JOURNAL OF ECONOMIC THEORY 4, 174–179 (1972)

The Optimal Structure of Market Areas

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Received January 22, 1971

The "optimality of hexagons" has been asserted in economics since at least the 1930's and the classic work of Christaller [4] and Lösch [7]. Lösch demonstrated, for linear demand curves of consumers uniformly distributed over the plane, and constant transport costs per unit product per unit distance, and with a given mill price (where consumers bear transport costs), that the hexagon realises a given total demand with minimum area amongst regular plane-covering polygons (triangle, square, hexagon). He then said [6, pp. 110–114]: "The honeycomb is therefore the most advantageous shape for economic regions ... it makes possible the largest number of independent enterprises." More recently, Beckmann [1, p. 46] justified his use of hexagons in the following way: "Of all systems of regular market areas that will cover a plane completely, the hexagonal one is most efficient in the sense of minimising the distance to be covered between supplier and demander per unit area when demand is given." Other writers appeal to packing theorems, e.g., Berry [2], who justifies his use of the system with producers at the points of a regular triangular lattice with the theorem that the densest packing of circles with given radius places their centres at the points of such a lattice. These arguments provide grounds for guessing that regular hexagons yield optimal structure with respect to most sensible objective functions but are neither conclusive nor general.

For a rigorous justification we must start with a specific, well-formulated problem. Suppose a large domain S of the plane is covered with uniform

* We are grateful to Andrew Cornford for introducing us.

density by identical consumers,[1] and we have a given number n of producers at isolated points P_i of S. Let the area of S which the i-th producer supplies be C_i and suppose the C_i together cover S and their interiors are disjoint. The benefit at any point supplied by producer i is a nonincreasing function, $\phi(r)$, say, of the distance r between the point and P_i. Our interpretation of ϕ depends on the circumstances. If we are interested in profit maximisation, then the greatest profit (with given marginal production cost) we can make from a consumer, with a given demand curve, is a nonincreasing function of our distance from him, under most sensible pricing assumptions; likewise, the contribution to net social benefit under various reasonable criteria, e.g., cost minimisation, consumer plus producer surplus, etc. The problem is to maximise $\sum_i (\iint_{C_\perp} \phi(PP_i)\, dP)$ where the integral runs over the points of C_i and is with respect to the usual plane measure, and the argument of ϕ is the distance r just described. We must take C_i, since ϕ is a nonincreasing function, as those points which are closer to P_i than to any other producer, i.e., C_j is to be the Dirichlet cell of P_i. That the regular hexagonal arrangement of market areas is optimal has been shown in a certain sense by Fejes Tóth [5] in 1953. His basic theorem states the following:

THEOREM 1 (Fejes Tóth). *Let $P_1, P_2, \ldots, P_n$ be n points in a convex hexagon S and let ϕ: $R^+ \to R^+$ be a nonincreasing function. For a point P put $m(P) = \min(PP_1, PP_2 \cdots PP_n)$. Then*

$$\iint_S \phi(m(P))\, dP \leqslant n \iint_h \phi(PO)\, dP \tag{1}$$

where h is a regular hexagon with area $h = S/n$ and centre O and the integral is with respect to the ordinary plane measure.

Naturally if ϕ is strictly decreasing and there are at least two points P_i, equality cannot be attained.

The worth of this theorem to us is that (loosely) as $n \to \infty$ (or, equivalently, the size of S tends to infinity with constant density[2] of points P_i in S), an "approximate" cover of S by regular hexagons has edge effects of decreasing importance so that we can say for an infinite plane that the best cover is hexagons. (It is possible to make this limiting notion precise.)

[1] Precisely, there is a positive number d such that there are Ad consumers in any subdomain of area A.

[2] We define the notion of density of points over the plane as follows. Suppose we are given a countable number of points in the plane $P_1, P_2, \ldots P_n, \ldots$. We say that this system of points has density $\rho > 0$ if there are two functions $\rho'(n)$ and $\rho''(n)$ ($n = 1, 2, \ldots$) such that $\rho'(n)$ and $\rho''(n) \to \rho$, and in every square side n there are at least $\rho'(n) \cdot n^2$ points and at most $\rho''(n) \cdot n^2$ points.

