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The Optimal Size of Market Areas

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1. INTRODUCTION

We begin with a brief resumé[1] of the theory of the determination of the size of market areas given by Lösch in his pioneering book [6]. The theory considers a uniform plane of consumers (density D) with the same demand function $f()$ for the good under consideration. Transportation costs are constant per unit product per unit distance t, and there are decreasing average costs of production $C()$. Suppose the producer sets a price p at the production point and consumers bear transport costs, so that the price at a distance u is $p + ut$ (known as mill pricing). The isolated producer observes a demand curve at the mill

$$H(p) = D \int_0^{2\pi} \int_0^{r(p)} f(p + ut)\, ud\, ud.$$

where $f(p + \mathrm{tr}(p)) = 0$. We suppose $C()$, $H()$ are such that the curves $p = H^{-1}(x)$ and $p = C(x)$ (where x is output) intersect (Fig. 1.). The producer chooses $p = p^*$ to maximise his profits which are positive. We now let other producers into the picture, each with market area radius $r(p^*)$, and pack the maximum number of circles onto the plane (i.e., their centres are at the points of a regular triangular lattice). Profits are still positive, and Lösch's claim was that new producers would spring up at the interstices, reduce the size of market areas below $r(p^*)$, push the demand curve $H^{-1}(p)$ down, and that this process would continue until profits were eliminated and the demand curves were tangential to the average cost curves, and the producers were charging the same price and situated at the centre of similar regular hexagons. Beckmann [1] has pointed out that this story cannot be taken too literally as a dynamic process since, e.g., it is possible to have a hexagonal system with positive profits but where it is not possible for new producers to come in with

[1] An excellent review of location theory has recently been given by Beckmann [1].

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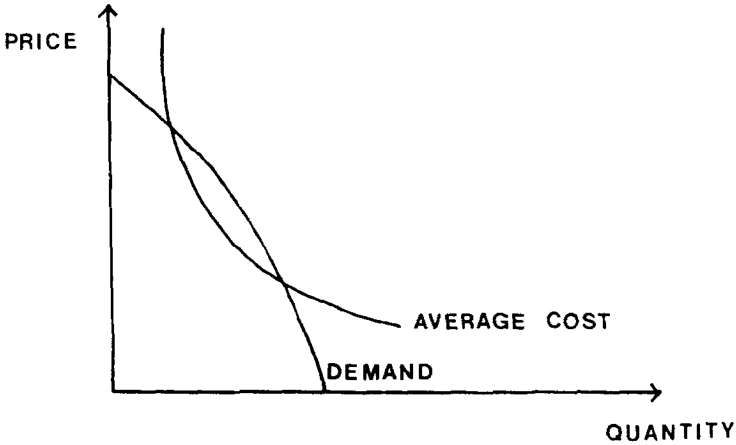

FIG. 1. Cost and demand curves observed by isolated producer.

market areas big enough to make positive profits. It also requires a non-invasion pricing policy by neighbors. However, we are used to this kind of tangency solution in imperfect competition theory. Indeed both Lösch and Chamberlin recognised location as a form of product discrimination (see, e.g., Lösch [6], p. 109n]). The interpretation of location as an index of differences can be very flexible. Let us therefore take the mill-pricing, hexagonal-market-area, zero-profit situation as one possible result from the planar competitive, free-market process. It seems to capture enough of some aspects of the problem to be worth study.[2]

It was recognised by Lösch[3] (but not by some others[4]) that the situation described above (called hereinafter the Löschian solution) is not optimal for reasons similar to the nonoptimality of the imperfect competition solution. We should like to know just how the Löschian solution differs from a socially optimum solution. We should also like to know how a socially optimal solution changes with variations in (*a*) objective function (*b*) allowable tools and (*c*) parameters. The optimal solution for a particular set of conditions is a distribution of producers over the plane, and the allocation of consumers to producers. Traditionally, it has been assumed, with Lösch, that the shape of market areas will be regular hexagons; and attention has been focused on the size of the hexagon. It is illegitimate, however, to divorce shape and size until the optimality of hexagons has been rigorously established (the arguments offered by

[2] Other pricing mechanisms can lead to solutions of the same type — see Beckmann [1, Chap. 3].

[3] Reference [6, p. 112].

[4] See, e.g., Berry [3, p. 72].

Lösch and subsequent writers have been suggestive rather than conclusive). That hexagons *are optimal* for the cases considered here is demonstrated in Bollobás and Stern [4]. Attention in this paper is focused on the appropriate size of the hexagon.

The easiest way to examine the issue of the appropriate size of the hexagon was to work out sizes of market area and levels of benefit under various particular assumptions about demand curves, tools, and objectives. A further simplification is to work with a market of one dimension, since once the appropriate shape has been established we are only interested in the size of market area. It is not surprising that results in one and two dimensions are very similar, since the question at issue is just the appropriate number of consumers in a producer's market area. The calculations presented in the text are, therefore, for one dimension. Many of the two-dimensional versions of these results are presented in the Appendix and it is easy to check that the conclusions we draw from the one-dimensional results carry over to two dimensions. Some of the calculations involved are fairly tedious and thus, occasionally, details have been omitted where the method of calculation has already been fully described.

