{\mathrm{a},t}$ for all $a\in\{\mathrm{y},\mathrm{m},\mathrm{o}\}$. Thus, it follows that $\lim_{\varepsilon_{t+1}\to 0}(\mathbb{E}_t c_{\mathrm{m},t+1}/c_{\mathrm{y},t})/(\mathbb{E}_t c_{\mathrm{o},t+1}/c_{\mathrm{m},t})=(\tilde{c}_{\mathrm{m},t+1}/\tilde{c}_{\mathrm{o},t+1})/\delta=1+x_{t+1}$ noting that $c_{\mathrm{y},t}=\delta c_{\mathrm{m},t}$.

For dynamic efficiency, note that both $c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1}=\delta$ and $\tilde{c}_{\mathrm{y},t+1}/\tilde{c}_{\mathrm{m},t+1}=\delta$ so the first requirement from Step 7 is satisfied both by the equilibrium with financial crises and the new allocation (A.18). The new allocation implies $\tilde{c}_{\mathrm{m},t+1}/\tilde{c}_{\mathrm{o},t+1}=\delta(1+x_{t+1})$. The equilibrium consumption levels are given by (27) and hence $c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}=\delta(1-\phi_{t+1})^{-1}$. Since $\lim_{\varepsilon_{t+1}\to 0}\mathbb{E}_t[(1-\phi_{t+1})^{-1}]=1+x_{t+1}$ using the arguments above, this means $\lim_{\varepsilon_{t+1}\to 0}\mathbb{E}_t[c_{\mathrm{m},t+1}/c_{\mathrm{o},t+1}]=\mathbb{E}_t[\tilde{c}_{\mathrm{m},t+1}/\tilde{c}_{\mathrm{o},t+1}]=\delta(1+x_{t+1})$. It follows that if the probabilities of financial crises are small, moving from the equilibrium $c_{a,t}$ to the new allocation $\tilde{c}_{a,t}$ has a negligible effect on whether the consumption smoothing and dynamic efficiency requirements for efficiency are satisfied.

The probability distribution of financial conditions ϕ_t in (33) also implies:

$$\lim_{\varepsilon_t \to 0} \mathbb{E}_{t-1}[\log(1-\alpha\phi_t)] = \lim_{\varepsilon_t \to 0} \left((1-\varepsilon_t)\log\left(1-\alpha\frac{x_t}{1+x_t}\right) + \varepsilon_t \log\left(\frac{\alpha(1-\varepsilon_t)x_t}{1+x_t} + \varepsilon_t\right) - \varepsilon_t \log \varepsilon_t \right)$$
$$= \log\left(1-\alpha\frac{x_t}{1+x_t}\right),$$

which makes use of the result $\lim_{\varepsilon \to 0} \varepsilon \log \varepsilon = 0$. Similar steps can be used to show $\lim_{\varepsilon_t \to 0} \mathbb{E}_{t-1}[\log(1-\phi_t)] = \log(1-x_t/(1+x_t))$. Using the equilibrium consumption levels (27) and comparing to the new allocation (A.18), it follows that $\lim_{\varepsilon_t \to 0} \mathbb{E}_{t-1}[\log c_{a,t}] = \mathbb{E}_{t-1}[\log \tilde{c}_{a,t}] = \log \tilde{c}_{a,t}$ for all $a \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}$. So, for small probabilities of financial crises, the welfare consequences of moving from the equilibrium to the new consumption allocation (A.18) are negligible. Since the new allocation satisfies risk sharing, but fails consumption smoothing (and possibly dynamic efficiency) to the same extent as the original equilibrium, this demonstrates that the welfare loss from the failure of risk sharing in the equilibrium with financial crises is negligible when the probability of a crisis is small. Any non-negligible welfare costs must therefore come from the failure of consumption smoothing (and possibly dynamic efficiency).

Now consider a deterministic economy where the values of all future variables are known with certainty. Suppose there is a proportional tax levied on the capital value of loans. The tax rate is κ_t at date t and the proceeds are rebated as a lump-sum transfer K_t to the group of taxpayers. In particular, if a loan of value $D_t = -Q_t B_{\mathrm{y},t}$ is made then a borrower receives an amount $(1-\kappa_t)D_t$ after the tax. The budget identity of the young in (4a) is replaced by:

$$C_{\mathrm{y},t} + \frac{V_t H_{\mathrm{m},t+1}}{P_t} + \frac{(1-\kappa_t)Q_t B_{\mathrm{y},t}}{P_t} = \frac{V_t H_{\mathrm{y},t}}{P_t} + \frac{K_t}{P_t}, \tag{A.19}$$

where the lump-sum transfer K_t is taken as given by an individual, but is equal to $-\kappa_t Q_t B_{\mathrm{y},t}$ in equilibrium. An alternative tax instrument applies to the interest income derived from making loans. The present value of the interest income resulting from a loan $D_t = -Q_t B_{\mathrm{y},t}$ is $-(1-Q_t)B_{\mathrm{y},t}$. If this transfer between borrowers and lenders is taxed at rate ι_t then borrowers receive an amount $(1-\iota_t(1-Q_t)/Q_t)D_t = (1-\iota_t i_t)D_t$ after tax using (8), so this instrument is equivalent to $\kappa_t = \iota_t i_t$ in terms of the nominal interest rate i_t.

