Optimum Saving with Economies of Scale [1,2]

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1. INTRODUCTION

There are many problems in economics where it is important to think of investment taking place in discrete lumps rather than as a continuous flow. These are usually problems where fixed costs are significant or, more generally, where there are economies of scale. An important example is the creation of a new centre of population which requires some large initial capital outlay. A second is that of an individual saver who makes investments at discrete points in time because there is some cost to the act of making an investment.

The model that we investigate in this paper was originally motivated by the first of these examples. It has some limitations as a model of timing and size of new centres, however, and these are discussed later. It should be viewed as a first approach to that problem, though we suspect that better models will be similar in flavour. It turns out, however, that a special case of the model does capture the problem of the individual saver.

The plan of the paper is as follows. Previous literature is discussed in this introduction. Then a growth model with economies of scale in the production of output, using capital but no labour, is presented. Conditions necessary for optimality are obtained, and some simple properties of the optimal path established. In Section 3 we show that in a simple special case the optimal policy is readily obtained, but takes the economy to infinite output in finite time. We then turn to the version appropriate to individual saving behaviour, with fixed transaction costs and a constant interest rate, showing how to identify the optimum policy and computing it for the iso-elastic utility case. Finally in Section 5 we introduce labour, retaining economies of scale, and solve the optimum growth problem for the Cobb-Douglas production function and iso-elastic utility. We conclude with some remarks on generalizations. A general existence theorem for optimum paths is given in an appendix. We would like to draw attention to the particular importance and difficulty of establishing which of the paths satisfying the necessary conditions actually is optimal. It is this problem together with that of proving existence of an optimum that make rigorous mathematics essential at some stages of the argument. Elsewhere we have not troubled to show that our arguments can be made rigorous.

[1] *First version received June* 1973; *final version accepted September* 1974 (*Eds*.).

[2] Work on this paper was done when all three of us were, at different times, visiting the Massachusetts Institute of Technology. We are grateful to their Department of Economics for hospitality. Valuable comments were received from participants in seminars at MIT, the Cowles Foundation, and elsewhere; and from the referees of a previous version, and John Flemming. The research of Dixit ande Stern was financed by a grant from the National Science Foundation. Computing assistance was provided by Mike O'Neill, of the Nuffield College Research Services Unit.

The best known study of the problems in development planning that arise from increasing returns is the work by Manne *et al.* [4]. The examples discussed in that volume were concerned with the choice of the size of plants and the time-intervals between the construction of plants, to minimize the present value of costs while meeting a given time profile of output. We shall call such a problem a Manne-type problem. Demand can reasonably be taken as exogenous in these models since they are designed to discuss capacity expansion in individual industries. For our purposes, we need a model where demand is endogenous, for the choice of the optimum time paths of output and consumption is the question at issue.

A model with increasing returns (in one sector) where output and consumption levels are endogenous is given in Weitzman [7]. Weitzman's model has two types of capital: α-capital and overhead or β-capital. Output can be consumed or invested, and is produced with a standard neo-classical production function of α-capital. This output is constrained, however, by the quantity of β-capital. Overhead capital is produced from the output good under conditions of increasing returns. Weitzman shows that the optimum path can be decomposed into two types of phases. In the first type, all investment goes to add to α-capital, and a standard Ramsey path is followed. In the second kind, a Manne-type problem is solved for investment in β-capital where the Ramsey path gives the output-demand profile and prices come from the marginal utility of consumption. During this phase, while savings accumulate for subsequent investment in β-capital, output and consumption are constant. The phases alternate indefinitely.

In our model we have just one sector. This sector produces output which can be consumed or saved. Investment shows increasing returns, at least for small levels, in that $g(x)$, the output flow from an investment of size x measured in units of accumulated output, has zero derivative at the origin. We assume that investments, once made, cannot be augmented. A full treatment of the problem of new centres would allow some, possibly limited and certainly more costly, additions to existing centres.

In the first version of the model, there is no labour. This is a serious drawback and detracts especially from the interpretation of the model in terms of new centres of population. More generally, it ignores the chief reason for expecting decreasing returns to investment in the real world, in association with lumpy investment in the manner considered in this paper. In a second version, we introduce labour. We had some trouble in finding a trick which would make a model incorporating labour manageable. Our model does not distinguish between ex-ante and ex-poste substitutability, but it does generate some interesting results, and is a useful vehicle for discussions of decentralization.

It is easy to see, and is proved below, that the optimum policy, for an objective of maximizing the integral of the utility of consumption, is to save for a time until an inventory of appropriate size has been accumulated, and then to make an investment. In our model, therefore, output is constant for a time and then jumps. This contrasts with the Weitzman case where output rises continuously in the Ramsey phase and is constant in the Manne-type phase. The reason we have used a model with increasing returns in the economy as a whole is our interest in new centres of population. Even so, discrete jumps in output are a little stark. If there were decreasing returns elsewhere in the model, or we allowed for many commodities, we would presumably have some smoothing.

It turns out that our model can, in a sense, be viewed as a limiting case of the Weitzman model. Although this helps in understanding the problem, it is of no assistance in finding the optimum policy. Further, we shall pay special attention to a specific example of a production function $g(x) = -\sigma+\rho x$ where σ and ρ are positive constants, which was not discussed by Weitzman. It is this example which gives us a model of the problem faced by the individual saver. Given a fixed cost of making an investment σ/ρ and an interest rate ρ an investment of size x yields him a stream $\rho(x-\sigma/\rho)$ indefinitely.

Flemming [2] has discussed the problem facing an individual who allocates his initial wealth to the purchase of a sequence of durable goods only one of which is held at a time and each of which yields a consumption stream for the time that it is held. Wealth not allocated to the durable good earns interest. The individual must decide how often to " trade-in " his old model for a new one and how large a new model to purchase. He obtains a second-hand price of ρ $(0<\rho<1)$ times the original purchase price of a good and maximizes a discounted stream of the utility of consumption. Flemming obtains, for the iso-elastic utility function, the optimum policy of a constant time between investments and a constant ratio between sizes of successive models.

A problem similar to our version of individual saving has been discussed by Baumol [1] and Tobin [6]. They both consider the optimum time sequence and amounts of the conversion of bonds into money to meet a steady flow demand for transactions. We are considering something like a mirror image: when and how much to invest from a flow of saving. They have a fixed cost of a withdrawal, while we have one for making an investment.

The main difference between our model and the Baumol-Tobin model is that in ours the flow of saving is endogenous, with the result that income, consumption and marginal utility can change over time. Baumol and Tobin have an exogenous and constant transaction flow, so they can consider just one of a sequence of uniform withdrawals, a procedure which is justified if the horizon is infinite.

2. THE MODEL WITHOUT LABOUR

There is one commodity. An investment x at a particular date yields an output stream $g(x)$ ever after. We assume $g(0) = g'(0) = 0$, implying that there are economies of scale at least for small investments, and that g increases with x and is differentiable for x such that $g(x)>0$. We start with a given output capacity y_0, and no accumulated savings from the past. We denote by $k(t)$ total savings since time zero (which may not all have been invested by time t). $y(t)$ is output at time t, and for convenience we take $y(t)$ to be left-continuous at jump points. x_t is the investment done at t: except at discrete points, x_t will be zero. A consumption path $c(t)$ is feasible if for all t,

$$0 \leqq c(t) \leqq y(t) \qquad \ldots(1)$$

$$y(t) = c(t)+\dot{k}(t) \qquad \ldots(2)$$

$$y(t) = y_0 + \sum_{0 \leqq t' \leqq t} g(x_{t'}) \qquad \ldots(3)$$

$$k(t) \geqq \sum_{0 \leqq t' \leqq t} x_{t'}. \qquad \ldots(4)$$

If the time interval between investments is not strictly positive, then the summation signs can be interpreted as appropriate integrals. We see below that it is never optimal to have continuous investment. Note that mere efficiency implies equality in (4) at each instant t when an investment is made; for if a plant of a particular size is to be constructed it may as well be constructed as early as possible, thus giving the benefit of its output for a longer period of time.