2. Linear Demand Curves

It has become customary in location theory, since Lösch, to work with linear demand curves whenever we become specific (see, e.g.,.[1] or [7]). Lösch,[5] interestingly, claimed that it is an "average" demand curve! Let us begin with such curves. Suppose we have demand functions $x = a - bp$, D people per unit length along an infinite line, total production costs $A + kx$ for output x, and transport costs t per unit product per unit distance. Our first maximand is total producer plus consumer surplus per unit distance along the line. We choose prices and the number of producers. We are free to choose the best price at each point; and the appropriate pricing tool is, with the maximand under consideration, price equal marginal cost,[6] i.e., price at distance r from a producers is $(k + rt)$. Throughout this paper, we assume that $a/b > k$ so that there is a positive demand at price equal to marginal production cost. It is clear that each consumer should be served by the nearest producer and that each market should be the same size. We therefore calculate the net benefit in a market radius R, $S(R)$. We divide by the size of market area $2R$ and choose R to maximise.

[5] Reference [6, p. 111*n*].
[6] See Hotelling [5].

The area under the demand curve for a price p (and demand $a - bp$) is $1/2b(a^2 - b^2p^2)$. Producer plus consumer surplus is just the area under the demand curve less total costs. Thus

$$S(R) = 2D \int_0^R \frac{1}{2b} (a^2 - b^2(k + tr)^2)\, dr$$
$$- \left[A + 2D \int_0^R (a - b(k + tr))(k + tr)\, dr\right]$$

for $k + Rt \leqslant a/b$ (see below) which gives

$$S(R) = \frac{D}{b} (a - bk)^2 R - D(a - bk)\, tR^2 + \frac{D}{3} bt^2R^3 - A. \qquad (1)$$

We now make the transformation $R = (c_2/c_3)R'$, where[7]

$$c_1 = \frac{1}{b} (a - bk)^2, \qquad c_2 = t(a - bk), \qquad c_3 = bt^2,$$

and we divide by $2R$ to obtain

$$2\rho B(R') = 1 - R' + \frac{1}{3} R'^2 - \frac{A'}{R'},$$

where

$$B = \frac{S}{2R}, \qquad \rho = \left(\frac{1}{c_1 D}\right), \qquad A' = \left(\frac{c_2}{c_1{}^2 D}\right) A.$$

For nonnegativity of demand we require $k + Rt \leqslant a/b$ or $R' \leqslant 1$. For $R' \geqslant 1$, $2\rho B = (\frac{1}{3} - A')/R'$, since demand is zero beyond $R' = 1$. Thus the largest relevant A' is 1/3, which corresponds to the largest relevant R', which is one, and to zero net benefits. (Since S increases as R' increases from zero to one and reaches its maximum at $R' = 1$, and we do not wish to have production if net benefits are negative.)

The first-order conditions for a maximum with respect to R' are ($R' \leqslant 1$)

$$A' = R'^2 - \frac{2R'^3}{3}. \qquad (2)$$

We can now sketch R' as a function of A'. We require the smaller[8] positive root of (2), and this increases from zero to one as A' increases from zero to 1/3.

[7] Compare [7, p. 280].

[8] It is easy to check the second order conditions, require the smaller root for given A'.

We wish to compare (2) with Löschian solution. The profit a producer can make in a market area radius R when he charges a mill price p is

$$\Pi(p, R) = 2D(p - k) \int_0^R (a - b(p + tr))\, dr - A$$
$$= 2D(p - k)[(a - bp)\, R - \tfrac{1}{2}btR^2] - A.$$

Maximisation with respect to p gives

$$p = \frac{k}{2} + \frac{a}{2b} - \frac{Rt}{4}$$

and maximised profits $\Pi^*(R)$

$$\Pi^*(R) = \frac{Dbt^2}{8} R^3 - \frac{Dt}{2} (a - bk)\, R^2 + D \frac{(a - bk)^2}{2b} R - A.$$

The Löschian solution has R such that $\Pi^*(R) = 0$. This gives

$$A' = \tfrac{1}{8}R'^3 - \tfrac{1}{2}R'^2 + \tfrac{1}{2}R' \tag{3}$$

where R', A' are as before[9]. We require the smaller root, for similar reasons to (2). The largest relevant A' is such that maximum possible profits are zero (max $A' = 0.1482$) and, again demand is zero at the edge (since $R' = \frac{2}{3}$ in this case.)

Similar calculations to those that gave (2) and (3) were performed for the case of uniform pricing. The socially optimal uniform price was found to be $p = k + Rt/2$ for a market area radius R. The optimum R as a function of A was found to be given by

$$A' = R'^2 - \tfrac{1}{2}R'^3 \tag{4}$$

where A' and R' are as before and we need the smaller root. Again this is only relevant, for A' small enough, that positive net benefits are possible (max $A' = 0.2963$ when $R' = 2/3$—see (9)—and at the edge consumer surplus equals transport cost plus marginal production cost.) The Löschian solution with uniform pricing (i.e., firms choose the profit maximising uniform price in their given area which is then reduced until profits are

[9] Equations (2) and (3) are given by Beckmann in [2]. The two-dimensional versions of Eqs. (2)–(5) and (7)–(10) were first calculated by the author in June 1969 and presented to a seminar in Cambridge in November 1969 (see Appendix).

zero) gave a profit maximising uniform price of $k/2 + a/2b + Rt/4$ for a market radius R. The zero profit condition gave

$$A' = \tfrac{1}{8}R'^3 - \tfrac{1}{2}R'^2 + \tfrac{1}{2}R'. \tag{5}$$

This is just the same as (3). That (3) and (5) are identical is a consequence of the linearity assumptions (the profit maximising uniform price is the average price in the profit maximising mill price situation.) The largest relevant A' is again 0.1482 when $R' = 2/3$ and price equals cost of supplying the most distant point.