Maximizing utility (1) subject to the new budget identity (A.19) implies the first Euler equation in (7b) is replaced by $(1-\kappa_t)Q_t/P_t C_{\mathrm{y},t} = \beta/P_{t+1}C_{\mathrm{m},t+1}]$. All other first-order conditions remain unchanged, except that conditional expectation operators are ignored because the economy is deterministic. Imposing the equilibrium lump-sum transfer $K_t = \kappa_t D_t$ in (A.19) implies the original form of the budget identity in (4a). Hence, apart from dropping expectation operators, it follows that all the equilibrium conditions in (14) are unaffected except for the first equation in (14d) being replaced by:

$$\frac{1-\kappa_t}{c_{\mathrm{y},t}} = \beta \frac{1+r_{t+1}}{(1+g_{t+1})c_{\mathrm{m},t+1}}. \tag{A.20}$$

The proofs of Step 1 and Step 2 are unaffected. Following the proof of Step 3 but using (A.20) leads to $i_t = (1-\kappa_t)/\beta\delta(1+\pi_{t+1}) - 1$ instead of the equation for i_t in (18). Comparing this with (21) shows that (22) is replaced by the equation $\phi_t = \kappa_{t-1}$. Conditional on this solution for ϕ_t, the proof of Step 4 still applies and the same solutions for consumption as a function of ϕ_t in (27) are valid. Since $c_{\mathrm{o},t+1}$ is decreasing in κ_t, the incidence of the tax on lending falls on savers.

In the financial crises equilibrium with small ε_t, the housing risk premium is given by $\xi_t = x_{t+1}$. By setting $\kappa_t = \xi_t/(1+\xi_t)$, the consumption allocation with $\phi_t = \kappa_{t-1}$ in the deterministic economy with taxes on lending is the same as (A.18) because $\phi_t = x_t/(1+x_t)$. It follows that the welfare consequences of having the equilibrium with financial crises are equivalent to a tax on lending in a deterministic economy with tax rate κ_t equal to $\xi_t/(1+\xi_t)$ in terms of the housing risk premium. Or equivalently, to a tax on the income derived from lending with tax rate ι_t equal to $(\xi_t/i_t)/(1+\xi_t)$ because $\iota_t = \kappa_t/i_t$ has the same effect as κ_t.

A.14 Proof of Result 6

Step 8 shows that setting the nominal interest rate $i_t = (1+\mathbb{E}_t\pi_{t+1})/\beta\delta - 1$ is a necessary condition for the consumption smoothing requirement of Pareto efficiency. This is the same interest rate policy that implements the equilibrium with financial stability from Result 4, so that equilibrium is the only Pareto-efficient allocation that can be implemented through monetary policy alone.

Now suppose that transfers are available. Using the new budget identities and following the same steps used to derive (14b), those budget equations are replaced by:

$$c_{\mathrm{y},t} + h_t = d_t, \quad c_{\mathrm{m},t} + d_t = 1 + (1-b_t)h_t + \tau_{\mathrm{m},t}, \quad \text{and } c_{\mathrm{o},t} = b_t h_t + \tau_{\mathrm{o},t}, \tag{A.21}$$

which uses the definitions $\tau_{a,t} = T_{a,t}/Y_t$ and the restriction $T_{\mathrm{y},t} = 0$. Dividing both sides of the fiscal budget constraint (37) by Y_t, the following constraint on $\tau_{\mathrm{m},t}$ and $\tau_{\mathrm{o},t}$ must be respected:

$$\tau_{\mathrm{m},t} + \tau_{\mathrm{o},t} = 0. \tag{A.22}$$

Since none of the first-order conditions is affected by the lump-sum transfers, the only change to the set of equilibrium conditions (14) is that (14b) is replaced by (A.21).

Now take any allocation that satisfies the sufficient conditions for Pareto efficiency stated in Step 7 from some starting date s onwards. It will be shown that there exist a sequence of transfers $\{\tau_{a,t}\}$ satisfying $\tau_{a,t} = \mathbb{E}_{s-1}\tau_{a,t}$ for all $t \geq s$ that support this allocation as a competitive equilibrium subject to monetary policy pursuing financial stability.