We wish to find a feasible path which " maximizes " an undiscounted integral of $u(c(t))$ over all future time, " maximization " being interpreted in the overtaking sense. In other words, a feasible path $c^*(t)$ is optimum if, for any other feasible path $c^0(t)$, there exists T_0 such that

$$\int_0^T u(c^*(t))dt \geqq \int_0^T u(c^0(t))dt, \quad \text{for all } T \geqq T_0. \qquad \ldots(5)$$

It will be recollected that in infinite-horizon growth models there are, broadly speaking, two ways of identifying the optimum path of the economy. One is to find a path satisfying the intertemporal first-order conditions for a maximum (the Euler conditions), and with known asymptotic behaviour, which can be proved directly to be an optimum path. Notice that it is not enough to find a path that satisfies the Euler conditions, and seems to be better than any other Euler path, because it is possible that no optimum exists: that is why one needs a direct proof that the identified path is optimal. The other is to prove by an alternative method that an optimum exists, and then identify the best Euler path. The first method relies on sufficient conditions for an optimum, the second on necessary conditions. The sufficiency method works quite well in optimum growth models with convex technology. For our models we have had to use the necessity method, essentially because first order conditions do not imply global maximization.

Naturally, we must make some assumptions if we are to guarantee existence in our model. The chief assumptions are:

Assumption A. u is increasing, strictly concave, satisfying $u(c) \to 0$ $(c \to \infty)$, and $u'(c) \to \infty$ $(c \to 0)$.

Assumption B. For any $y_0 > 0$, there exists a feasible path with convergent utility integral.

Assumption C. For all y_0 there exists $\delta > 0$ such that all paths starting from y_0 and making the first investment at a time sooner than δ can be overtaken.

Assumption A implies that the utility function is always negative, and therefore that for every path the utility integral either diverges to $-\infty$ or converges. Assumption B then assures us that the utility integral has a finite supremum for the model, so that nearly optimal paths exist. In an appendix we prove that these assumptions in fact ensure that an optimal path exists. We suspect that Assumption C holds automatically if $g'(0) = 0$, but have not been able to prove it. It clearly holds if $g(x)$ is zero over an interval to the right of $x = 0$, which is the model of Section 4.

Assumption B may seem to be a rather awkward one, but it is generally easy to check: for example, one might look at a path resulting from saving a constant proportion of output and investing once a year, or one obtained by keeping x constant. We shall justify it in particular cases below.

To derive necessary conditions for the optimum, we look at development as a sequence of periods, in each of which saving is accumulated, but investment is made only at the end of the period. Since utility is not discounted, and is strictly concave, optimum consumption must be constant throughout a period. Defining

t_i = length of ith period,

y_i = output during ith period,

x_i = size of investment at the end of ith period,

c_i = consumption during ith period,

we have

$$x_i = t_i(y_i - c_i) \qquad \text{...(6)}$$

because it is clearly inefficient to carry any savings over instead of incorporating it in the current investment (one could otherwise have invested sooner). Also

$$y_{i+1} - y_i = g(x_i); \qquad \text{...(7)}$$

and we seek maximization of

$$\sum_{i=0}^{\infty} t_i u(c_i) \qquad \text{...(8)}$$

subject to the requirement that $\Sigma t_i = \infty$ or $y_i \to \infty$. The replacement of the integral by the sum is justified rigorously in the appendix, in the course of the proof of the main theorem. The associated requirements say that either our sequence of investments stretches over the indefinite future or we reach infinite output in finite time. We have as yet no guarantee that $\Sigma t_i = \infty$ and we shall have to consider this point carefully below.

Substituting from (6) for t_i in (8), it can be seen that the maximand becomes

$$\Sigma x_i \frac{u(c_i)}{y_i - c_i}. \qquad \text{...(9)}$$

It follows at once that for each i, c_i must maximize $u(c)/(y_i - c)$:

$$u(c_i) + u'(c_i)(y_i - c_i) = 0. \qquad \text{...(10)}$$

This equation, known as the Keynes-Ramsey equation, was derived by Ramsey for the optimum rate of saving in an economy with continuous investment. Since it is essentially a rule for an economy without change other than that brought about by capital accumulation, we should not be surprised to see that it remains valid.

Now consider the effect of varying y_i, x_i and x_{i-1} simultaneously while leaving everything up to $(i-1)$ and after $(i+1)$ unchanged. This can be done in such a way that the feasibility conditions (7) continue to hold. The changes must satisfy

$$-dy_i = g'(x_i)dx_i, \quad dy_i = g'(x_{i-1})dx_{i-1}. \qquad \text{...(11)}$$

Then the effect on (9) is

$$\frac{u(c_i)}{y_i - c_i} dx_i + \frac{u(c_{i-1})}{y_{i-1} - c_{i-1}} dx_{i-1} - \frac{x_i u(c_i)}{(y_i - c_i)^2} dy_i.$$

Assumption C ensures that t_i, and therefore x_i, cannot be zero and thus the first-order condition holds with equality. Using (6) and (10), the condition is

$$\frac{u'(c_i)}{g'(x_i)} - \frac{u'(c_{i-1})}{g'(x_{i-1})} + t_i u'(c_i) = 0. \qquad \text{...(12)}$$

(10) and (12) are the first-order conditions (corresponding to the Euler conditions in more orthodox calculus of variations). Following the example of standard optimum growth analysis, we expect that the optimum path will be that solution of (10) and (12) which has the smallest initial t_0 (and x_0), subject to being feasible for all time. It can be verified in particular cases that as t_0 increases and the subsequent path satisfies (10) and (12), all terms in the series $\Sigma t_i u(c_i)$ become smaller (i.e. more negative), and we presume that this is very generally true.

We must now clarify the possibility that Σt_i converges. The assumptions we have made are by no means sufficient to exclude the possibility of infinite output in finite time.

The following two lemmas give a condition sufficient to exclude such explosion, and indicate a property one may expect to hold in many such cases.

Lemma 1. *If there exists k such that $g(x)/x \leqq k$ for all $x > 0$, $\sum_0^\infty t_i$ can be finite only if y_i tends to a finite limit.*

Proof. Since negative consumption is impossible, $x_i \leqq t_i y_i$. Therefore, using our hypothesis on g,

$$y_{i+1} \leqq (1 + kt_i)y_i.$$

Multiplying such inequalities together,

$$\begin{aligned} y_I &\leqq y_0 \prod_{i=0}^{I-1} (1 + kt_i) \\ &\leqq y_0 \prod_{i=0}^{I-1} e^{kt_i} \\ &= y_0 \exp\left(k \sum_0^{I-1} t_i\right), \end{aligned}$$

which is bounded by hypothesis. Then $\{y_i\}$ is a bounded monotone sequence, and therefore has a finite limit.

For the next lemma we specialize to an iso-elastic utility function and show that if x_i tends to a finite limit as i tends to infinity, then this limit must be where average productivity is maximum. Some reader will, we hope, show that this holds more generally.[1]

The iso-elastic utility function bounded above by zero will be of the form

$$u(c) = -c^{-n}, \quad n>0.$$

For it, the Keynes-Ramsey equation becomes

$$c_i = (1-\beta)y_i, \qquad \text{...(13)}$$

where $\beta = 1/(1+n)$, so $0<\beta<1$, and (6) becomes

$$x_i = \beta t_i y_i. \qquad \text{...(14)}$$

These will be of much use later.