We are now in a position to compare the size of market areas arising from Löschian and optimal solutions for the case of mill pricing and uniform pricing as functions of A'. These are illustrated in Fig. 2 where Graphs II(i)–(iv) give A' as a function of R' for the 4 curves where (i)–(iv)

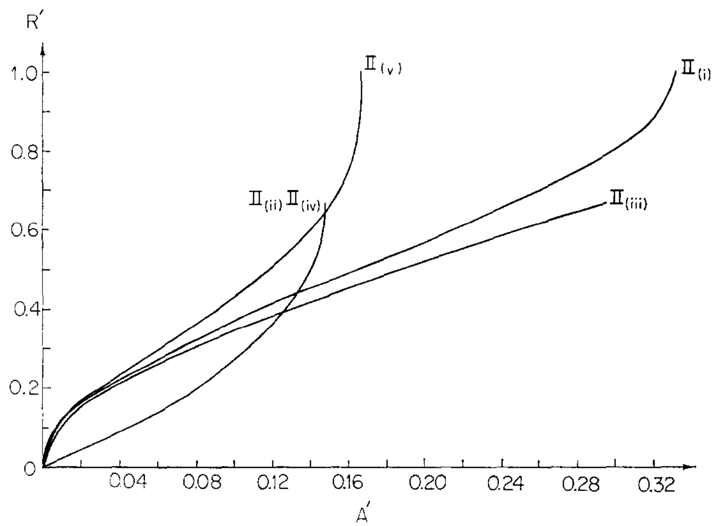

FIG. 2. The size of market areas against levels of fixed costs for linear demand curves.

Legend:

Graph II(i)	Unconstrained maximisation of consumer plus producer surplus.
Graph II(ii)	Löschian mill-pricing free-entry solution.
Graph II(iii)	Maximisation of consumer plus producer surplus using uniform pricing.
Graph II(iv)	Uniform pricing free-entry solution.
Graph II(v)	Maximisation of consumer plus producer surplus when profits are constrained to be nonnegative.

(Definitions of R' and A' and methods of calculation are given in Section 2.)

correspond to Eqs. (2)–(5) in correct order. When interpreting them we should note that

$$A' = \frac{b^2 t}{D(a - bk)^3} A \qquad \text{and} \quad R' = \frac{bt}{(a - bk)} R.$$

We can therfore interpret movements to the right along the horizontal axis as increases in A or decreases in D. Some conclusions from these graphs are given in Section 6.

In both the socially optimal solutions (mill and uniform pricing) the firm makes a loss equal to A. We should like to know how much of the difference between socially optimal and Löschian solutions is a result of allowing the firm to make losses. We therefore calculated the optimal size of market area under the constraint that the producer should make zero profit. We allow the best price at each point, i.e., we maximise

$$S(p_r, R) = 2D \int_0^R \frac{1}{2b} (a^2 - b^2 p_r^2)\, dr$$

by choice of p_r subject to the constraint

$$A + 2D \int_0^R (a - bp_r)(k + rt)\, dr = 2D \int_0^R p_r(a - bp_r)\, dr.$$

We take a Lagrange multiplier λ for the constraint, and differentiating with respect to p_r we find that

$$p_r = \frac{\lambda}{(1 + 2\lambda)} \left(\frac{a}{2b} + k + rt\right).$$

Substituting p_r in $S(p_r, R)$ and dividing by $2R$ and choosing R to maximise (remembering λ is a function of R), we obtain

$$A' = \tfrac{2}{3} R'(3 - 3R' + R'^2)(L - L^2) \tag{6}$$

where

$$L = \frac{4R'^2 - 9R' + 6}{2R'^2 - 6R' + 6}.$$

We find that $\lambda > 1$ ($L = (\lambda/(2\lambda - 1))$ and we discover $\frac{1}{2} < L < 1$) for all relevant A'; so we see that we have in fact maximised producer plus consumer surplus subject to the constraint of making nonnegative profit; it is always optimal to have precisely zero profit. We plotted (6) as Graph II (v) in Fig. 2. It is only relevant for A' such that the constraint can be satisfied, (the maximum such A' is $\frac{1}{6}$) and it is again simple to check that just one root of (6) is relevant. The graphs of equations (6), (11), (14), (15), (16) were plotted using the Oxford University KDF 9 computer.