A "uniqueness-in-the-limit" sort of result is the consequence of a proof by Bollobás of Theorem 1.[3]

Consequence of proof of Theorem 1 (*Bollobás*).

Suppose the points P_i have positive density ρ (see footnote 2 for precise definition). Let h be a regular hexagon area $h = 1/\rho$. Let $C(n, q)$ be the proportion of q-gons amongst the Dirichlet cells of the P_i in S_n[4]. Suppose $\lim_{n\to\infty} C(n, q)$ exists $\forall q \geqslant 3$ and $= c_q$. (Naturally, $\sum_q c_q = 1$.) Let R_q be regular q-gons (where, for convenience, we denote the area by the same letter) such that the average integral of $\phi(r)$ over the perimeter of R_q is independent of q and that $\sum_q c_q R_q = h$. Denote $\iint_{R\perp} \phi(PO)\, dP$ by p_q, where O is the centre of R_q. Let

$$a_n = \left[\frac{\iint_{S\perp} \phi(m(P))\, dP}{\rho n^2}\right]$$

where $m(P)$ is the minimum, for $P_i \in S_n$, of PP_i.

Then,

$$\limsup_{n\to\infty} a_n \leqslant \sum_q c_q p_q \leqslant \iint_h \phi(PO)\, dP$$

We have strictness in the right hand inequality (if ϕ is not constant on the perimeter of h) unless $c_6 = 1$ and $c_q = 0$ for $q \neq 6$.

In other words, for maximum average profit over the plane the limit of the proportion of hexagons (in the above sense) must be unity. An examination of the outline of the proof of Theorem 1 given in the appendix will reveal that this *is* a consequence of the proof. The left-hand inequality follows from lemmas 1 and 2 of the appendix, the strictness of the second inequality is a consequence of lemma 4 and a proposition similar to lemma 3, stating that $\sum_q c_q(q - 6) \leqslant 0$.

Now if the domain S of Theorem 1 is the union of n hexagons of a regular hexagonal tessalation, by choosing the P_i as centres of the hexagons, we have $\iint_S \phi(m(P))\, dP = n \iint_h \phi(PO)\, dP$. One solution to the problem, as opposed to the limiting nature of the above result, would be to prove that the left hand side is bounded above by the right hand side for any point system, and if ϕ is "not too constant", then equality holds only when the P_i are as we have just described. This result is rather more difficult to prove than Theorem 1 and, as Theorem 2 here, was obtained by Imre [6] in 1964. It is also a corollary of a result by Bollobás [3].

[3] Bollobás independently proved Theorem 1 for S any convex polygon with not more than six sides.

[4] Sn is a square with side n.

THEOREM 2 (Imre). *Let S be the union of n hexagons of a regular hexagonal tessalation consisting of hexagons congruent with a regular hexagon h (area h). Let $P_1, P_2, \ldots, P_n$ be arbitrary points in S. Then*

$$\iint_S \phi(m(P))\, dP \leqslant n \iint_h \phi(PO)\, dP.$$

Furthermore, if $\phi((\sqrt{3}/2)r - \epsilon) > \phi(r + \epsilon)$ for any $\epsilon > 0$ where r is the radius of the circumcircle of h, equality holds if and only if the P_i are in the centres of the hexagons of S. In other words, if the area in which we want to place n producers can be dissected into n regular hexagons area h, then the unique *best possible arrangement is when the producers are at the centres of the regular hexagons into which the area has been dissected (provided it costs more to transport r than $(\sqrt{3}/2)r$)*[5].

This theorem seems the best we can hope for if the number of producers to be arranged in the domain is given. At an easier level the reader can amuse himself by constructing simple geometrical arguments to show that (i) a system with producers at the points of a plane lattice cannot be optimal unless the producers are at the centres of regular hexagonal market areas, or (ii) square or triangular market areas are not optimal. In both cases we can find a hexagonal figure of appropriate area (plane-covering when replicated) whose points are uniformly closer to the producer than in the market area under consideration.

The application of this theorem is as follows. Suppose we have fixed costs of production (the same for each producer) so that production takes place at discrete points. Suppose we also have the same constant marginal costs for every producer, and transportation costs which increase with distance but which do not depend on direction. If we then stipulate a density of producers, we know from the above theorems that we must place them in regular hexagonal market areas (since we shall usually have a maximand in the class covered by these theorems). We can then find the size of regular hexagon that maximises net benefit per unit area under the benefit criterion that interests us (see, e.g., Stern [9]). In other words, we separate structure and size.