Lemma 2. *For the case* $u(c) = -c^{-n}, n>0$, *if on the optimum path* $x_i \to \bar{x}$, *then* $\bar{x}$ *satisfies* $\bar{x}g'(\bar{x})/g(\bar{x}) = 1$.

Proof. If $x_i \to \bar{x}$ then $(y_{i+1}-y_i)$, and hence y_i/i, converges to $g(\bar{x})$ by (7). By (14),

$$it_i \to \bar{x}/[\beta g(\bar{x})]. \qquad \text{...(15)}$$

But (12) implies that

$$\lim_{i\to\infty} [it_i g'(x_i)] = \lim_{i\to\infty}\left[i\,\frac{g'(x_i)}{g'(x_{i-1})}\,\frac{u'(c_{i-1})}{u'(c_i)} - 1 \right].$$

With $u(c) = -c^{-n}$ and $x_i \to \bar{x}$, the right-hand side limit is $\lim_{i\to\infty} [i(1-1/i)^{1/\beta}-1]$, which equals $1/\beta$, since $c_i/i \to (1-\beta)g(\bar{x})$. By (15) the left hand limit is $\bar{x}g'(\bar{x})/[\beta g(\bar{x})]$, and we have completed the proof.

Before turning to specific production functions, we suggest a dual approach to our model and its relation to the work of Weitzman [7]. It should be noted that in writing our maximand as $\Sigma t_i u(c_i)$, we had to be careful to ensure that it equals $\int_0^\infty u dt$, since otherwise the problem has a formal solution $t_i = 0$ for all i. This is obviously not a solution to our original problem and is ruled out by Assumption C. Passing from an integral of utility to the sum of $t_i u(c_i)$ is valid if Σt_i diverges, or if $c(t)$ can be infinite after time Σt_i, i.e. if $y_i \to \infty$. These requirements were constraints on our maximization.

We should presumably have a corresponding difficulty if we attempted to write the problem in some cost-minimization form: minimize $\Sigma p_i x_i$ where the p_i are appropriate prices for plants. Formally we have a solution $x_i = 0$ for all i, unless we rule it out through the constraints. Weitzman used cost-minimization to determine investment in his β-capital (overhead capital the production of which shows increasing returns to scale) and derived the prices from marginal utilities on the target path—the Ramsey path which would be followed if the productivity of α-capital were unrestricted by the necessity for β-capital. The target path constrains the x_i.

Our model might be viewed as a limiting case of Weitzman's model where the productivity of α-capital tends to infinity. This does not help, however, in the solution to our problem, since if we take the limit as the productivity of α-capital tends to infinity we lose

[1] Presumably the case of asymptotic iso-elasticity is not difficult.

the Ramsey-type target path. It might be claimed that the target for a cost-minimization form of our model is simply infinite output. Two intuitive arguments in favour of such a view would be that the " constraint " analogous to $\Sigma t_i = \infty$ is $\Sigma x_i = \infty$, and that infinite output is the limit, as the productivity of α-capital tends to infinity, of the Ramsey path in Weitzman's model.

3. THE CONSTANT ELASTICITY CASE

From now on we specialize to the case of the iso-elastic utility function. In this section the production function, too, will have constant elasticity, i.e. $g(x) = Kx^{\varepsilon}$, $\varepsilon > 1$. This has an average product which tends to infinity with output and thus we have to entertain the possibility of infinite output in finite time (see Lemma 1).

Using (13), equation (12) becomes

$$\left(\frac{y_i}{y_{i-1}}\right)^{n+1} = \left(\frac{x_{i-1}}{x_i}\right)^{\varepsilon-1}\left\{1+(n+1)\varepsilon K\,\frac{x_i^{\varepsilon}}{y_i}\right\}. \qquad \text{...(16)}$$

Since $y_{i+1} - y_i = Kx_i^{\varepsilon}$, this can be expressed most conveniently as a recursion relation for the expansion coefficient

$$\alpha_i = \frac{y_{i+1}}{y_i} - 1. \qquad \text{...(17)}$$

Thus

$$\left(\frac{x_{i-1}}{x_i}\right)^{\varepsilon-1} = \left(\frac{y_i - y_{i-1}}{y_{i+1} - y_i}\right)^{(\varepsilon-1)/\varepsilon} = \left\{\frac{\alpha_{i-1}}{\alpha_1(1+\alpha_{i-1})}\right\}^{(\varepsilon-1)/\varepsilon}$$

and (16) becomes

$$\alpha_{i-1}^{-\gamma}(1+\alpha_{i-1})^{n+1+\gamma} = \alpha_i^{-\gamma}\{1+(n+1)\varepsilon\alpha_i\}, \qquad \text{...(18)}$$

where

$$\gamma = (\varepsilon - 1)/\varepsilon.$$

There is a unique non-zero $\bar{\alpha}$ satisfying $\bar{\alpha}_i = \alpha_{i-1} = \bar{\alpha}$. In fact, $\alpha_i < \alpha_{i-1}$ when α_{i-1} is small, $\alpha_i > \alpha_{i-1}$ when α_{i-1} is large, and where $\alpha_i = \alpha_{i-1} = \bar{\alpha}$, it is readily shown that

$$\frac{d\alpha_i}{d\alpha_{i-1}} = \frac{1+(n+1)\varepsilon\bar{\alpha}}{1+\bar{\alpha}} > 1.$$

The graph of α_i as a function of α_{i-1} is shown in Figure 1. It is clear from the graph that for all paths other than $\alpha_i = \bar{\alpha}$ (all i), $\alpha_i \to 0$ or ∞ as $i \to \infty$. Both of these can, we believe, be rejected. The optimum policy is

$$y_{i+1} - y_i = \bar{\alpha} y_i. \qquad \text{...(19)}$$

The policy of increasing capacity by a constant fraction at each investment was found by Srinivasan [5] and Weitzman [7] for the same production function.

We can now see that infinite output is reached in finite time, as follows. The policy $\alpha_i = \bar{\alpha}$ implies $y_i = (1+\bar{\alpha})^i y_0$, $g(x_i)/y_i = \bar{\alpha}$, and hence $t_i = D(1+\bar{\alpha})^{-i\gamma}$, where D is a constant, using $g(x_i) = Kx_i^{\varepsilon}$ and $x_i = \beta t_i y_i$. But $\gamma = (\varepsilon - 1)/\varepsilon > 0$, so Σt_i converges, and of course $y_i \to \infty$. This does not contradict the existence theorem: there is no problem with the convergence of the utility integral.

This case is instructive, but we should perhaps be circumspect about a production function which yields infinite output in finite time. We turn now to a production function

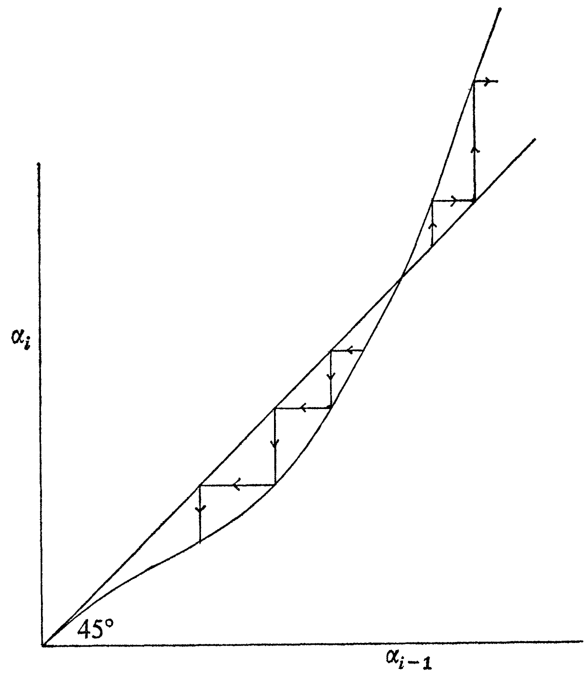

FIGURE 1

with bounded average product so that Lemma 1 applies and output goes to infinity only asymptotically.

