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Fairly Good Plans*

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"The complicated analyses which economists endeavour to carry through are not mere gymnastic. They are instruments for the bettering of human life." (A. C. Pigou, Preface "Economics of Welfare," 3rd ed.) For this reason, one expects economists to be particularly interested in assessing the relative value of different directions for their research. This paper is intended as a contribution to that task. We take a simple model and show how much of an increase in welfare might be available if subtle theory were substituted for crude as the basis for economic policy, or if more elaborate model formulation replaced simple calculations. The simplicity of the model makes it easier to develop our methods. More important, as we shall argue, the analysis of simple models is essential if we are to understand the corresponding situation for more complex models of the economy.

In order to make the discussion intelligible, we measure welfare in commodity units (as do users of cost-benefit techniques). There would be little point in a claim that, for example, expropriation of privately owned industry would yield 23 million social utils. Such a measurement does nothing to suggest how much of our enthusiasm and research should be allocated to the proposal in question. If instead we claim that, in terms of the social valuation used, it would be as good as a 5% increase in the national income, we can compare with our own memory or forecast of of the gains from other policies, and convey the magnitude of our claim

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to others.[1] With commodity units, it should then be possible to discuss whether there is good enough reason to pursue a line of research further, to elaborate an existing economic model, or to press a new economic policy upon the authorities. Commodity units for welfare allow the comparisons implicit in such questions to be expressed in a way that is invariant to the forms of welfare function that may have seemed appropriate to each particular context.

Specifically, we use a natural calibration of alternative growth paths in the simplest of the growth models, the one-good malleable-capital model. By means of this calibration, we discuss, first, the worth of optimum growth theory, and, second, the use of finite-time-horizon optimizing models in actual planning.

Economists working in many different areas have their own version of the problem here illustrated in the case of optimum growth theory. For example, the econometrician wants to know whether he should make a large effort to obtain a precise estimate of a parameter about which he already has some information. The cost-benefit analyst frequently neglects considerations, while retaining an impression of their probable magnitude: he wants to know whether the difference in benefits and costs is sufficiently large to justify his neglect of awkward items. The proponent of pet ideas might consider whether the likely gains from adopting his idea justify the effort of turning it into a precise and detailed proposal. The practical planner has to worry whether the existing model is good enough for the job at hand, or should be replaced by a more complicated one.

It might be contended that a "superoptimizing" model, incorporating costs of reasearch and delay, is necessary if questions of this type are to be answered. But such an approach begs the question. If it is not worth while developing a more complicated model, it is certainly not worth while developing a superoptimizing model, containing within it, as part of its range of choice, that more complicated model.[2] In our view, evidence relevant to the question whether research activity should be extended in particular directions can be obtained by considering the corresponding

[1] In effect, our discussion in this paper relies chiefly on the possibility of guessing the opportunity-cost (in commodity units) of research effort of average probable effectiveness. Commodity units also have the advantage that they indicate when welfare differences are "small." One can imagine this being useful when testing the acceptability of a particular objective function. If we feel that one simple economic state is much better than the other, while the proposed objective function gives it only a very small relative advantage, we shall have good reason to change the function.

[2] Unless an essentially similar process of model construction will be undertaken a number of times (e.g., for different countries). It may sometimes be reasonable to construct a superoptimizing model for a single representative case when many applications are possible.

issue in a much simpler case. That simple case then simulates the real research problem. It is obvious that a knowledge of the best policy in the simple case does not tell us, with perfect certainty, what to do in the more complex situation. Nothing can do that. But it seems right that evidence from simple cases should not be ignored.

An example will illustrate our point. Suppose we are concerned with a multisector planning problem of some complexity, for which an infinite horizon is believed to be appropriate. It may be impracticable, or, in the current state of knowledge, impossible to calculate optimum growth for such a model. We may, nevertheless, be able to calculate optimum growth for the case of a finite time-horizon, provided that the number of periods is not too great. Is there some crude rule, specifying a time-horizon and terminal capital requirements, which will lead to a "satisfactory" approximation to the true optimum; or would we do better to devote further research effort to discovering ways of computing the infinite-horizon optimuum? Clearly this question cannot be answered directly without rendering the question pointless. The only way of obtaining evidence relevant to it is to consider the corresponding question for a simpler model where direct answer is possible without excessive effort. For example, we can pose the question in the context of a simple one-good model, and use our welfare calibration to measure the welfare loss from using finite-horizon calculations.

We do this particular exercise in Section 5 below. Our first example of welfare comparisons compares simple and complicated savings policies, and is relevant to the general question, unanswerable with perfect precision and certainty, whether in certain cases increased refinement of economic calculation is desirable. These are only examples of a rather general way of tackling a large class of problems. We see the task as similar to the use of small-scale models (of bridges, ships, and aeroplanes, for example) by engineers. If the model bridge does not fall down, the real bridge may; but the engineer may still have been well advised to build the model first.

We would not claim that research has in all fields been pushed so far that estimates of the kind discussed in this paper can be made. Probably it can be done in public economics and international economics as readily as in growth theory; but labor economics and the theory of the firm, for example, seem less ready for such treatment. Neither do we claim that the direct and obvious social consequences of an economic theory necessarily consitute the major part of its claim to be an "instrument for the bettering of economic life." Development of a theory may affect quite remote fields, and it may improve the economist's own grip on real problems. Some theoretical work may provide a very satisfactory use for one's leisure hours. However, the "natural" next steps in the development of the logical structure should not always have first claim on the economist's research effort. The value of his services might be increased by calculated consideration of when to stop and what to do next.