3. The Level of Benefits for Linear Demand Curves

Apart from comparing the size of market areas in the various cases, we wish to compare the level of net benefits. We should like to know how much is lost by, e.g., constraining producers to make nonnegative profits or constraining ourselves to uniform pricing. These results are discussed in Section 6. Meanwhile, we briefly describe how benefits are calculated as a function of A'. Suppose we charge a mill price p; the consumer plus producer surplus in a market-area radius R is

$$S(p, R) = 2D \int_0^R \frac{1}{2b}(a^2 - b^2(p + tr)^2)\,dr - \left[A + 2D \int_0^R (a - b(p + tr))(p + tr)\,dr\right].$$

If we put $p = k$, we obtain the $S(R)$ expression for the mill pricing case given previously (1). If we then divide it by $2R$ and let R be given as a function of A by (2), we have the level of benefits B (consumer plus producer surplus per unit length) as a function of A. Simplifying this gives

$$B' = \tfrac{1}{2}(R' - 1)^2 \tag{7}$$

where R' is given as a function of A' by (2) and $B' = [b/(a - bk)^2D]B$.

Similarly for given R we can put in the profit maximising mill price $p = (k/2) + (a/2b) - (Rt/4)$ in $S(p, R)$ and divide by $2R$ and let R be given as a function of A by (3) to obtain the level of benefits B as a function of A for the Löschian mill price solution. This operation gives

$$B' = \frac{1}{8}\left(\frac{7R'^2}{12} - R' + 1\right), \tag{8}$$

where R' is given as a function of A' by (3), and B' is as above.

The equivalent equations to (7) and (8) for the case of uniform pricing are

$$B' = \tfrac{1}{2}(\tfrac{3}{4}R'^2 - 2R' + 1), \tag{9}$$

where R' is given by (4) (the optimal uniform pricing solution), and

$$B' = \frac{1}{8}\left(\frac{R'^2}{4} - R' + 1\right) \tag{10}$$

where R' is given by (5) (the Löschian uniform pricing solution).

The level of benefits as a function of A are given for the "optimal subject to nonnegative profits" case by

$$B = \tfrac{1}{2}L^2(1 - R' + \tfrac{1}{3}R'^2), \tag{11}$$

where L and R' are as in (6).

We are now in a position to graph the level of benefits as a function A' for the five cases examined for Fig. 2. These are given as Fig. 3 where Graphs III(i)–(v) are Eqs. (7)–(11) in correct order. These graphs are discussed in Section 6.

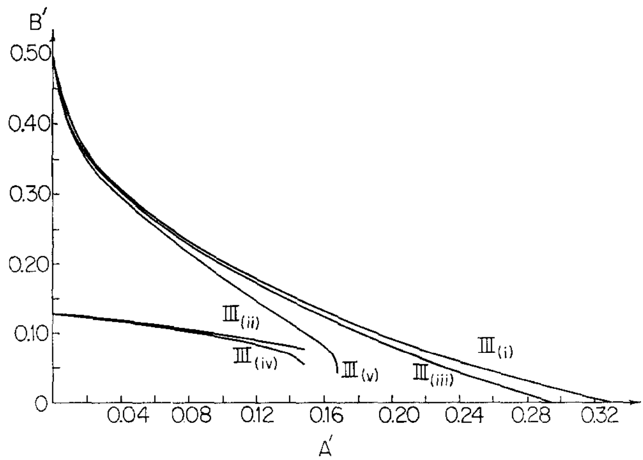

FIG. 3. The level of benefits against fixed costs for linear demand curves.

Legend:

Graph III(i) Unconstrained maximisation of consumer plus producer surplus.

Graph III(ii) Löschian mill-pricing free-entry solution

Graph III(iii) Maximisation of consumer plus producer surplus using uniform pricing.

Graph III(iv) Uniform pricing free-entry solution.

Graph III(v) Maximisation of consumer plus producer surplus when profits are constrained to be nonnegative.

(Methods of calculation are given in Section 3.)

4. Size of Market Areas for the Demand Function $x = (1/p)$

We examine a different demand curve for three reasons. First, we should like to know whether some of the more striking results of Fig. 2 carry over to different demand curves. Second, results from a different demand curve can help us interpret the results we obtained for the linear demand curve. Third, we should like to know whether the use of consumer surplus as a criterion is misleading. The criterion which we find more attractive is maximisation of the integral of utilities along the line per unit length. In our previous socially optimal solution, consumers at the edge of a market area were worse off than those close in; but no account was taken of this (i.e., no different weighting of welfare) when adding their consumer surplus to that of consumers closer to a producer. There is no simple utility function which gives linear demand curves.

To obtain specific results we have to work with a specific utility function and the first one I took was $u(x, m) = \alpha \log x + \beta \log m$ where $\alpha + \beta = 1$. This is the utility function of each consumer, and consumers are distributed along an infinite line with unit density and have a stock of the second good $\bar{m}$. We are thinking of either a two-good economy where the two goods are x, m, respectively, or (rather crudely) x as the good in question and m as money. Production requires $A + kx$ units of good 2 to produce x units of good 1 (or if we think of m as money we just interpret these as production costs.) The second good is not produced and is perfectly mobile. Transportation costs are t units of good 2 per unit of good 1 per unit distance. A consumer facing a price p for 1 in units of 2, has a demand $(\alpha\bar{m}/p)$ whith is derived from his maximisation of utility subject to the constraint $px + m = \bar{m}$. When he faces a price p, (indirect) utility $v(p) = F - \alpha \log p$, where $F = \alpha \log \alpha + \beta \log \beta + \log \bar{m}$, and v is obtained by substituting the demand functions $x(p)$ and $m(p)$ in $u(x, m) = \alpha \log x + \beta \log m$.