4. THE FIXED COST CASE

We retain the constant elasticity utility function $u(c) = -c^{-n}$, but turn to the case

$$g(x) = \begin{cases} -\sigma+\rho x & \sigma/\rho \leqq x \\ 0 & 0 \leqq x < \sigma/\rho \end{cases} \quad \text{...(20)}$$

We now also have the interpretation of the model as that of the individual saver facing a given fixed cost σ/ρ of making an investment, and a marginal return ρ on investments. Equation (12) becomes

$$(y_i/y_{i-1})^{1/\beta} = 1+\rho t_i \quad \text{...(21)}$$

and the accumulation equation

$$y_i = y_{i-1}-\sigma+\rho\beta t_{i-1}y_{i-1}. \quad \text{...(22)}$$

Before discussing the solution of these equations, let us note that the optimum x_i can be written as a function of y_i:

$$x_i = \frac{\sigma}{\rho} h(y_i/\sigma). \quad \text{...(23)}$$

From now on we set $\sigma = \rho = 1$ by choice of units of time and commodities.

The character of the solutions of (21) and (22) can be best appreciated if we obtain a difference equation for t. Eliminating y_i from (21) and (22) we obtain

$$t_i = \{1+\beta t_{i-1}-y_{i-1}^{-1}\}^{1/\beta}-1. \quad \text{...(24)}$$

Therefore $t_i \geqq t_{i-1}$ if and only if

$$y_{i-1} \geqq \{1+\beta t_{i-1}-(1+t_{i-1})^{\beta}\}^{-1} \qquad \ldots(25)$$

At the same time, the need to have $x_i \geqq 1$ means that

$$t_{i-1} \geqq 1/(\beta y_{i-1})^{-1}. \qquad \ldots(26)$$

Given y_0, and having chosen t_0, we can generate a sequence (y_i, t_i) satisfying (22) and (24). The possible sequences are shown in Figure 2. The lower curve shows the effect of the inequality (26); any sequence that crosses into the region below it yields an infeasible policy. The upper curve shows where t_i would be equal to t_{i-1}; any sequence that crosses into the region above this curve would remain there for ever.

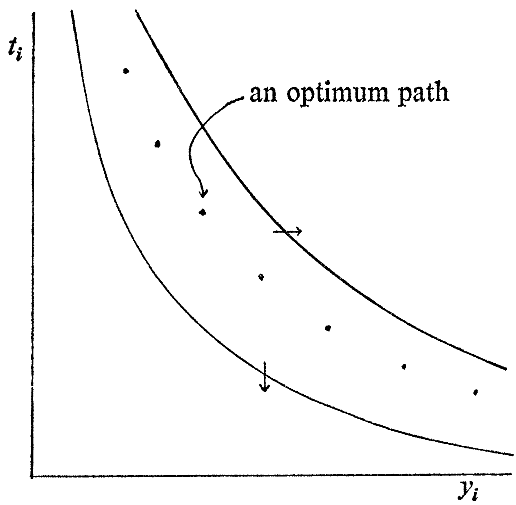

FIGURE 2

In Lemma 3 we prove some more properties of these sequences, and then characterize the optimum policy in the theorem that follows.

Lemma 3. (a) *On any sequence that is forever feasible,* $y_i \to \infty$. (b) *Comparing sequences for any given i,* y_i *and* t_i *are increasing functions of* y_0 *and* t_0. (c) *If* $y'_0 \geqq y_0$, $t'_0 \geqq t_0$, *and the sequences* (y_i, t_i) *and* (y'_i, t'_i) *starting respectively from* (y_0, t_0) *and* (y'_0, t'_0) *are both feasible, then for each i,*

$$t'_{i+1}-t_{i+1} \geqq t'_i - t_i.$$

Proofs. (a) For a feasible policy, (26) and (22) show that (y_i) is an increasing sequence. If it does not tend to infinity, it must then have a finite limit $\bar{y}$. Then (21) shows that $t_i \to 0$, while (22) shows that $t_i \to 1/(\beta\bar{y})$, which is a contradiction.

(b) This is obvious by induction from (22) and (24).

(c) From part (b) of this lemma and the feasibility condition, we conclude

$$t'_i - 1/y'_i \geqq t_i - 1/y_i \geqq 0.$$

Now consider the function $f(z) = (1+z)^{1/\beta}$. For non-negative z its derivative is bounded below by $1/\beta$, and therefore for $z' \geqq z \geqq 0$, we have

$$f(z')-f(z) \geqq (1/\beta).(z'-z).$$

In the present case this becomes, from (24),

$$t'_{i+1}-t_{i+1} \geqq \frac{1}{\beta}\{(\beta t'_i-1/y'_i)-(\beta t_i-1/y_i)\}$$

$$=(t'_i-t_i)+\frac{1}{\beta}(1/y_i-1/y'_i).$$

But, by part (b), $y'_i \geqq y_i$. This completes the proof.

It now remains to examine the effect on total utility of the choice of t_0 for a given y_0. It will be seen that t_0 should be chosen as small as possible subject to the resulting sequence being forever feasible. Thus the optimum path is the lowest that never hits the lower curve in Figure 2. We will also show that any higher choice of t_0 yields a path that crosses the upper curve. Thus the optimum is the unique path channelled between the two curves. These results are proved in the following theorem.

Define $t^*(y_0) = \inf\{t_0 \mid \text{resulting } (y_i, t_i) \text{ satisfy (26) for all } i\}$. Clearly $t^*(y_0)$ exists and is positive for each y_0. Then

Theorem 1.

(a) *$t^*(y_0)$ is the optimum choice of t_0 given y_0.*

(b) *Any choice $t_0 > t^*(y_0)$ will yield a sequence that crosses the upper curve in Figure 2.*

(c) *$t^*(y_0)$ is a decreasing function. t^* tends to zero as y_0 tends to infinity.*

Proof.

(a) Consider the sequence starting from t_0. The utility in period i is $t_i u(c_i)$, which is proportional, in the iso-elastic case, to

$$\begin{aligned}-t_i y_i^{-n} &= -(y_i^{1+n}y_{i-1}^{-1-n}-1)y_i^{-n} \quad \text{by (21)}\\ &= y_i^{-n}-y_i y_{i-1}^{-1-n}\\ &= y_i^{-n}-(y_{i-1}-1+\beta t_{i-1}y_{i-1})y_{i-1}^{-1-n} \quad \text{by (22)}\\ &= -\beta t_{i-1}y_{i-1}^{-n}+(y_i^{-n}-y_{i-1}^{-n})+y_{i-1}^{-1-n}.\end{aligned}$$

Summing from $i = 1$ to $i = I$, we can write

$$\sum_0^I(-t_i y_i^{-n}) = \beta\sum_0^I(-t_i y_i^{-n})-t_0 y_0^{-n}+\beta t_I y_I^{-n}+y_I^{-n}-y_0^{-n}+\sum_0^{I-1} y_i^{-1-n}$$

or

$$(1-\beta)\sum_0^I(-t_i y_i^{-n}) = -t_0 y_0^{-n}-y_0^{-n}+\beta t_I y_I^{-n}+y_I^{-n}+\sum_0^{I-1} y_i^{-1-n}. \qquad \ldots(27)$$

The series on the left-hand side consists of negative terms, therefore it is decreasing. Further, it is bounded below by $(-\beta t_0 y_0^{-n}-y_0^{-n})$. Therefore it converges, and it is a necessary condition of this convergence that $t_I y_I^{-n}\to 0$. By part (a) of Lemma 3, so long as the sequence is forever feasible, $y_I\to\infty$ and thus $y_I^{-n}\to 0$. We can then take limits in (27) to write

$$(1-\beta)\sum_0^\infty(-t_i y_i^{-n}) = -t_0 y_0^{-n}-y_0^{-n}+\sum_0^\infty y_i^{-1-n}.$$

The left-hand side is a positive multiple of total utility. The right-hand side for fixed y_0 is a decreasing function of t_0 and of all the y_i, each of which is an increasing function of t_0 by part (b) of Lemma 3. This proves the result. Continuity considerations show that the sequence starting from $t^*(y_0)$ cannot cross the upper curve (nor the lower one). Therefore, on the optimum sequence, t_i decreases and tends to 0.