1. Balanced Growth Equivalents

Consider a one-sector growth model in which balanced growth is possible, the natural rate of growth being α. Consider an arbitrary feasible consumption path c, such that consumption at time t is c_t. Suppose there exists a welfare ordering, or valuation, of consumption paths. If c is equivalent, in welfare terms, to the balanced growth path yielding consumption $\gamma e^{\alpha t}$ at t, we shall call γ the *balanced growth equivalent* (BGE) of c. For such a model, the BGE is the natural commodity measure of welfare. Changes in it brought about the adoption of new policies or techniques of policy-making show the percentage increase in consumption forever that is equivalent to the policy change contemplated.

We realize that for many microeconomic decisions 0.01 % of national income for 1 year, let alone forever, is of great importance; but we are concerned here with the comparison of rather broad economy-wide policies such as the overall savings rate, fiscal policy, or the control of foreign trade. In our opinion, one should look for at least a 1 % increase in the BGE before elaborating such policies into detailed proposals. We base our view of "at least 1 % "on the impression that there are many such areas of economic research, as yet imperfectly explored, where gains of at least this magnitude are available: for example international monetary arrangements, or the form of taxation. Our guesses at the possible gains from such policies will not be shared by everyone; but we think that most readers will be able to produce examples which in their view can produce gains of this size. On the other hand, there are many lines of research very actively pursued (perhaps rightly) that seem to offer gains of little greater than 1 % of the national income.[3]

The social welfare function for a growth model may not be a function at all, but rather an ordering and not necessarily a complete one. In the simplest case, where preferences regarding consumption in different time periods are independent, it is usual to order consumption paths by saying

[3] For example, most work on macroeconomics is intended to enable the economy to utilize available capacity more fully. Casual inspection of the British data (cf. [7]) suggests to us that few such policies could hope to obtain gains averaging, over time, much more than 1 % of total output per year.

that c is at least as good as c' if for any $\epsilon > 0$, there exists T_0 such that for all $T \geqslant T_0$,

$$\int_0^T u(c_t, t)\,dt \geqslant \int_0^T u(c_t', t)\,dt - \epsilon. \tag{1}$$

In terms of this ordering, c and c' are equivalent if c is at least as good as c', and c' is at least as good as c. This may not be a complete ordering, for example if u is independent of t. Thus there may be no balanced growth path equivalent to a specified consumption path c. The path yielding consumption $(\gamma e^{\alpha t})$ is equivalent to c if and only if

$$\int_0^\infty [u(c_t, t) - u(\gamma e^{\alpha t}, t)]\,dt = 0, \tag{2}$$

meaning that the integral converges and is zero. This definition of welfare equivalence follows at once from the definition of the welfare ordering given in (1).

We take, as an example, the one-good model with constant returns to scale and labor-augmenting technological change at rate g:

$$c_t + \dot{k}_t = e^{gt} f(e^{-gt} k_t), \qquad k_t \geqslant 0. \tag{3}$$

Here k_t is interpreted as the capital stock at t. The production function f is concave, increasing, and zero at zero. We omit population growth for the moment so that $\alpha = g$. We show in the Appendix that all paths in this model have a BGE in the case of bounded, concave u.

2. The Balanced Growth Equivalent of Optimum Growth

In this paper, we calculate balanced growth equivalents for various paths in the one-good neoclassical model (3). Since the questions at issue are the welfare gains from following fully optimum policies, compared with various, theoretically less ideal, alternative policies, we restrict attention to cases where an optimum path exists. In this section, we show how the BGE of the optimum path—which is of course the largest possible BGE for a feasible path—can be calculated. When the instantaneous valuation function has the special form

$$u(c, t) = e^{-\rho t} u(c) = -\frac{1}{\nu} e^{-\rho t} c^{-\nu}, \tag{4}$$

it is particularly easy to compute the BGE of optimum growth, once the optimum policy has itself been computed. To derive the relationship we require, we use the homogeneity property of the maximized valuation integral that is implied by the homogeneity of the instantaneous valuation function (4).

Suppose the valuation integral can converge.[4] Then the maximum of the valuation integral, which is the total valuation resulting from following the optimum policy, is a well-defined function of the initial capital stock and the initial date. We define

$$W(k_0, t) = \max \int_t^\infty e^{-\rho(\tau - t)} u(c)\, d\tau, \tag{5}$$

where the maximum is taken over feasible consumption paths beginning at time t when the capital stock is k_0.

W is homogeneous of degree $-\nu$ in k_0 and e^{gt}. To prove this, multiply k_0 and e^{gt} by the same factor a. We see from the production relationships (3) that this change enables any feasible consumption path to be multiplied by this factor, period by period: total valuation is then multiplied by a factor $a^{-\nu}$. Consequently the optimum consumption policy is a homogeneous function of degree 1 in k_0 and e^{gt}; and W is homogeneous of degree $-\nu$ in the two variables, as asserted. We write

$$W(k_0, t) = e^{-\nu g t} w(k_0 e^{-gt}). \tag{6}$$

It is clear from (5) that, on the optimum path,

$$W(k_0, t_0) - e^{-\rho(t_1 - t_0)} W(k_{t_1}, t_1) = \int_{t_0}^{t_1} u(c_\tau)\, e^{-\rho(\tau - t_0)}\, d\tau. \tag{7}$$

Dividing by $t_1 - t_0$, and letting $t_1 \to t_0$, we obtain

$$W_k(k_0, t_0)\, \dot{k}_t + W_t(k_0, t_0) - \rho W(k_0, t_0) = -u(c_{t_0}). \tag{8}$$

Furthermore, one expects that the marginal valuation of consumption should be equal to the marginal valuation of capital on the optimum path, since we are, at the optimum, just indifferent between an increase in immediate consumption and an increase in present capital: i.e.

$$W_k(k_0, t_0) = u_c(c_{t_0}). \tag{9}$$

This can be proved rigorously by calculus of variations methods.