As before, we choose p_r to maximise

$$\int_0^R v(p_r)\, dr$$

for a given R, and then divide this maximised value (now just a function of R) by $2R$ and find the R which maximises net benefit per unit length as a function of fixed costs. The constraint is now that we have enough of the good 2 to produce the demand for good 1 (or under the second interpretation, a cost-covering constraint.) The constraint is thus

$$\int_0^R px(p)\, dr = A + \int_0^R (k + rt)\, x(p)\, dr. \tag{12}$$

We have now, for convenience, taken the fixed cost to be $2A$. The first-order condition for maximisation with respect to p_r, if we take a Lagrange multiplier λ for the constraint, is

$$v'(p) + \lambda(px'(p) + x - kx'(p) - rtx'(p)) = 0 \tag{13}$$

This gives p as a function of λ, and r, and we substitute back in (12) to eliminate λ and obtain the p_r function (for given R)[10]. In the particular case under discussion, we have[11] from (13), $p_r = \lambda\bar{m}(k + rt)$ and from (12) $\lambda = \alpha R/(\alpha\bar{m}R - A)$. We must have $\alpha\bar{m}R > A$ to cover our fixed costs. We substitute the expression for p back in the maximand, divide by R, maximise with respect to R, and we finally obtain

$$A = [R^2 - R\log(1 + R)]/[2R - \log(1 + R)] \tag{14}$$

after putting $\alpha\bar{m} = 1$ and $k = t$ (which we can do by choice of units.)

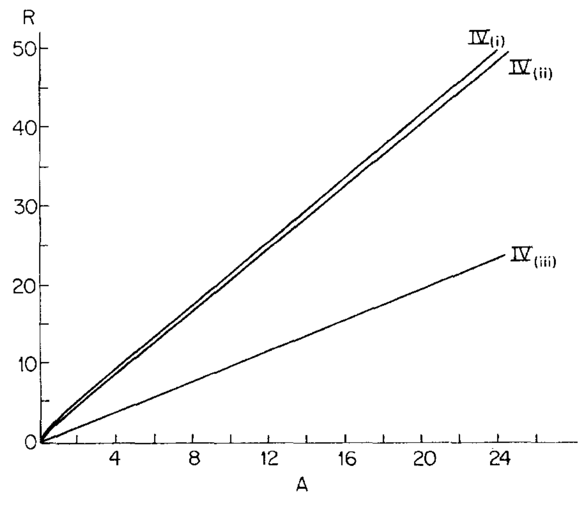

FIG. 4. Size of market areas against levels of fixed costs for the demand curve $x = (p)^{-1}$.

Legend:

Graph IV(i) Utility integral maximisation using utility function $u = \alpha\log x + \beta\log m$.

Graph IV(ii) Utility integral maximisation using utility function $= -e^{(-100u/\alpha)}$.

Graph IV(iii) Löschian free-entry solution.

(Methods of calculation are given in Section 4.)

[10] $p = 0$ is not a possible solution since revenue is fixed but costs increase as $1/p$; thus the constraint could not be met with $p = 0$.

[11] It is easy to check the second-order conditions.

This expression is valid for all A as it is always possible to choose a pricing system such that costs are covered (viz., the one given) and has a unique solution R for given A since the r.h.s. of (14) is monotonic. This curve is plotted in Fig. 4 as Graph IV (i).

Part of the object of this exercise was to check on the accuracy of consumer surplus as an approximation to utility changes. It is easy to check that for the precise utility function chosen consumer surplus is, in fact, equal to a constant less $\alpha \log p$ for a price p. In other words, consumer surplus is not an approximation here; it is exact[12]. However, for a non-trivial monotonic transformation of the utility function, consumer surplus will cease to be exact and the demand function will remain the same. The monotonic transformation taken was $u_n() = -e^{-(n/\alpha)()}$. The price derived from (13) was

$$p^{n+1} = \frac{\lambda}{n}(\bar{m} - 1)^{(n/\alpha - 1)}(k + rt).$$

The curve corresponding to (14) was found to be

$$A = R - \left[\frac{nR[(1 + R)^{(2n+1)/(n+1)} - 1]}{(2n + 1)R(1 + R)^{n/(n+1)} - (1 + R)^{(2n+1)/(n+1)} + 1}\right] \tag{15}$$

after a similar process of calculation. $n = 0$ corresponds to the case given in (14). This curve was calculated for the following values of n: 0.01, 0.1, 1.0, 10.0, 100.0. In each case, the value of R required for a given A was smaller than that derived from Eq. (14) and the value of R required for a given A decreased as n increased. However, the differences were extremely small and only the $n = 100.0$ case is graphed (as Graph IV(ii) in Fig. 4). Equation (15) is valid for all A.

We wish to compare these values of R as a function of A with the Löschian solution. In this case the profit maximising price is infinite since each person spends $\alpha\bar{m}$ on the x good regardless of price. The revenue from a market radius R is just $2\alpha\bar{m}R$, and costs are $2A$ at the profit maximising output of zero. The Löschian zero profit solution is therefore just $R = A$ (when we put, as previously, $\alpha\bar{m} = 1$). It is valid for all A. We can now compare the curves for three cases: (i) consumer surplus maximising (ii) utility integral maximising, and (iii) the Löschian solution. This is discussed in Section 6. The Löschian solution is a little bizarre here but can be instructive.