(b) Suppose the choice $t^*(y_0)$ produces a sequence (y_i^*, t_i^*) while a choice $t_0 > t^*(y_0)$ results in (y_i, t_i). Then, for all i, we have by part (*c*) of Lemma 3,

$$t_i - t_i^* \geqq t_{i-1} - t_{i-1}^* \dots \geqq t_0 - t^*(y_0)$$

and therefore

$$t_i \geqq t_i^* + t_0 - t^*(y_0) \geqq t_0 - t^*(y_0) > 0,$$

which shows that t_i is bounded below by a positive number. The sequence must therefore cross the upper curve, which is asymptotic to the horizontal axis.

(c) Suppose $y_0' > y_0$ and $t^*(y_0') > t^*(y_0)$. Then, using part (c) of Lemma 3 repeatedly for the resulting sequences, we have

$$t_i' - t_i \geqq t^*(y_0') - t^*(y_0) > 0.$$

But $t_i' \to 0$, and thus t_i must eventually become negative, which is impossible. The last part of (c) is already proved, since $t_i \to 0$ on any optimum path.

Having thus identified the optimum policy, we can calculate it. We have done this by starting from an estimate of optimal t_i when y_i is very large, and using equations (21) and (22) to calculate t and y for successively lower values of i. By starting from slightly different large values of y, one can equally map out the whole optimal policy showing t or x as functions of y. The computation can be done by taking the " initial " t_i as $1/(\beta y_i)$, the minimum possible length of the (Ramsey) saving period. It is then clear (cf. Figure 2) that one can get a good approximation to the optimum. In fact we used the asymptotic form of the optimal policy, which allows faster computation. We now derive that asymptotic form.

Lemma 4. *In the model of this section,*

$$t_i^2 y_i \to \frac{2}{\beta} \qquad \dots(28)$$

and

$$\frac{x_i^2}{y_i} \to 2\beta \qquad \dots(29)$$

as $i \to \infty$ *on the optimum path.*

Proof. We shall not work directly with $t_i^2 y_i$, but with a new variable

$$z_i = x_i(x_i - 1)/y_i$$
$$= \beta^2 t_i^2 y_i - \beta t_i.$$

First we establish a difference equation for z_i, in which it is convenient to use the variable

$$u_i = \beta t_i - y_i^{-1},$$

which is non-negative (by (26)) and tends to zero as $i \to \infty$ (by the Corollary to Lemma 3).

From (24) we have

$$t_{i+1} = (1 + u_i)^{n+1} - 1,$$

and from (22),

$$y_{i+1} = y_i(1 + u_i)$$
$$= (z_i - u_i)u_i^{-2}(1 + u_i),$$

as can readily be checked from the definitions of z and u. Thus,

$$z_{i+1} = \beta\{(1+u_i)^{n+1} - 1\}[\beta\{(1+u_i)^{n+1} - 1\}(z_i - u_i)u_i^{-2}(1+u_i) - 1].$$

Expanding $(1+u_i)^{n+1}$, and using the definition $\beta(n+1) = 1$, we obtain

$$z_{i+1} = \{u_i + \tfrac{1}{2}nu_i^2 + 0(u_i^3)\}[\{u_i + \tfrac{1}{2}nu_i^2 + 0(u_i^3)\}u_i^{-2}(1+u_i)(z_i - u_i) - 1],$$

where the " order " notation $0(u_i^m)$ means a function of i that is equal to u_i^m times a bounded function of i. Multiplying out, we find that

$$z_{i+1} = z_i + u_i\{(n+1)z_i - 2 + 0(u_i) + z_i 0(u_i)\}. \quad \text{...(30)}$$

In using this equation to demonstrate the asymptotic behaviour of z_i, we shall need two auxiliary results:

(i) z_i is bounded.

(ii) $\sum_1^\infty u_i = \infty$.

The first is deduced from the inequality (25), which tells us that

$$y_i\{1 + \beta t_i - (1+t_i)^\beta\} \geqq 1. \quad \text{...(31)}$$

In the interval $0 \leqq t \leqq 1$, the second derivative of $(1+t)^\beta$ is less than $-\beta(1-\beta)2^{-2+\beta}$ (β being less than one). Therefore, by the second-order mean value theorem,

$$(1+t)^\beta \leqq 1 + \beta t - \beta(1-\beta)2^{-3+\beta}t^2.$$

Applying this to (31), we deduce that

$$t_i^2 y_i \leqq 2^{3-\beta}/\{\beta(1-\beta)\}$$

once $t_i \leqq 1$ (as it must be eventually). Therefore $z_i = \beta^2 t_i^2 y_i - \beta t_i$ is bounded, as asserted.

To prove that Σu_i is a divergent series, we note that

$$u_i = (x_i - 1)/y_i = (y_{i+1} - y_i)/y_i \geqq \log(y_{i+1}/y_i).$$

Summing,

$$\sum_0^I u_i \geqq \log(y_{I+1}/y_0) \rightarrow \infty.$$

Returning to the difference equation (30), we deduce first that z_i tends to a finite limit. If z_i does not tend to $2/(n+1)$, there is a positive number ε such that $(n+1)z_i - 2 > \varepsilon$ infinitely often or $(n+1)z_i - 2 < -\varepsilon$ infinitely often. Consider the first possibility, and let i_0 be such that the terms $0(u_i) + z0(u_i)$ in (30) are less than ε in absolute value for all $i \geqq i_0$. Then for some i, say $i_1, \geqq i_0$, $(n+1)z_i - 2 > \varepsilon$, and (30) implies that z_i is then increasing, and must continue to do so for all i. A similar argument shows that z_i is eventually decreasing if $(n+1)z_i - 2 < -\varepsilon$ infinitely often. Therefore the sequence z_i tends to a limit. The fact that z_i is bounded implies that the limit is finite. Therefore $\Sigma(z_{i+1} - z_i)$ is a convergent series. Yet, if $\lim\{(n+1)z_i - 2\} \neq 0$, the divergence of Σu_i implies that $\Sigma u_i\{(n+1)z_i - 2 + 0(u_i)\}$ diverges. It follows that only one limit for z_i is possible:

$$z_i \rightarrow \frac{2}{n+1} = 2\beta. \quad \text{...(32)}$$

Since

$$z_i = \beta^2 t_i^2 y_i - \beta t_i \text{ and } t_i \rightarrow 0,$$

(32) implies (28), and also (29).

Although we do not use it in computation, it is interesting also to derive an approximation for the optimal policy when y_i is small.

Lemma 5. *In the model of this section, on the optimum path*

$$x_i \sim 1 + \beta^{-\beta/(1+\beta)} y_i^{1/(1+\beta)} \quad \text{...(33)}$$

$$t_i \sim \beta^{-1} y_i^{-1} + \beta^{-(1+2\beta)/(1+\beta)} y_i^{-\beta/(1+\beta)} \quad \text{...(34)}$$

as $y_i \rightarrow 0$.