[4] This is possible if and only if $\nu g + \rho > n$ where $u(c, t) = L_t(c_t/L_t)^{-\nu} e^{-\rho t}$ and $\dot{L}_t/L_t = n$ where L_t is population.

In the particular case we are now concerned with, Eqs. (8) and (9) become:

$$w'ke^{-gt_0} - (\nu g + \rho)\, w - gk_0e^{-gt_0}\, w' = \frac{1}{\nu}\, c^{-\nu}e^{\nu g t_0}, \tag{10}$$

$$w' = c^{-\nu-1}e^{(\nu+1)gt_0}, \tag{11}$$

where everything is evaluated at k_0 and t_0. Eliminating w' from (10) and (11), using (3), and writing k and t for k_0 and t_0 now, we obtain

$$(\nu\alpha + \rho)\, w = -(ce^{-gt})^{-\nu} \left[\frac{\nu + 1}{\nu} - \frac{e^{gt}f - gk}{c}\right]. \tag{12}$$

Thus when the optimum policy, giving ce^{-gt} as a function of ke^{-gt}, is known, the resulting total valuation can be calculated at once.

Denote the BGE of the optimum path by $c_b{}^*$. Then

$$\begin{aligned} W(k, t) &= -\frac{1}{\nu}\int_0^{\infty} (c_b{}^*e^{gt})^{-\nu}\, e^{-\rho t}\, dt \\ &= -[1/\nu(\nu g + \rho)](c_b{}^*)^{-\nu}. \end{aligned} \tag{13}$$

Combining (6), (12), and (13), we obtain for $t = 0$,

$$c_b{}^* = c\left[1 + \nu - \frac{(f - gk)}{c}\right]^{-1/\nu}, \tag{14}$$

where c is optimum initial consumption. Graphs giving values of c for certain particular models are given in Ref. [5].

When the instantaneous valuation function u is not homogeneous in consumption, the balanced growth equivalent can be obtained by integration of Eq. (9).

The formula just given extends immediately to the case of a constant rate of population growth n. We take an instantaneous valuation function $e^{-\rho t}L_t u(c_t/L_t)$, where $L_t = L_0e^{nt}$, and $u(c) = -c^{-\nu}/\nu$. The valuation integral converges if $\nu g + \rho > n$. Then the relation between the immediate optimum consumption policy and the BGE of continued optimum growth is given by

$$c_b{}^* = c\left[1 + \nu - \nu\left(\frac{f - (g + n)\, k}{c}\right)\right]^{-1/\nu}. \tag{15}$$

Finally, we consider briefly the case $\nu g + \rho = n$—the only case in which an optimum policy exists, but the valuation integral does not converge. In the narrow sense defined earlier, no BGE exists in general.

Let us compare a consumption path (c_t) in this model with balanced growth paths, in which consumption per head is γe^{gt}. Let consumption per head on the given path be $z_t e^{gt}$. The integral of valuation-differences for a period T is

$$-\frac{1}{\nu}\int_0^T e^{-(\rho-n)t}[(z_t e^{gt})^{-\nu} - (\gamma e^{gt})^{-\nu}]\,dt$$

$$= -\frac{1}{\nu}\int_0^T [z_t^{-\nu} - \gamma^{-\nu}]\,dt,$$

on the assumption that $\rho + \nu g = n$. It is clear, then, that if z_t tends to a limit as $t \to \infty$, the integral of valuation-differences will be negative for all large T if $\gamma < \lim z_t$; and positive for all large T if $\gamma > \lim z_t$. Thus, $\lim z_t$ is the BGE (wide sense). This term is defined in the Appendix.

The extended definition brings out clearly the peculiarity of this case, that the valuation of the path depends essentially upon its long run asymptotic behavior. The maximum possible BGE (wide sense) is, of course, golden rule consumption per efficiency unit of labor $\bar{z}$: this is the BGE (wide sense) of the optimum path, regardless of initial conditions.

3. The Balanced Growth Equivalent of Harrodian Paths

A convenient example of a simple growth plan for the economy is that implied by specifying a particular saving–income ratio, and maintaining it forever. A policy of this kind might be easier for the economy to follow that the more subtle variations in the saving–income ratio implicit in a full optimization. If the savings ratio is s, and production is described, as before, by (3), the path of the economy is determined by the differential equation

$$\dot{k}_t = s e^{gt} f(e^{-gt} k_t), \tag{16}$$

where, to simplify formulas, we again leave out population growth. This is easy to integrate in most special cases. In particular, if production is Cobb–Douglas—$f(x) = x^b (0 < b < 1)$—we have the solution

$$k_t^{1-b} = k_0^{1-b} + (s/g)(e^{(1-b)gt} - 1). \tag{17}$$

From this, we can obtain $c_t = (1 - s)\, k_t{}^b e^{(1-b)gt}$. The BGE which we denote by c_{bs} is obtained from

$$\int_0^\infty [u(c_t, t) - u(c_{bs} e^{gt}, t)]\,dt = 0, \tag{18}$$

which, in the case $u(c, t) = -e^{-\rho t}c^{-\nu}/\nu$ becomes

$$(c_{bs})^{-\nu} = (\rho + \nu g)(1 - s)^{-\nu} \int_0^\infty \exp[-(\rho + \nu(1 - b)\, g)\, t]\, k_t^{-\nu b}\, dt. \tag{19}$$

By changing the variable of integration to $y = (g/s)\, k_t^{1-b} e^{-(1-b)gt}$, we obtain the formula

$$\left(\frac{c_{bs}}{k_0^{\,b}}\right)^{-\nu} = p(1 - s)^{-\nu}\, h^q(1 - h)^{-p} \int_h^1 y^{-q}(1 - y)^{p-1}\, dy, \tag{20}$$

where $q = \nu b/(1 - b)$, $p = (\rho + \nu g)/[(1 - b)\, g]$, $h = gk_0^{1-b}/s$. The integral can be evaluated numerically, since the integral in (20) is an incomplete beta function. It is simple to differentiate (20) with respect to s to find an expression for the maximum c_{bs} which can be evaluated numerically. It will be noted that the particular policy

$$s = \alpha k_0^{1-b}, \tag{21}$$

corresponds to beginning growth at the natural rate of growth now and continuing it forever. [(21) is just the Harrod–Domar formula. Alternatively, this remark can be verified from (17).] Formula (20) is not valid when (21) holds; but since the policy yields a balanced growth path, the BGE must be the initial level of consumption, namely

$$k_0^{\,b} - \alpha k_0\,.$$

We shall see that optimum growth usually provides a BGE substantially greater than this. But if the saving ratio s is chosen with more deliberation, it will usually be possible to obtain a BGE quite close to the maximum attainable.