[12] This is one of the cases noted by Samuelson [8].

5. Size of Market Areas for the Demand Function $x = (1/p^2)$

Since the Löschian solution for Section 4 was a little odd, and since we require further insights into the structure of the problem, the Löschian solution and a utility integral maximising solution for a utility function which gives a demand function $x = (1/p^2)$ were calculated. The utility function used was $u(x, m) = 2x^{1/2} + m$. We again assume individuals are distributed along the line with unit density and they maximise utility subject to the constraint $px + m = \bar{m}$. The indirect utility function is now $v(p) = \bar{m} + 1/p$. Production and transport costs are as in Section 4, and we wish to maximise an integral of utilities per unit length subject to a production constraint as before. The pricing system[13] found from Eq. (13) was $p = [2\lambda/(\lambda + 1)](k + rt)$. The size of the market area was found by a process similar to that used for Eq. (14) and was given by

$$tA = \frac{1}{4}\log(1 + R)\left[1 - \left(\frac{R}{2(1 + R)\log(1 + R) - R}\right)^2\right] \tag{16}$$

after putting $k = t$ (choice of units). This is valid for all levels of tA, since it is always possible to price so as to cover costs. The mark-up $(2\lambda/\lambda + 1)$ increases from one to two as tA increases from zero to infinity. R is unique, given tA, since the r.h.s. is monotonic.

We also derived the Löschian solution. The profit a millpricing producer makes from a market radius R is

$$-2A + 2(p - k)\int_0^R \frac{1}{(p + rt)^2}\,dr.$$

Maximisation with respect to p gives $p = k + (k(1 + Rt/k)^{1/2}$. Substituting this back into the profit expression we find that his maximised profit is

$$-2A + \frac{2}{t}\left[\frac{(1 + R)^{1/2} - 1}{(1 + R)^{1/2} + 1}\right]$$

after putting $k = t$. The Löschian solution is thus

$$tA = \frac{(1 + R)^{1/2} - 1}{(1 + R)^{1/2} + 1}. \tag{17}$$

This equation is valid[14] for $tA < 1$. Equations (16) and (17) are plotted

[13] $p = 0$ is not a possible solution since revenue increases as $1/p$ but costs increase as $1/p^2$; thus the constraint could not be met with $p = 0$.

[14] As $tA \to 1$ the market area needed to make nonnegative profit tends to infinity (the pricing system is less flexible than that allowed in the optimal solution).

as Graphs V (i) and (ii) in Fig. 5. Conclusions are drawn in Section 6.

It should be noted that in this case also consumer surplus reflects utility changes precisely. Indeed this is the second case given in Samuelson [8]. It was very difficult to calculate further utility integral maximising curves for demand functions other than $x = 1/p$ and $x = (1/p)^2$. The reason for this difficulty can be seen by examination of Eq. (13).

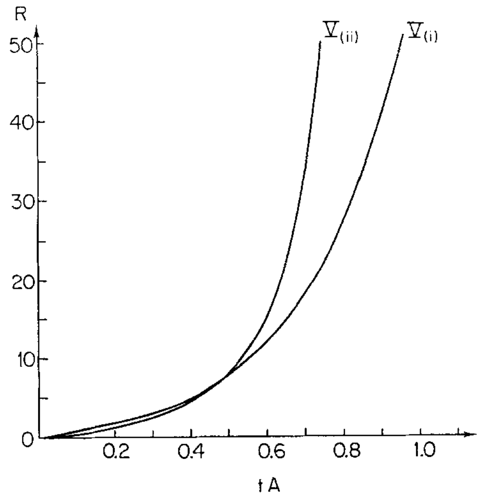

FIG. 5. Size of market areas against levels of fixed costs for the demand curve $x = (p)^{-2}$.

Legend:

Graph V(i) Utility integral maximisation using utility function $u = 2x^{\frac{1}{2}} + m$.

Graph V(ii) Löschian free-entry solution.

(Methods of Calculation are given in Section 5.)

6. CONCLUSIONS

Our first conclusion from an examination of the graphs is that the Löschian size of market area is not unambiguously smaller than an optimal size market area. This is perhaps in contradiction to the conclusion we might expect from imperfect competition theory. In fairness it should be stressed that the correct conclusion from imperfect competition theory is that firms are too small in relation to output (i.e., firms produce at an average cost above the minimum) rather than in relation to the size of market area. However, it is sometimes claimed that because of an imperfect competition type of process we have too much variety at output levels that are too low. We can, of course, use distribution along a line as an index of, e.g., tastes, political conviction, or size of feet (although an infinite line is not always appropriate.) The number of firms is then the number of products made, and transport costs can be viewed as conversion costs, special ordering costs, or just consumer discomfort (in the case of shoes.) If we use this interpretation we can see that the first part of the claim just mentioned would not always be justified; the answer will frequently depend on the level of fixed costs. In some cases the claim will only be true for fixed costs below a certain level.