Proof. Since $y_i \to 0$,

$$x_i - 1 = y_{i+1} - y_i \to 0.$$

Then (21) implies

$$y_{i+1}^{n+2} y_i^{-n-1} = y_{i+1} + t_{i+1} y_{i+1} \to \frac{1}{\beta}.$$

Therefore

$$(y_i + x_i - 1) y_i^{-(n+1)/(n+2)} \to \beta^{-1/(n+2)}$$

from which it follows that

$$(x_i - 1) y_i^{-(n+1)/(n+2)} \to \beta^{-1/(n+2)}.$$

This is just (33) in different notation. (34) follows directly from (33).

The computations presented in Figure 3—which gives optimum x as a function of y for several values of β—show that (29) is a good approximation even for quite small values

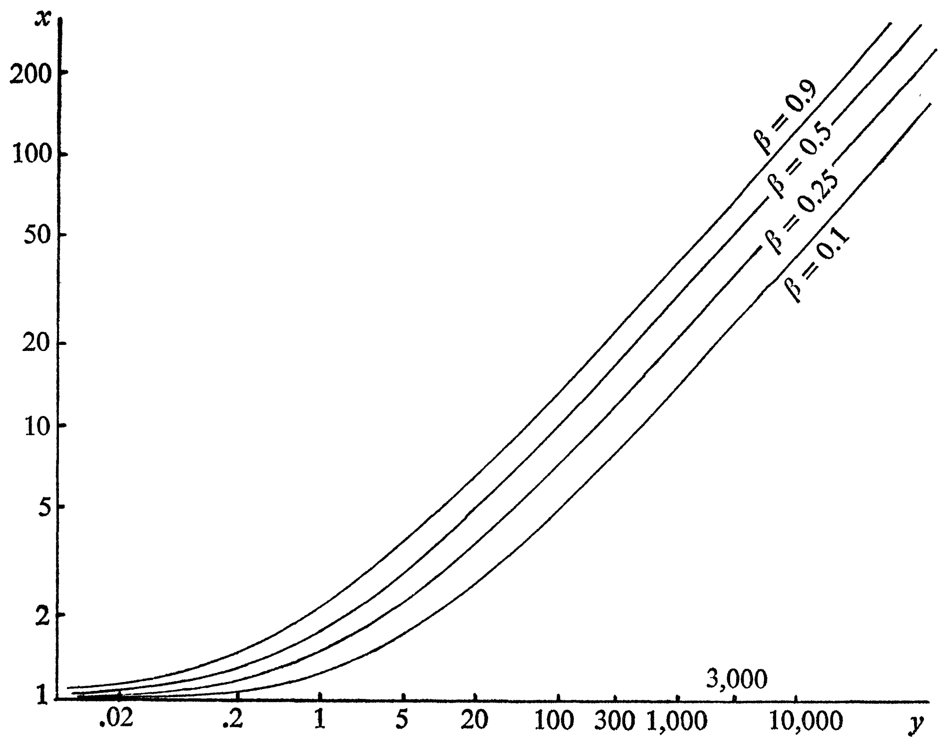

FIGURE 3

of y.[1] For example, when $\beta = 0{\cdot}5$, the approximation is correct within 1 per cent for all $y \geqq 593$, and correct within 2 per cent for all $y \geqq 124$. The lower approximation (33) is less useful. For the same case, it is correct within 2 per cent for all $y \leqq 0{\cdot}04$.

Reverting to the case of general σ and ρ, where, it will be recollected, the optimum policy is

$$x = \frac{\sigma}{\rho} h\left(\frac{y}{\sigma}, \beta\right), \qquad \text{...(35)}$$

[1] $y = 100$ is quite small for the problem, because it means that the minimum investment size is one per cent of the output that would be produced over a period such that the rate of return is 100 per cent—say ten years or more.

we note a number of properties of the optimum. First, we write the approximations, for large y

$$x \sim \rho^{-1}\sqrt{2\sigma\beta y}, \quad t \sim \rho^{-1}\sqrt{\frac{2\sigma}{\beta y}}. \qquad \ldots(36)$$

The properties of x and t suggested by (36) are confirmed by the accurate computations:

(i) *For fixed parameters, x increases as y increases, while t decreases.*

We have already proved the second of these results. The first can be proved in the same way, by considering difference equations in x_i and y_i rather than t_i and y_i. The interested reader will readily prove the proposition by means of a phase diagram.

(ii) *For fixed y, σ and ρ, x is an increasing function of β, and t a decreasing function.*

Thus, in an economy that wishes to save at a higher rate, the next investment is undertaken sooner, but not in proportion, so that the investment is bigger.

(iii) *For fixed y and β, x and t are increasing functions of σ and decreasing functions of ρ.*

(iv) *For fixed y and β, x and t decrease when σ and ρ increase in the same proportion.*

This tells us what happens if ρ is greater while the minimum size of an investment is unchanged.

5. INTERPRETATION AS AN INDIVIDUAL SAVING PROBLEM

The result in the previous section that $h(y)$ behaves like $\sqrt{y}$ for large y is rather striking since it is close to the result of Baumol [1] and Tobin [6], and the early analysis by Whitin [9] of inventory problems. We have used a full infinite horizon optimizing formulation taking into account the effects of savings on future income, and there is no *a priori* reason to expect results similar to these stationary state models. As an aid to understanding the similarity we give a brief description and extension of the Baumol-Tobin inventory type model that yields the square root results.

Suppose we need a flow of a per day for transactions purposes, that there is a cost of withdrawal k, and interest opportunity cost of holding cash of r. We want to choose a withdrawal period to minimize costs per unit time subject to meeting the flow demand. The cost per withdrawal period is k, and the interest cost is $arT^2/2$ where the period is T units of time long. Thus we choose T to minimize $k/T+arT/2$, giving optimal T as $\sqrt{2k/ar}$ and withdrawal quantity $\sqrt{2ak/r}$.

We compare $\sqrt{2ak/r}$ with our approximation to $h(y)$ for large y, viz. $\sqrt{2\sigma\beta y/\rho}$. Now k corresponds to σ/ρ, the fixed cost of investment, a to βy, the flow of savings, and r to ρ, the rate of return on investments. We thus have an *exact* parallel.

One might think at first that the Baumol-Tobin model is a limiting case of our model, but this is not so. The formal structure remains different in the limit in our model, for (i) income is rising with a temporal growth rate tending to $\beta\rho$, (ii) the time between investments is not constant, but is falling as $y^{-1/2}$, (iii) we have a consumption rate of discount in our model, for marginal utility is falling as income rises, whereas there is no discounting in the Baumol-Tobin formulation. The lack of discounting in their model is odd since there is an opportunity cost of holding money.