4. Balanced Growth Equivalents in Particular Examples

We have computed BGEs for optimum growth and Harrodian paths in a number of particular cases that we think may have realistic features.[5] The aim is to discover (i) how close to the optimum an economy might get by choosing a single policy parameter instead of following a fully optimum strategy; (ii) whether substantial changes in saving policies bring about important changes in welfare. This should provide relevant

[5] We are very grateful to D. M. G. Newbery for programming and carrying out these calculations. Extensions of these computations may be found in his paper [6].

evidence for deciding how much benefit economists are likely to provide by the detailed analysis of fully optimum growth policies, and whether we—or the public—need be very concerned about changes of a few percentage points in the aggregate saving of an economy.

Examples in which the optimum path would in any case follow the policy of keeping the savings ratio fairly constant are not of much interest here. We have chosen examples in which the initial optimum savings ratio is substantially different from the long run optimum ratio. Their main features are shown in Table I.[6] In each case

$$\rho = 0, \qquad g = 0.03.$$

TABLE I

				Optimum path			Best Harrodian path	
n	b	ν	k_0/y_0	Initial s	Final s	$c_b{}^*/y_0$	$\bar{s}$	c_{bs}/y_0
0	0.375	1.0	3	0.29	0.19	0.993	0.23	0.990
0.02	0.375	1.0	3	0.27	0.31	0.970	0.32	0.969
0.02	0.375	1.0	1.4	0.40	0.31	1.246	0.33	1.244
0	0.375	1.5	3	0.22	0.15	0.944	0.19	0.943
0.02	0.375	1.5	3	0.29	0.25	0.896	0.26	0.895
0.02	0.375	1.5	1.4	0.32	0.25	1.085	0.28	1.083
0	0.5	1.0	3	0.37	0.25	1.139	0.32	1.136
0.02	0.5	1.0	3	0.45	0.41	1.233	0.43	1.233
0.02	0.5	1.0	1.4	0.47	0.42	1.819	0.43	1.817
0	0.5	1.5	3	0.29	0.20	1.019	0.26	1.018
0.02	0.5	1.5	3	0.36	0.33	1.008	0.35	1.008
0.02	0.5	1.5	1.4	0.37	0.33	1.332	0.36	1.331

Three cases, providing a cross section of possibilities, are further illustrated in Fig. 1, where the ratio of c_{bs} to the initial value of output y_0 is shown for the whole range of possible savings ratios. These are:

Case I: $n = 0,\quad b = 0.5,\quad \nu = 0.5,\quad k_0/y_0 = 3.$

Case II: $n = 0.01,\quad b = 0.25,\quad \nu = 0.5,\quad k_0/y_0 = 3.$

Case III: $n = 0,\quad b = 0.375,\quad \nu = 1.5,\quad k_0/y_0 = 3.$

[6] Rough checks suggest that the values in Table I are not all completely accurate in the last decimal place. The inaccuracies are not large enough to affect our conclusions.

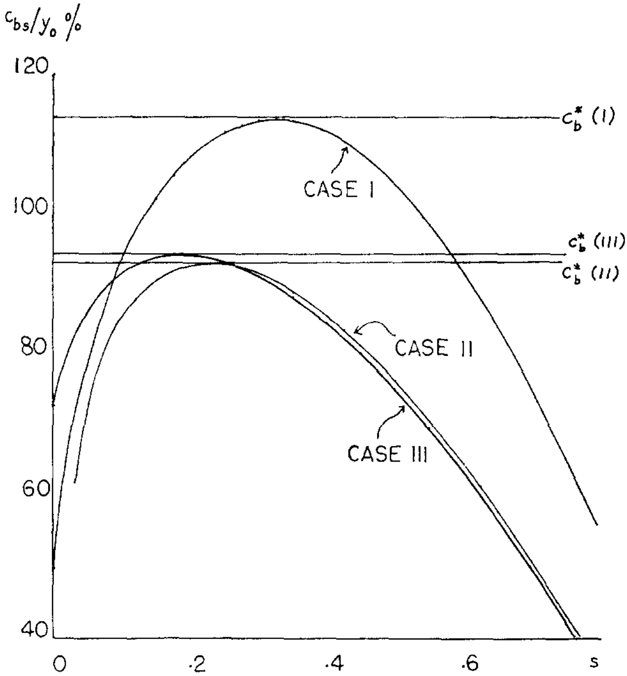

FIGURE 1

The curves have one maximum and it is striking that this is very close to the BGE of optimum growth. It is also of interest that the absolute value of the gradient of the curve is small over a fairly wide range near the optimum—i.e., the top of the curve is quite flat and in this range small changes in s have little effect on the BGE. In Case I, if we choose a savings ratio between the initial optimum savings ratio and the asymptotic optimum savings ratio ($0.25 \leqslant s \leqslant 0.37$) we are within one half of a percentage point of the BGE of optimum growth. For case III, choosing a savings ratio between the initial and asymptotic optimum ratios brings us within one percentage point of the BGE of optimum growth. Not surprisingly, the savings ratio giving maximum BGE (among Harrodian paths) is roughly midway between the initial optimum and the asymptotic optimum. It should be noted that these two examples use valuation functions that yield quite low optimum savings ratios (as compared to lower values of ν). The particular choice of valuation $\nu = 1$ may be interpreted as a rather egalitarian point of view. It means that we would value an extra unit of consumption given to generation A for times as much as a unit of consumption given to generation B if generation B was consuming twice as much as generation A (if $\nu = 3/2$, we value the extra unit 5.7 times as much). In general, high ν and low b give low optimum savings ratios [5]. In the light of these graphs, we can compare different policies that suggest themselves rather naturally:

(i) *Golden Rule Paths.* These are paths which save all competitively imputed profits and have the highest long run consumption per head. In this case they are paths where $s = b$. In case I, the BGE for $s = 50\%$ is 105 % compared with an optimum BGE of 113.3 % and a BGE of over 113 % for $s = 30\%$. In Case III, the BGE of $s = 37.5\%$ is 85 % compared with an optimum of 93.8 % and a BGE of over 93 % for $s = 18\%$. It must be concluded therefore that the golden rule policy has very little to commend it—the welfare loss as compared to the best Harrodian path is very large. In our welfare measure and examples, it is the equivalent of throwing away 8 % of GNP for ever, even if only Harrodian paths are considered.

(ii) *A Policy of Balanced Growth.* In cases I and III, this means a savings ratio of 9 % and thus a BGE of 91 %. As is to be expected this gives a less serious welfare loss where the economy begins with a capital–output ratio close to the asymptotic optimum one. In case I, the loss is over 22 %; but in case III the loss is nearly 3 %. Thus in the former case the balanced growth policy does much worse than the golden rule; whereas in the latter case it does much better—this is not very surprising since optimum savings ratios are higher in the former case. Certainly for low ν and high b the balanced growth policy involves very considerable welfare losses.

(iii) *Constant Savings Ratio at the Asymptotic Optimum Ratio.* In cases I and III saving at the long run optimum rate loses less than 1 % in BGE from the optimum. However this Harrodian path has a savings ratio at the lower end of the range which is close to the best Harrodian path and it is possible to do a little better by raising the savings ratio a few percentage points.

(iv) *The Harrodian path* which gives maximum BGE is in case I less than 0.3 % from the optimum and in case III less than 0.9 % from the optimum. Thus by fairly careful choice of constant savings ratio, we can have a path fairly close (in the relevant sense) to the optimum.

Before drawing conclusions from these remarks we should consider how to interpret differences in BGE between paths. It should of course be remembered that the welfare difference, although expressed as a percentage of GNP, is crucially dependent on which valuation function we choose. However, we could explain to someone who did not share our welfare judgements, the implications of our welfare judgements in language he could understand. In our view, welfare differences of more than 2% in BGE are certainly not small.

It does seem, from this point of view, that we can do rather well, as compared to the optimum, with constant savings ratios if they are chosen in the right range (roughly—between initial and asymptotic optimum ratios); and very well if we choose the best Harrodian path. If it is considered too difficult to adjust the savings rate all the time to the optimum growth rate, then following a cruder policy will not do much harm, provided our guesses about the savings ratio are of the right order of magnitude. The above results do show that thumb-rules are likely to be unsatisfactory and we do need some sort of optimum growth analysis to enable us to choose the right range. But, since very crude approximations are apparently satisfactory, one would need very good reasons to justify developing more "realistic" models in order to calculate the optimum savings rate.

These remarks are, of course, based on a few examples using a very simple unit-elasticity-of-substitution production function. It was natural to expect, however, that results concerning the satisfactory nature of simple policies would carry over to the case where the elasticity-of-substitution is less than one, since then there are smaller improvements available from fine adjustments. Calculations made by Newbery [6] confirm this expectation.

It is interesting to note (from Table I and Fig. 1) that if the present generation errs on the side of selfishness and saves less than the initial optimum would demand, then, provided it does save as much or more than the long run optimum (and continues to do so) overall welfare is not reduced very much, although there is a redistribution, as compared to the optimum, in favor of earlier generations. For some values of the parameters, this range of tolerable policy error may be fairly large.

We conclude that: (i) Harrodian paths can do well compared with the optimum provided that some care is taken in choosing the savings ratios; (ii) if such care is taken then small changes in savings ratios give very small changes in BGE—this is not true outside the range of savings ratios that do well.

5. Finite-Horizon Models

As a further illustration of the uses of the balanced growth equivalent, we consider the following problem. Many economists believe that a computable optimizing model must have a finite time-horizon, and nearly all computed planning models that have been published possess this feature, e.g., Sandee [8]. We are not sure that finite-horizon optimizations are in fact the best simplification of the planning problem, but it is true that the difficulties both of formulating and calculating an infinite-horizon model are formidable. It is worth asking, therefore, how much may be lost by relying on calculations based on a finite horizon. It is possible to discuss this issue explicitly for a one-good model, such as the one we use in this paper. Evidence about the desirable length of planning horizon, and the desirable method of setting up terminal conditions, obtained from studying this simple model, is relevant to the choice of n-sector model in practice. Better evidence for this decision can no doubt be obtained, at the cost of a more complicated analysis, from the theory of models with more than one sector. The analysis for a one-sector model is not, for that reason, irrelevant, although it ought to be superseded.