The possibility of the crossing of the Löschian solution curve and the optimal solution curve does not depend on the allowance of negative profits in the optimal solution; it also occurs when profits are constrained to zero (see Figs. 2 and 5). Neither does it depend on the free choice of pricing systems allowed for an optimal solution in most cases; it occurs also with the constraint to uniform pricing (see Fig. 2). The reason it occurs is that, for high levels of fixed cost, firms need a large market area to break even and this can cause a Löschian solution to have larger market areas than optimum. We should note, however, that if we allow firms to charge a profit maximising price at each point, then the Löschian solution will result in smaller market areas than the socially optimum solution with profits constrained to be nonnegative. The proof is simple. Consider a Löschian solution in this case with fixed cost A and market area radius R. The firm is using a pricing system which maximises profit from this area and profit is zero. Any other pricing mechanism yields negative profit with this R and A. Therefore, the constrained socially optimal solution must have a larger market area in order to satisfy the nonnegative profit constraint. We can tentatively say that curve crossing is less likely, the less elastic the demand. Curve crossing occurred for the $x = (1/p^2)$ case, but not the $x = 1/p$ case; and it occurred towards the higher levels of A (and so higher average prices) for the linear demand curve case (the elasticity increases with price.) The reason is that for more inelastic demand curves producers have a higher degree of monopoly and can make higher profits from smaller areas (e.g., Graph IV (iii)) and so the Löschian process results in smaller areas for a given fixed cost A. Thus there is a greater tendency for the Löschian solution to be systematically smaller than the optimal one and curve crossing is less likely.

The Löschian Graphs of R against A are convex; see Figs. 4 and 5. The reason is that the value of R shown is that needed to give a profit level A, before fixed costs. Since the extra consumers are added at the edge where potential profit is smaller, we need proportionately more of them to achieve the same increment in profits. Formally, let $\pi(R)$ be the profit before fixed cost obtainable from a market size R. The Löschian solution gives $R(A)$ from $\pi(R) = A$. $R(A)$ is convex if $\pi(R)$ is concave. $\pi'(R) = z(R)$ where $z(R)$ is the profit to be made from a consumer at a distance R (unit density of consumers, say) $z'(R)$ is negative; so $R(A)$ is convex. The graphs tend to be "more convex", the greater the elasticity (compare IV(iii) and V(ii)) since the greater the elasticity, the smaller $z'(R)$. That is, with more elastic demand curves the profit to be obtained from a consumer falls off more rapidly with distance. We see this effect also with the R against A curves for the socially optimal case; they tend to be "more convex" when demand is more elastic and increments to consumer surplus are lower and we need "proportionately more consumers at the edges" to achieve given increments of net benefits at the edges. Formally, let the net benefit (before fixed cost) obtainable from a market size R be $b(R)$. Then for the optimal solution curve, R maximises $B(R) = (b(R) - A)/R$; thus $A = b(R) - Rb'(R)$ and $(dA/dR) = -Rb''(R)$. This last expression helps us understand the $R(A)$ curve for the optimal solution. For very low R we have very high (dR/dA). For R near the point where demand falls to zero, $b''(R)$ is very small (in absolute magnitude), since $b'(R)$ is just $y(R)$, the net benefit to be obtained from a consumer distance R. Again $y(R)$ falls more rapidly, the more elastic is demand and $R(A)$ is more convex.

In the case studied here (see Section 4 and Fig. 4) the consumer surplus result was very close to the utility integral result. The consumer surplus criterion gave bigger market areas for a given A than the areas from the other utility functions used in Section 4. The reason is that the monotonic transformations used were concave, and so gave greater weighting to the utility increments to those consumers furthest away from producers who have least welfare and thus made for smaller optimal market areas than consumer surplus. As the "concavity" (n) increases (i.e., egalitarianism increases) R decreases for these reasons. Presumably if a convex monotonic transformation had been used, we should have had larger optimal market areas than those found from consumer surplus. Similar considerations lead us to suppose that the introduction of lump sum taxation as a tool would lead to larger market areas for a given A, since we should then have less reason to worry about consumers at the edges being worse off.

The last set of conclusions concerns the level of benefits (Fig. 3). Mill pricing is superior to uniform pricing for both the Löschian and optimal cases, but the differences seen small. The welfare loss from constraining our firms to make nonnegative profits seems small for small levels of fixed costs but larger for larger fixed costs (compare Graphs III(i) and III(v).) The large welfare loss from the Löschian solution (compared with the optimal one Graph III(i)) is neither due to the constraint to make nonnegative profits nor, probably, to the wrong size of market area, but is due to overall levels of price that are too high (compare Graphs III(i) (ii), and (v)). The loss from the Löschian solution (compared with the optimal) decreases with levels of fixed cost. There are large areas of fixed cost which would not give production at all under the Löschian solution but where it is possible to have positive net benefits.

It is interesting to try to think of example of industries where market areas are either too big or too small. Perhaps an example of one with market areas too big might be the automobile industry; because fixed costs are so high we have too few genuinely different types of motor car on offer. An example of one where market areas are too small might be plumbing—the fixed cost of setting up as a local plumber is small. From inspection of Fig. 2 it seems that fixed costs need to be very large for Löschian solutions to yield market areas bigger than optimum.

Finally, it should be noted that most of these conclusions are drawn from calculations based on particular examples. They do seem reasonable from intuitive economic arguments but it would be pleasant to obtain more general demonstrations. For some of these conclusions, however, it seems that this will be fairly difficult, e.g., those concerning relative benefit levels and the occurence of curve crossing.