We incorporate discounting and growth into the Baumol-Tobin model to see if the square root result remains. The opportunity cost of holding money and the fixed costs of withdrawals are discounted at rate r and the flow demand increases at rate g. If a is flow demand now we withdraw $a(e^{gT}-1)/g$. We then have a per-period cost of

$$k+\int_0^T e^{-rt}ra(e^{gT}-e^{gt})/g\,dt,$$

i.e.

$$\psi(a, T) = k + \frac{ar}{g} e^{gT} \left\{ \frac{1-e^{-rt}}{r} - \frac{e^{-gT}-e^{-rT}}{r-g} \right\}. \qquad \ldots(37)$$

Now consider the choice of T. This leads to a functional equation

$$V(a) = \min_{T} [\psi(a, T) + V(ae^{gT})e^{-rT}] \qquad \ldots(38)$$

in the standard dynamic programming framework and obvious notation. The optimum T satisfies

$$\psi_T(a, T) + V'(ae^{gT})age^{gT}e^{-rT} - rV(ae^{gT})e^{-rT} = 0 \qquad \ldots(39)$$

$$V(a) = \psi(a, T) + V(ae^{gT})e^{-rT} \qquad \ldots(40)$$

and differentiating (40) using the envelope theorem

$$V'(a) = V'(ae^{gT})e^{gT}e^{-rT} + \psi_a(a, T). \qquad \ldots(41)$$

If we substitute from (40) and (41) into (39) we obtain

$$rV(a) - agV'(a) = \psi_T(a, T) + r\psi(a, T) - ag\psi_a(a, T). \qquad \ldots(42)$$

Now suppose it turns out that both rT and gT are small. Then from a Taylor expansion of (40) we have, to first order,

$$\begin{aligned} V(a) &= \psi(a, T) + [V(a) + V'(a)agT][1-rT] \\ &= \psi(a, T) + V(a) - T[rV(a) - gaV'(a)] \end{aligned}$$

and, using (42)

$$T\psi_T(a, T) = (1-rT)\psi(a, T) + agT\psi_a(a, T). \qquad \ldots(43)$$

Now we can differentiate (37) to find ψ_a and ψ_T, expand them and ψ in Taylor series, and compare leading terms in (43). This leads to $T = \sqrt{2k/(ar)}$. In other words, the square root formula carries over unchanged when we allow constant growth and discounting, as long as T is small. Of course g must satisfy suitable convergence conditions, but its exact level is immaterial.

We cannot, however, assume directly that rT and gT are small, for T is a choice variable and depends on r and g. Thus, the above method does not tell us in advance of a solution to the problem whether the square root results carry over to allow growth and discounting.

We should note two things about the consumption rate of discount in our model. First, inside a period all marginal increments to consumption are equally valuable. Second, the temporal growth rate of consumption tends to a constant as time goes to infinity, and this implies, with an iso-elastic utility function, a constant discount rate. It is presumably the asymptotic constancy of the growth rate and the discount rate, and the asymptotic decline in the period, that is helping to give a result close to that of Baumol and Tobin. But even when their model is extended to allow growth and discounting as done here, one important difference remains, for both these are endogenous in our model and exogenous in the above extension of their model.

We have emphasized the differences between our model and the Baumol-Tobin model that persist even asymptotically. It remains interesting, however, to see that a full specification of the individual saver problem yields asymptotically similar results to those of a simpler specification. For finite values of output, of course, the two models are quite different. Yet, as we have remarked above, the square root approximation is really extremely good, certainly within the range that seems relevant for the individual saving problem.

6. A MODEL WITH LABOUR

We introduce labour into the model by assuming that a lump of investment of size x will when combined with labour l produce output $g(x)l^{\alpha}$, where $\alpha<1$ and g is concave. This form is, of course, rather special. We allow labour used with the investment lump to be varied freely over time, in such a way that one would want to use all investments for ever (though to a diminishing extent): establishing the investment in no way restricts the possibilities of combining labour with it. We shall make assumptions about g (stated in Lemma 6) sufficient to avoid the possibility that output can explode (become infinite in finite time). One further assumption is needed:

$$xg'(x)/g(x)>1-\alpha. \qquad \ldots(44)$$

This ensures that $g(\lambda x)(\lambda l)^{\alpha}$ increases faster than λ as λ increases. It follows that lumpy investment is desirable, for if investment were being done very frequently, halving the frequency would double the lump and allow double the labour to be applied to the lump, thus more than doubling output per lump and increasing aggregate output.[1]

The labour force is constant. If the constant is taken to be unity, output at the $(i+1)$th stage (after the ith investment lump) is

$$y_{i+1}=\max\left\{\sum_{-\infty}^{i} g(x_j)l_j^{\alpha}\;\middle|\;\sum_{-\infty}^{i} l_j=1\right\}. \qquad \ldots(45)$$

The sum is shown from $j=-\infty$ so as to include all past investments. Of course $x_j=0$ for sufficiently small j. Working out the maximum, we have

$$\alpha g(x_j)l_j^{\alpha-1}=\mu, \quad \text{a Lagrange multiplier.}$$

Since $\Sigma l_j=1$,

$$\mu^{\xi}=\alpha^{\xi}\Sigma g(x_j)^{\xi} \qquad \ldots(46)$$

where ξ is written for $1/(1-\alpha)$. Therefore

$$\begin{aligned} y_{i+1}&=\Sigma g(x_j)\cdot\{\mu^{-\xi}\alpha^{\xi}g(x_j)^{\xi}\}^{\alpha}\\ &=(\mu^{-\xi}\alpha^{\xi})^{\alpha}\Sigma g(x_j)^{\xi}, \quad \text{since } 1+\xi\alpha=\xi\\ &=\{\Sigma g(x_j)^{\xi}\}^{1-\alpha}, \quad \text{by (46).} \end{aligned} \qquad \ldots(47)$$

Thus

$$y_{i+1}^{\xi}-y_i^{\xi}=g(x_i)^{\xi}. \qquad \ldots(48)$$

This is a very convenient form of production constraint to use in the maximization problem: it simply generalizes the case of Section 2, which corresponds to $\xi=1$. We note first that, if g is suitably restricted, the economy cannot explode.

Lemma 6. *For the model of this section, if g is such that there exists a positive constant K, and a positive number δ not greater than one, such that*

$$g(x)\leqq Kx^{\delta} \quad (\text{all } x), \qquad \ldots(49)$$

then $y_i\to\infty$ only if $\Sigma t_i=\infty$.

Proof. By (44), $d(\log g(x))/dx>(1-\alpha)/x$. Integrating between $x=x_0$ and $x=x_1$, we obtain

$$g(x_1)>g(x_0)(x_1/x_0)^{1-\alpha}. \qquad \ldots(50)$$

[1] This intuitive argument is our justification for believing that Assumption C will continue to hold.

Comparing (49) and (50), we see that $1-\alpha \leqq \delta$, i.e. $\delta\xi \geqq 1$. Then, writing $z_i^\delta = y_i$, we have the inequality

$$\begin{aligned}(z_{i+1}-z_i)^{\delta\xi} &\leqq z_{i+1}^{\delta\xi}-z_i^{\delta\xi}\\ &= g(x_i)^\xi\\ &\leqq g(t_iy_i)^\xi, \quad \text{since } c_i \geqq 0\\ &\leqq K(t_iy_i)^{\delta\xi}, \quad \text{by (49).}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{z_{i+1}-z_i}{z_i} &\leqq K^{1/(\delta\xi)}y_i^{1-1/\delta}t_i\\ &\leqq K't_i, \quad K' \text{ a constant} \qquad \text{...(51)}\end{aligned}$$

for all large enough i, since $y_i \to \infty$, $\delta \leqq 1$. Summing, for large i, and using the inequality $\log(z_{i+1}/z_i) \leqq (z_{i+1}-z_i)/z_i$,

$$\begin{aligned}K'\Sigma t_i &\geqq \Sigma(\log z_{i+1}-\log z_i)\\ &= \infty.\end{aligned}$$

This proves the lemma.

This is a natural generalization of Lemma 1.