Various methods for setting up the terminal conditions in a finite-horizon model have been proposed in the literature—for instance, the achievement of a given growth rate at the end of the planning period (e.g., Chakravarty [1]), fixing an overall growth-rate for the plan (e.g., Manne [4]) and achieving Von Neumann proportions (e.g., Stoleru [9] and Chakravarty [2]). Some of these methods lack an economic rationale. The particular method we shall consider is based on the tendency of optimum paths, in many kinds of models, asymptotically to balanced growth (Gale [3]). This suggests that one first computes the balanced growth state that would be optimal if one were already on it: if that state is unique, the calculation is not likely to present serious difficulties. One then finds the path that will maximize total utility over a finite period T, subject to the constraint that the path should reach the optimum balanced growth path (OBGP) at T, e.g., Stoleru [9] and Chakravarty [2].

We shall discuss how large T would need to be if we wanted to make sure of getting "reasonably near" to maximum welfare by using this simplified planning calculation. We note first that the planning horizon must be at least as long as the minimum time necessary for the economy to reach OBGP. We then go on to consider how long it would take to reach the OBGP if a constant saving ratio s were used until the OBGP was reached. It will then be possible to put an upper bound to the time horizon if we are to get within 1 % of the full optimum.

For the Cobb-Douglas model with labor-augmenting technical progress, no population growth, no discounting, and a homogeneous utility function, the capital-output ratio and savings ratio on the OBGP are (see e.g., [5])

$$\frac{b}{(\nu + 1)g} \quad \text{and} \quad \frac{b}{\nu + 1}, \tag{22}$$

respectively. If a Harrodian path with saving ratio s is followed, it is easy to calculate that, at time t, the capital-output ratio,

$$k_t/y_t = \frac{s}{g} + \left(k_0^{1-b} - \frac{s}{g}\right) e^{-(1-b)gt}. \tag{23}$$

Therefore, using (22), we see that the time taken, on this path, to reach the OBGP is (when the argument of the logarithm is positive)

$$T_s = \frac{1}{(1-b)\,g} \log \left(\frac{s - gk_0^{1-b}}{s - [b/(\nu+1)]}\right). \tag{24}$$

The minimum time to the OBGP, which we shall call $T_{\min}$, is T_1 if $k_0^{1-b} < b/(\nu+1)g$ (as will usualy be the case), and T_0 if the opposite inequality holds.

It is a routine matter to calculate the BGE for the path obtained as a result of saving s of output until the OBGP is reached, and then continuing along the OBGP itself (Table II). The situation is illustrated by the following particular case:

$$\rho = 0, \quad g = 0.03, \quad n = 0, \quad b = 0.5, \quad \nu = 1, \quad k_0/y_0 = 3.$$

TABLE II

s	BGE/y_0	T_s (years)
0.25	1.116	∞
0.30	1.137	large
0.35	1.136	62
0.40	1.124	47
0.50	1.076	33

$T_{\min} = 12.7$ years

$c_b{}^*/y_0 = 1.139$

In Fig. 2, we show this case, and one other ($\rho = 0$, $g = 0.03$, $n = 0$, $b = 0.375$, $\nu = 1.5$, $k_0/y_0 = 3$), which provides rather shorter time-horizons. These cases are case I and case III of Fig. 1. In each case, we indicate on the graph for T_s as a function of s the range of values for which a BGE within 1 % of $c_b{}^*$ and within 3 % of $c_b{}^*$, is possible. The interpretation of these results is that a time-horizon T_s is satisfactory if the BGE is close to, say within 1 % of, the maximum BGE. These cal-

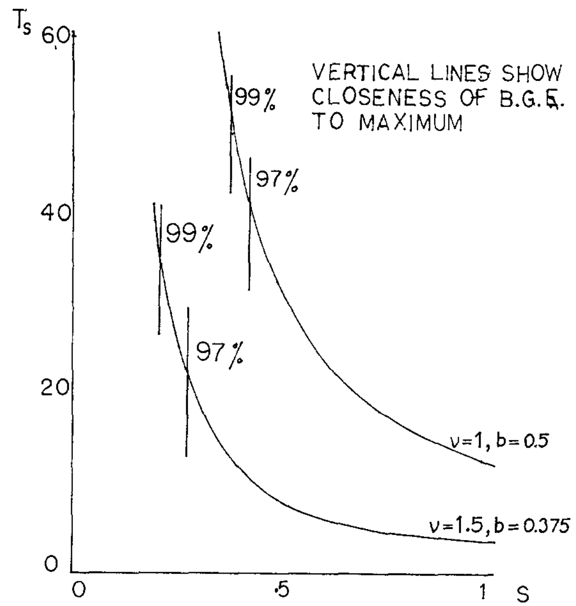

FIGURE 2

culations do not establish that a shorter time-horizon could not also yield a plan that is satisfactory, if the full finite-horizon optimum were computed. We think, however, that the result would not usually be much improved if an optimum initial path were substituted for the Harrodian initial path.

While the time-horizons deduced by this method are rather long, several points should be borne in mind.[7] First, that one reason for the long time-horizons required is the Cobb–Douglas assumption, which presumes a rather large range of technical possibilities, and therefore, very often, an OBGP capital-output ratio very different from that currently ruling. Naturally when the range of techniques allowed is small—often the case in programming models—these large differences are less likely to occur. From this point of view, however, the calculations suggest that it may be important to extend the formulation of planning models to include a richer variety of techniques and to extend the time-horizon so as to allow time for their exploitation. Secondly, it may be possible to devise terminal conditions that, even with a short time-horizon, are less likely to divert the computed plan far from the optimum. It may be noted, for example, that the use of the shadow prices corresponding to the OBGP

[7] This paragraph owes much to discussions with Peter Hammond.

as a means of valuing terminal capital would lead to a path having the opposite fault to the plans we have considered, in that they tend to reduce saving below optimum rather than increase it to above optimum. This suggests that a suitable compromise would be greatly superior to either; but we have not explored the possibility further. Finally, and probably most important, we have supposed that the computed plan will be followed for the rest of time. In fact, a finite-horizon computation would surely be used only for a time, and then a new plan, based on further computation, would be adopted. Such a "rolling-plan" procedure is presumably superior, perhaps far superior, to the one we have assumed; and may well be satisfactory even with a rather short planning horizon. We have not been able to think of any easy way of computing the consequences of such a planning procedure, and cannot guess at its importance.