This separation is not always possible with marginal production costs which decrease with output. It is easy to provide a counterexample to the

[5] In the case where net benefits (ϕ) are constant between $(\sqrt{3}/2)\, r$ and r we should not expect a uniqueness result. In this case small variations in the boundary may not decrease the maximand. However, the hexagonal structure will still be one of the solutions. Thus the Mills and Lav [8] speculation on dodecagons, etc., is misplaced. If demand is zero at some points of the edge, we have market areas which are hexagons but the corners are not served (the unserved area is given by a circular "cut-off" with radius given by zero demand). See Appendix to Stern [9].

optimality of hexagons if we *a priori* fix the density of producers. We just consider a structure of regular hexagons with fixed density. Transfer a small area at the boundary of two hexagons A and B from service by A to B. There will be a production cost saving in the case of diminishing marginal costs of production. It is then possible to choose transport costs so small that this production cost saving is not offset by the increase in transport costs. This indicates that the inclusion of non-constant marginal production costs complicates the problem considerably.

In a subsequent paper we shall show that increasing marginal production costs are sufficient to ensure the optimality of the regular hexagonal arrangement, when optimisation includes the choice of the number of producers (and the criterion is the minimisation of total costs). We shall also show that if the production cost function is non-convex, there exists a transportation cost function for which the regular hexagonal arrangement is not optimum (see also Bollobás [3]).

APPENDIX

We give a brief sketch of the proof (by Bollobás) of Theorem 1. We give this version of the proof (slightly different from that of Fejes Tóth) in order to justify "the consequence" given in the text. The proof proceeds by way of four lemmas (we use the same notation as that of the text).

LEMMA 1. *Let X be a point of Y_k, a k-gon. Let K_k be the regular k-gon of the same area and let O be its centre.*

Then[6]

$$\iint_{Y_k} \phi(XP)\, dP \leqslant \iint_{K_k} \phi(OP)\, dP.$$

LEMMA 2. *Let K_i and K_j be regular polygons centre O. Denote the average integral of $\phi(OP)$ when P goes over the perimeter of K_i by $\eta(K_i)$. If $\eta(K_i) < \eta(K_j)$, then, by increasing K_i and decreasing K_j (so that $K_i + K_j$ is fixed), we increase $\int_{K_i} + \int_{K_j}$ where the integrand is $\phi(OP)$ and the integral runs over the area indicated.*

LEMMA 3. *Suppose S is a convex polygon with q sides, and $s(p)$ of the polygons C_i have p sides (C_i are the Dirichlet cells of the P_i, which are points of S.) Then $q - 6 \leqslant \sum_{p=3}^{\infty} (6 - p)\, s(p)$.*

By considering degenerate polygons we can take equality.

[6] By choosing Y_k equal to K_{k-t} (a degenerate k-gon) we have $\int_{K_p} \leqslant \int_{K_q}$ if $p \leqslant q$ and K_p and K_q have equal area.

LEMMA 4. *Let K_t $(t = p, q, s)$ denote a regular t-gon with centre O. Denote the radii of the inscribed and circumscribed circles by ρ_t and r_t, respectively. Suppose $p < q < s$, $\rho_p \leqslant \rho_s$ and $r_s \leqslant r_p$.* If

$$(s - q) K_p + (q - p) K_s = (s - p) K_q ,$$

then $(s - q) \int_{K_p} + (q - p) \int_{K_s} \leqslant (s - p) \int_{K_q}$, where the integrals are as in Lemma 2.

The proof now goes as follows. Remember, we are looking for a bound on the sum of integrals for the structure we are given. We proceed by steps, each step increasing the sum of integrals. Step 1: Replace an irregular polygon by a regular one of the same area (Lemma 1). Step 2: Expand and contract the regular polygons until the $\eta(K_i)$ are all equal (Lemma 1). The polygons now satisfy the conditions for Lemma 4. Lemma 3 ensures that we can apply the convexity conditions of Lemma 4 to replace all the polygons by hexagons by considering pairs of polygons with numbers of sides r, s where $r > 6$ and $s < 6$. This is Step 3.

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