The risk-sharing conditions in Step 7 require $c_{\mathrm{m},t}/c_{\mathrm{y},t} = \mathbb{E}_{s-1}c_{\mathrm{m},t}/\mathbb{E}_{s-1}c_{\mathrm{y},t}$ and $c_{\mathrm{o},t}/c_{\mathrm{m},t} = \mathbb{E}_{s-1}c_{\mathrm{o},t}/\mathbb{E}_{s-1}c_{\mathrm{m},t}$ to hold for all $t \geq s$, so it follows that $c_{\mathrm{m},t}/c_{\mathrm{y},t}$ and $c_{\mathrm{o},t}/c_{\mathrm{m},t}$ are $s-1$ measurable. The allocation must also satisfy the resource constraint, which is equivalent to (13), and which can be rearranged to obtain $c_{\mathrm{y},t} = 1/(1+(c_{\mathrm{m},t}/c_{\mathrm{y},t}) + (c_{\mathrm{o},t}/c_{\mathrm{m},t})(c_{\mathrm{m},t}/c_{\mathrm{y},t}))$. This implies $c_{\mathrm{y},t}$ is $s-1$ measurable for all $t \geq s$, and since the ratios $c_{\mathrm{m},t}/c_{\mathrm{y},t}$ and $c_{\mathrm{o},t}/c_{\mathrm{m},t}$ are also $s-1$ measurable, so must be $c_{\mathrm{m},t}$ and $c_{\mathrm{o},t}$ too.

Now suppose there exists an equilibrium where this efficient consumption allocation prevails. Rearranging the budget identity of the old from (A.21) gives $b_t h_t = c_{\mathrm{o},t} - \tau_{\mathrm{o},t}$. The restriction $\tau_{\mathrm{o},t} = \mathbb{E}_s\tau_{\mathrm{o},t}$ and the efficiency requirement of $s-1$ measurability for $c_{\mathrm{o},t}$, an equilibrium that implements the allocation requires that $b_t h_t$ is $s-1$ measurable for all $t \geq s$. Using the first accounting identity in (14a), $(1+r_t)/(1+g_t) = b_t h_t/d_{t-1}$, from which it follows that the equilibrium must have $(1+r_t)/(1+g_t)$ be $t-1$ measurable for all $t \geq s$ (because $b_t h_t$ is $s-1$ measurable). These measurability conditions for $(1+r_t)/(1+g_t)$ together with those for the consumption allocation mean that the bond Euler equations (14d) can be stated as:

$$\frac{c_{\mathrm{m},t+1}}{c_{\mathrm{y},t}} = \beta\frac{1+r_{t+1}}{1+g_{t+1}}, \quad \text{and} \quad \frac{c_{\mathrm{o},t+1}}{c_{\mathrm{m},t}} = \beta\frac{1+r_{t+1}}{1+g_{t+1}}. \tag{A.23}$$

These equations reveal that $(1+r_t)/(1+g_t)$ must be $s-1$ measurable for all $t \geq s$ owing to the measurability requirements on the consumption allocation (A.23 shows this for $t > s$, while the case of $t = s$ follows from the earlier finding of $t-1$ measurability).

The first accounting identity in (14a) can be rearranged to write $d_t = (b_{t+1}h_{t+1})/((1+r_{t+1})/(1+g_{t+1}))$, and thus $s-1$ measurability of d_t is inherited from the same property of $b_{t+1}b_{t+1}$ and $(1+r_{t+1})/(1+g_{t+1})$. The budget identity of the young in (A.21) requires $h_t = d_t - c_{\mathrm{y},t}$, and $s-1$ measurability of h_t can be deduced from the same property of l_t and $c_{\mathrm{y},t}$. By combining the second accounting identity in (14a) and the ex-post Fisher equation (14e) to eliminate γ_t:

$$1+\pi_t = (1+i_{t-1})\left(\frac{1+g_t}{1+r_t}\right)\frac{h_t}{h_{t-1}}.$$

As h_t and $(1+r_t)/(1+g_t)$ must be $s-1$ measurable for all $t \geq s$, it follows from the equation above that π_t must be $t-1$ measurable for all t from s onwards. Therefore, it is necessary that monetary policy is used to achieve financial stability as in Result 4 to support an efficient allocation.