Although we are not in a position to refute the finite-horizon methods of plan computation now in use, we have shown how consideration of relative BGEs could be used to establish that the time-horizon employed in a particular planning model is satisfactory. We conclude that the use of short time-horizons requires special justification, and should not be lightly adopted.

6. Final Remarks

This paper is intended only as a first approach to the evaluation of models. We are interested in models of an economy that are simple enough to be used and complex enough to be realistic in the relevant respects. Not all extensions in the direction of greater realism are worth making. Unaided intuition is becoming an increasingly unreliable judge of the "unrealism" of this or that assumption. We have tried to show how a more formal setting of the question and measurement of the possible benefits is possible and can be used to influence model development. The model we have used was chosen entirely for its analytical convenience. It has realistic features, but more complex models can be handled, and would throw more reliable light on these issues of research strategy and model formulation. Nevertheless, it is interesting to see how useful this very simple model can be in making important points. The most important point that has emerged is the degree of insensitivity of welfare to the exact savings policy pursued by the economy. It seems to us doubtful that more complicated models can greatly improve economic advice on the desirable level of investment in any economy.

Our second illustration of the use of commodity measures of welfare concerned a more subtle matter, that of assessing the worth of further complicating an already complicated model, as for example by extending the time-horizon of a many-sector planning model. The purpose of such a planning model is to give quite detailed advice on the comparative advantage of different industries and the direction of their development, not just to recommend an aggregate saving rate. The only reason for using the simple-optimum-saving model was as an analogue to the much more complicated planning calculation. The simple analogue has the advantage that one can compute the effect of changing an aspect of the formulation (in this case the time-horizon). One can then be guided by the results of that computation when deciding on the formulation of the large model, where the extension cannot be set up and analyzed without already assuming that it is worth doing. If it seems odd to the reader that one should use a simple one-sector model to guide the construction of a many-sector model, we would ask him whether he has good reason to use relatively untutored intuition to guide that construction instead.

There are two further remarks we should like to make. In the first place, our neglect of uncertainty may be of some importance for the results. It is clear, for example, that when there is great uncertainty about the productivity of the economy, a policy of saving a fixed proportion of expected national income may be quite unsatisfactory, unless there is a good foreign capital market to use. The trouble with simple policies of this kind is that they have insufficient flexibility. It is to be expected therefore that the difference between the BGE for the optimum and the maximum BGE for a Harrodian path will be greater when there is uncertainty about future technology. It seems unlikely that aggregate uncertainty is in fact so great as to modify our results substantially, but the techniques for verifying this conjecture are not available. In the case of finite time-horizons, greater uncertainty is not necessarily a reason for employing a shorter time-horizon, except to the extent that it makes any formulation of a model more difficult (which it may or may not do). We do not know how uncertainty would affect the results of Section 5, but there is no reason to think that uncertainty justifies a shorter time-horizon.

Finally, we recognize that theories with application to low-income countries have an overwhelmingly greater claim on the economist's attention than those whose sole application is in high-income countries. The BGE is not an appropriate measure for assessing such claims. But few interesting economic theories are relevant to rich countries alone, and in many cases it is likely to be quite hard to discriminate among applications in this way. We feel the techniques discussed in this paper are more useful for discussing the elaboration of economic models in particular contexts. In these cases at least, they may help the economist to decide when to stop worrying.

APPENDIX: EXISTENCE OF BALANCED GROWTH EQUIVALENTS

We use the notation of the text and make the assumption:

$$f'(0) > \alpha > f'(\infty). \tag{A.1}$$

We introduce the variables $z = ce^{-\alpha t}$ and $x = ke^{-\alpha t}$ so that production possibilities are described by

$$z_t + \dot{x}_t = f(x_t) - \alpha x_t, \qquad x_t \geqslant 0. \tag{A.2}$$

If the instantaneous valuation function is of the form $u(c)$, concave, increasing and bounded above, the T-period valuation integral is $\int_0^T u(z_t e^{\alpha t})\,dt$. It is known that in this case, an optimum policy exists if u is bounded above and concave, and $\alpha > 0$ (Von Weizsäcker [10]). The valuation integral may nevertheless be unbounded as $T \to \infty$, even for the optimum path. For convenience, one takes the least upper bound of u to be zero, so that the integral either converges, or diverges to $-\infty$. Suppose that there exists a path for which the integral converges. We shall show that in such a case, all paths have a BGE.

Consider a balanced growth path, along which consumption is $\gamma e^{\alpha t}$. The T-period valuation integral is

$$\int_0^T u(\gamma e^{\alpha t})\,dt = \frac{1}{\alpha}\int_\gamma^{\gamma e^{\alpha T}} u(\zeta)\,\frac{d\zeta}{\zeta}, \tag{A.3}$$

which either tends to a finite limit for all such $\gamma > 0$ as $T \to \infty$ or diverges for all such γ. We show that the former is the case, by showing that the existence of a path yielding a finite valuation integral implies that the balanced growth path $\bar{z}e^{\alpha t}$, where $\bar{z}$ is the maximum value of $f - \alpha x$ [finite by assumption (A.1)], has finite valuation integral.