APPENDIX

For the two-dimensional situation we must decide on the appropriate shape of market areas. The application of the theorems of [4] to the case of constant marginal cost is immediate[15]. For a given specification of the density of producers the maximum benefit or profit at a point *is* a non-increasing function of its distance from the nearest producer with all the maximands used here. We know therefore that regular hexagons are correct for our given density. We therefore calculate the net benefit per unit area for a given hexagon and maximise with respect to the radius of the hexagon to obtain the socially optimal solution curves. For the Löschian solution curves we calculate the profit inside a given hexagon and find the size of hexagon by equating this to zero[16]. The results of these calculations were as follows: we number the cases with an asterisk and they have just the same assumptions as those used for the unstarred equations in the main body of the text. The accuracy is to 3 s.f.

[15] That hexagons are always optimal with marginal costs diminishing with output is less clear. See [4].

[16] For the linear demand curve case this had already been calculated by Mills and Lav [7].

$$A' = -0.963U'^4 + 1.216U'^3 \quad (2)^*$$
$$A' = 0.427U'^4 - 1.216U' + 0.866U'^2 \quad (3)^*$$
$$A' = -0.850U'^4 + 1.216U'^3 \quad (4)^*$$
$$A' = 0.427U'^4 - 1.216U'^3 + 0.866U'^2 \quad (5)^*$$
$$B' = 1.926U'^2 - 3.648U' + 1.732 \quad (7)^*$$
$$B' = 0.323U'^2 - 0.607U' + 0.433 \quad (8)^*$$
$$B' = 1.707U'^2 - 3.648U' + 1.732 \quad (9)^*$$
$$B' = 0.213U'^2 - 0.607U' + 0.433 \quad (10)^*$$

We have the same demand, production and transportation of functions as in Sections 2 and 3, and $B' = \rho B$ where B is benefits per unit area and

$$\rho = \frac{3.464b}{(a-bk)^2 D}, \qquad A' = \frac{Ab^3t^2}{D(a-bk)^4}, \qquad U' = \frac{bt}{(a-bk)} U,$$

and the values of U' as functions of A' in (7)*, (8)*, (9)*, (10)* are as in (2)*, (3)*, (4)*, (5)*, respectively. By comparing the starred and unstarred equations we see that the qualitative results for one and two dimensions are really very similar.

One difference between working in one and two dimensions is that, for a small range of values of A', "edge" effects arise with hexagons which do not arise in one dimension. In the discussion of Eqs. (2)–(5) it was noted that the largest relevant value of R' was where the benefit (respectively, profit) maximising choice of price gave zero net increment to benefit (profit) at the furthest point of the market. For hexagons, however, the net increment to benefits (profits) does not vanish at all points of the boundary simultaneously. Thus, when A' is so large that it is only possible to provide positive benefits (profits) with a large market area it is possible that demand will be zero at the "corners" of the hexagon (in the case of uniform pricing). The market area is then a hexagon with the corners "cut off" by a circle with radius given by zero demand (mill pricing) or zero benefit (profit) increment (uniform pricing). (This is contrary to the speculation of Mills and Lav [7] who speculated that dodecagons might be appropriate in certain circumstances[17]). The largest relevant A' is such that the maximum benefit (profit) that can be provided, with no limitation on area, is equal to zero. In this case the market areas are circular. The above equations (2)*–(10)* cease to be valid when A' is such that demand, or benefit (profit) increments, fall to zero at the farthest point of the edge

[17] It should also be noted that Mills and Lav did not observe the non-negativity conditions in drawing their graphs. Their curve for the hexagon (given by (3)* above) is therefore not valid for values of U, A larger than the U_*' and A_*' given below.

(call this A_*'). The largest A' for which production will take place (when market area is unconstrained and circular) is slightly larger. These two values of A' for Eqs. (2)*–(5)* (and consequently for (7)*–(10)*) are as follows (where the corresponding U' are U_*', the inscribed radius of the hexagon, U'_{**} the radius of the circle)[18]:

$$(2)^* \qquad \begin{aligned} A_*' &= 0.248 & A'_{**} &= 0.262 \\ U_*' &= 0.866 & U'_{**} &= 1.000 \end{aligned}$$

$$(3)^* \qquad \begin{aligned} A_*' &= 0.106 & A'_{**} &= 0.111 \\ U_*' &= 0.622 & U'_{**} &= 0.750 \end{aligned}$$

$$(4)^* \qquad \begin{aligned} A_*' &= 0.166 & A'_{**} &= 0.221 \\ U_*' &= 0.622 & U'_{**} &= 0.750 \end{aligned}$$

$$(5)^* \qquad \begin{aligned} A_*' &= 0.106 & A'_{**} &= 0.111 \\ U_*' &= 0.622 & U'_{**} &= 0.750 \end{aligned}$$

The other important difference between one and two dimensions is that the Löschian solution curve is no longer convex (the proof of the convexity of Graphs II(ii) and (iv) was given in Section (6)). The reason is that we are using linear dimension (U') as a measure of market area whereas R' in one dimension also gives, directly, the number of consumers in the market area. The graph of (3)* is initially concave and then convex (the eventual convexity arising for the reasons discussed in Section 6).

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