The next step is to show that there exists an equilibrium supported by a particular scheme of transfers that implements an efficient allocation. Since $c_{a,t}$ and h_t must all be $s-1$ measurable in such an equilibrium, the housing Euler equation (14c) is equivalent to:

$$\frac{h_t}{c_{\mathrm{y},t}} = \beta\theta + \beta\frac{h_{t+1}}{c_{\mathrm{m},t+1}} = \beta\theta + \frac{\beta c_{\mathrm{y},t+1}}{c_{\mathrm{m},t+1}}\frac{h_{t+1}}{c_{\mathrm{y},t+1}}. \tag{A.24}$$

Using the measurability conditions for the consumption allocation, the first dynamic efficiency requirement (with $\limsup_{t\to\infty}$ and strict inequality) from Step 7 can be stated as $\limsup_{t\to\infty} \beta c_{\mathrm{y},t+1}/c_{\mathrm{m},t+1} < 1$. Taking (A.24) as a difference equation in $h_{t+1}/c_{\mathrm{y},t}$, dynamic efficiency there exists a unique bounded solution for $h_t/c_{\mathrm{y},t}$, with h_t given by:

$$h_t = \beta\theta\left(1 + \frac{\beta c_{\mathrm{y},t+1}}{c_{\mathrm{m},t+1}} + \frac{\beta c_{\mathrm{y},t+1}}{c_{\mathrm{m},t+1}}\frac{\beta c_{\mathrm{y},t+2}}{c_{\mathrm{m},t+2}} + \cdots\right)c_{\mathrm{y},t}.$$

This value of h_t is well defined, and together with the consumption allocation, satisfies (14c). Letting $d_t = c_{\mathrm{y},t} + h_t$, the budget identity of the young in (A.21) is satisfied.

Let the real return on bonds r_t be $r_t = (1+g_t)c_{\mathrm{o},t}/\delta c_{\mathrm{m},t-1}$ for all $t \geq s$ using the consumption allocation and the exogenous g_t, which satisfies the second equation in (A.23) by construction. Since the consumption allocation is efficient, it must satisfy the consumption smoothing requirement $c_{\mathrm{m},t+1}/c_{\mathrm{y},t} = c_{\mathrm{o},t+1}/c_{\mathrm{m},t}$ for all $t \geq s$, which means that the first equation in (A.23) also holds with the constructed path of r_t. As (A.23) is equivalent to (14d) at an efficient consumption allocation, (14d) is satisfied. Finally, let $b_t = (1+r_t)d_{t-1}/(1+g_t)h_t$ for all $t \geq s$, which implies the first accounting identity from (14a) holds by construction.

Using the constructed values of b_t and h_t and the efficient consumption allocation, set the net transfer to the old equal to $\tau_{\mathrm{o},t} = c_{\mathrm{o},t} - b_t h_t$, and the net transfer to the middle-aged equal to $\tau_{\mathrm{m},t} = b_t h_t - c_{\mathrm{o},t}$. Note that this satisfies the budget identity of the old in (A.21) and the fiscal budget constraint (A.22) because $\tau_{\mathrm{o},t} = -\tau_{\mathrm{m},t}$. The allocation necessarily satisfies the resource constraint, so $c_{\mathrm{m},t} = 1 - c_{\mathrm{y},t} - c_{\mathrm{o},t}$, and by substituting for the consumption of the young and old using the first and third budget identities from (A.21) it follows that $c_{\mathrm{m},t} = 1 - d_t + h_t - b_t h_t - \tau_{\mathrm{o},t}$. Together with (A.22) this implies the second budget identity of the middle-aged in (A.21) must hold.

Now specify an arbitrary path of nominal house-price inflation such that $\pi_t = \mathbb{E}_{t-1}\pi_t$ with probability one for all $t \geq s$. Let goods-price inflation be $\gamma_t = (1+\mathbb{E}_{t-1}\pi_t)h_{t-1}/(1+g_t)h_t - 1$ in terms of the constructed path of h_t and the exogenous g_t. This implies the second accounting identity in (14a) holds by construction. The final equilibrium condition to verify is (14e), the ex-post Fisher equation, which requires $1 + i_{t-1} = (1+r_t)(1+\gamma_t)$. Using the constructed value of γ_t, $(1+r_t)(1+\gamma_t) = (1+\mathbb{E}_{t-1}\pi_t)(h_{t-1}/h_t)(1+r_t)/(1+g_t)$, which is $t-1$ measurable for all $t \geq s$ because $\mathbb{E}_{t-1}\pi_t$ is known at date $t-1$ and $(1+r_t)/(1+g_t)$ and h_t are known to be $s-1$ measurable for all $t \geq s$. There exists a well-defined path of nominal interest rates that supports the equilibrium with financial stability and transfers.

A.15 Proof of Step 9

[To be added.]

A.16 Proof of Result 7

[To be added.]