Let z_t be a path having finite valuation integral. Then

$$z_t \leqslant \bar{z} - \dot{x}_t. \tag{A.4}$$

Also, from Eq. (A.2) and the requirement $z_t \geqslant 0$, x_t is bounded above by a number $\hat{x} = \max[x_0, \bar{x}]$ where $f(\bar{x}) - \alpha\bar{x} = 0$. Consequently,

$$\begin{aligned}
&\int_0^T [u(z_t e^{\alpha t}) - u(\bar{z}e^{\alpha t})]\,dt \\
&\quad\leqslant \int_0^T (z_t - \bar{z})\, e^{\alpha t} u'(\bar{z}e^{\alpha t})\,dt \text{ (by the concavity of } u) \\
&\quad\leqslant -\int_0^T \dot{x}_t e^{\alpha t} u'(\bar{z}e^{\alpha t})\,dt \\
&\quad= x_0 u'(\bar{z}) - x_T e^{\alpha T} u'(\bar{z}e^{\alpha T}) \\
&\qquad + \alpha \int_0^T x_t[e^{\alpha t}u'(\bar{z}e^{\alpha t}) + \bar{z}e^{2\alpha t}u''(\bar{z}e^{\alpha t})]\,dt \\
&\quad\leqslant x_0 u(\bar{z}) + \alpha \int_0^T x e^{\alpha t} u'(\bar{z}e^{\alpha t})\,dt \text{ (dropping negative terms)} \\
&\quad= A + \frac{\hat{x}}{z}\, u(\bar{z}e^{\alpha T}) \text{ where } A \text{ is a constant independent of } T \\
&\quad\leqslant A.
\end{aligned}$$

We conclude that $\int_0^T u(\bar{z}e^{\alpha t})\,dt$ is bounded below, and therefore convergent.

The valuation integral of a typical balanced growth path is

$$U(\gamma) = \frac{1}{\alpha} \int_\gamma^\infty u(\zeta) \frac{d\zeta}{\zeta}. \tag{A.5}$$

This is clearly a continuous function of γ. Hence we can obtain the BGE of any particular path whose valuation integral is finite by finding the the value of γ that makes Eq. (A.5) equal to the valuation integral of the given path. We can find such a γ, because $U(\gamma) \to 0$ as $\gamma \to \infty$ (since Eq. (A.5) is convergent) and $U(\gamma) \to -\infty$ as $\gamma \to 0$ (since for fixed A, $U(\gamma) \leqslant \int_\gamma^A u(\zeta)\,(d\zeta/\zeta) \leqslant u(A) \int_\gamma^A d\zeta/\zeta \to -\infty$). Thus $U(\gamma)$, which is certainly continuous, is a monotonically increasing function that takes all negative values.

If the valuation integral of the path diverges, one wants to say that the BGE is 0. In general, we define the BGE (wide sense) as a number $\bar{\gamma}$ corresponding to a path c if

$$\begin{aligned} \bar{\gamma} &= \inf\{\gamma\colon (\gamma e^{\alpha t}) \text{ is at least as good as } c\} \\ &= \sup\{\gamma\colon c \text{ is at least as good as } (\gamma e^{\alpha t})\}. \end{aligned}$$

It is clear that this definition generalizes the earlier definition (2). The BGE (wide sense) surely is the balanced growth that best reflects the welfare provided.

If no path has a finite valuation integral, it is still true—in the present case with u bounded and $\alpha > 0$—that every path has a BGE. To prove this, we show that $\int_0^T [u(z_t e^{\alpha t}) - u(\bar{z}e^{\alpha t})]\,dt$ tends to a finite limit or to $-\infty$ as $T \to \infty$. If this difference tends to a finite limit we can find γ such that

$$V(\gamma) = \int_0^\infty [u(\gamma e^{\alpha t}) - u(\bar{z}e^{\alpha t})]\,dt = \frac{1}{\alpha} \int_\gamma^{\bar{z}} u(\zeta) \frac{d\zeta}{\zeta} \tag{A.6}$$

is equal to this limit, and this is the BGE. The same kind of arguments as before show that V takes all values.

If the difference tends to $-\infty$, the BGE (wide sense) is zero. We can write

$$z_t = \bar{z} - \dot{x}_t - a_t, \tag{A.7}$$

where $a_t \geqslant 0$. Therefore

$$\begin{aligned} &\int_0^T [u(z_t e^{\alpha t}) - u(\bar{z}e^{\alpha t})]\,dt \\ &\qquad = -\int_0^T u'(\bar{z}e^{\alpha t})(\dot{x}_t + a_t)\, e^{\alpha t}\,dt - \int_0^T b_t\,dt, \end{aligned} \tag{A.8}$$

where $b_t \geqslant 0$. Write $a_t e^{\alpha t} u'(\bar{z} e^{\alpha t}) + b_t = m_t \geqslant 0$. Then the right side of (A.8) becomes

$$[-x_t u'(\bar{z}e^{\alpha t})\, e^{\alpha t}]_0^T + \alpha \int_0^T x_t[e^{\alpha t} u'(\bar{z}e^{\alpha t}) + \bar{z}e^{2\alpha t} u''(\bar{z}e^{\alpha t})]\, dt - \int_0^T m_t\, dt$$

$$= [-x_t u'(\bar{z}e^{\alpha t})\, e^{\alpha t}]_0^T + \alpha \int_0^T x_t e^{\alpha t} u'(\bar{z}e^{\alpha t})\, dt - \int_0^T m_t'\, dt, \tag{A.9}$$

where $m_t' \geqslant 0$. We consider the three expressions in Eq. (A.9). The first tends to a limit or $-\infty$ since x_t is bounded and $u(y)y \to 0$ as $y \to \infty$ (easily checked using the concavity of u and $\lim_{y\to\infty} u(y) = 0$). The second tends to a finite limit since it increases with T and is bounded above by $-\bar{x}u(\bar{z})/\bar{z}$. The third tends to a limit or $-\infty$ since m_t' is positive. Thus $\int_0^T [u(z_t e^{\alpha t}) - u(\bar{z}e^{\alpha t})]\, dt$ tends to a limit or $-\infty$ as required.

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