The first-order conditions for the maximization of

$$\Sigma t_i u(c_i) = \Sigma x_i \frac{u(c_i)}{y_i - c_i}$$

are obtained as before by maximizing $u(c)/(y_i-c)$ with respect to c_i and considering changes in x_{j-1}, x_j and y_j (holding other x's and y's constant) that keep (48) satisfied for all i. This latter change yields

$$\frac{u(c_{i-1})}{y_{i-1}-c_{i-1}}\cdot\frac{y_i^{\xi-1}}{g(x_{i-1})^{\xi-1}g'(x_{i-1})} - \frac{u(c_i)}{y_i-c_i}\cdot\frac{y_i^{\xi-1}}{g(x_i)^{\xi-1}g'(x_i)} = \frac{x_iu(c_i)}{(y_i-c_i)^2}. \qquad \text{...(52)}$$

Maximization of $u(c)/(y_i-c)$ yields the Keynes-Ramsey equation

$$u(c_i)+(y_i-c_i)u'(c_i) = 0. \qquad \text{...(53)}$$

Combining (52) and (53), we get a slightly neater equation,

$$\left\{\frac{u'(c_{i-1})}{g(x_{i-1})^{\xi-1}g'(x_{i-1})} - \frac{u'(c_i)}{g(x_i)^{\xi-1}g'(x_i)}\right\} y_i^{\xi-1}(y_i-c_i) = x_iu'(c_i). \qquad \text{...(54)}$$

We shall not discuss the general problem further, but turn to the simple special case where

$$u(c) = -c^{-n}, \quad g(x) = x^\delta. \qquad \text{...(55)}$$

(Units of measurement for commodities are so chosen that the multiplicative factor in g is unity.) For this case we can give a complete analysis. It turns out to be easier to use dynamic programming methods rather than the first-order conditions (54).

Theorem. *If* $(n+1)\delta > 1$, $\delta\xi > 1$, *and* $\delta < 1$, *the model* (55) *has an optimum policy, which is given by*

$$c_i = \frac{n}{n+1}y_i; \qquad \text{...(56)}$$

and

$$y_{i+1} = \gamma y_i, \qquad \text{...(57)}$$

or, equivalently,

$$x_i = \varepsilon y_i^{1/\delta}, \quad t_i = (n+1)\varepsilon y_i^{1/\delta - 1}, \qquad \text{...(58)}$$

with $\varepsilon = (\gamma^{\xi}-1)^{1/(\delta\xi)}$, *where* γ *is defined by*

$$\gamma^{\{(n+1)\delta-1\}/\delta}+\{(n+1)\delta-1\}\gamma^{-\xi} = (n+1)\delta, \quad \gamma>1. \qquad ...(59)$$

Proof. The condition $\delta\xi>1$ is just (44). $\delta<1$ means that Lemma 6 applies. If $(n+1)\delta>1$, the economy is valuation-finite, i.e. has a path for which the utility integral converges. For example, the path defined in the theorem, which we are going to prove optimal, gives a utility integral proportional to

$$-\Sigma y_i^{1/\delta-n-1} = y_0^{1/\delta-n-1}(1-\gamma^{1/\delta-n-1})^{-1},$$

since

$$1/\delta-n-1<0 \quad \text{and} \quad \gamma>1.$$

It follows that we can define $V(y_0)$, the supremum of the utility integral when initial output is y_0, and

$$V(y_0) = \sup\{-xy_0^{-n-1}+V(y_1) \mid y_1^{\xi} = y_0^{\xi}+x^{\delta\xi}\}, \qquad ...(60)$$

since the maximum utility obtainable before the first new investment is (using the Keynes-Ramsey rule, (56)),

$$\frac{x}{y_0-c}u(c) = -xu'(c) = -Axy^{-n-1},$$

for some positive constant A, which we absorb into the definition of V.

Using the argument of the existence theorem in the appendix, we can show that an optimum policy $x(y_0)$ exists, such that

$$x(y_0) \quad \text{maximizes} \quad -xy_0^{-n-1}+V(y_1). \qquad ...(61)$$

Then, in our particular example, we can obtain the form of V explicitly. Suppose $(c_0, c_1, c_2, ...)$ and $(x_0, x_1, x_2, ...)$ is the optimum path starting from y_0. Then $(\lambda c_0, \lambda c_1, \lambda c_2, ...)$ and $(\lambda^{1/\delta}x_0, \lambda^{1/\delta}x_1, \lambda^{1/\delta}x_2, ...)$ is a feasible path starting from λy_0. The utility integral on this second path is $\lambda^{1/\delta-n-1}$ times $V(y_0)$ hence $V(\lambda y_0) \geqq \lambda^{1/\delta-n-1}V(y_0)$. A similar argument beginning with an optimum path starting from λy_0 gives the inequality the other way and it follows that, V is homogeneous of degree $1/\delta-n-1$ in y_0, i.e.

$$V = -Ay_0^{1/\delta-n-1}. \qquad ...(62)$$

This means that V is differentiable, so that (61) implies

$$y_0^{-n-1} = V'(y_1)\delta x^{\delta\xi-1}y_1^{1-\xi}. \qquad ...(63)$$

At the same time, since by (60)

$$V(y_0) = -xy_0^{n-1}+V(y_1), \qquad ...(64)$$

$$V'(y_0) = (n+1)xy_0^{-n-2}+V'(y_1)y_0^{\xi-1}y_1^{1-\xi}. \qquad ...(65)$$

Combining (63) and (65), we obtain

$$(n+1)\delta x^{\delta\xi}y_0^{-n-1-\xi}+y_0^{-n-1} = \delta x^{\delta\xi-1}y_0^{1-\xi}V'(y_0)$$
$$= A((n+1)\delta-1)x^{\delta\xi-1}y_0^{1/\delta-n-1-\xi}, \quad \text{by (62)}.$$

Thus

$$(n+1)\delta+(y_0/x^{\delta})^{\xi} = A((n+1)\delta-1)(y_0/x^{\delta})^{1/\delta}, \qquad ...(66)$$

while, from (64), using (62), we obtain

$$A(y_0/x^{\delta})^{1/\delta} = 1+A(y_0/x^{\delta})^{1/\delta}\{1+(x^{\delta}/y_0)^{\xi}\}^{-\frac{(n+1)\delta-1}{\delta\xi}}. \qquad ...(67)$$

Eliminating A from (66) and (67), and writing $\varepsilon^{\delta} = x^{\delta}/y_0$, we have

$$(n+1)\delta+\varepsilon^{-\delta\xi} = (n+1)\delta-1+\{(n+1)\delta+\varepsilon^{-\delta\xi}\}(1+\varepsilon^{\delta\xi})^{-\frac{(n+1)\delta-1}{\delta\xi}},$$

which can be rewritten in terms of $\gamma = (1+\varepsilon^{\delta\xi})^{1/\xi}$:

$$\gamma^{\{(n+1)\delta-1\}/\delta}+\{(n+1)\delta-1\}\gamma^{-\xi} = (n+1)\delta. \qquad \ldots(68)$$

By its definition γ must be greater than one, and it is readily shown that (68) has one and only one root greater than one: for the left-hand side is equal to $(n+1)\delta$ when $\gamma = 1$, decreases at first as γ increases above one, and then increases, tending to infinity for large γ.

It follows that

$$\frac{y_1}{y_0} = \left(1+\frac{x^{\delta\xi}}{y_0^{\xi}}\right)^{1/\xi} = \gamma, \qquad \ldots(69)$$

and this must hold at all subsequent stages of the optimal development too. This proves the theorem.

Several features of the optimal policy and path call for comment.

(i) As $i \to \infty$, $t_i \to \infty$.

This is clear from (58), since $y_i \to \infty$, and $\delta < 1$. The situation is thus quite different from the model discussed in Section 4. The reason is that we now have decreasing returns to capital alone, and thus diminishing productivity of capital over time.

(ii) Proportional increments in output are the same from every investment.

This feature is special to the homogeneous case we have analysed.

(iii) The sufficient condition for existence of an optimum path, $(n+1)\delta > 1$, is the same in form as Weizsäcker's sufficient condition for the Ramsey model with homogeneous utility and production functions [8]. It is to be presumed that no optimum policy exists when $(n+1)\delta < 1$.

(iv) Some values for γ are given in Table I.

TABLE I

Optimal proportional increments in output for the case
$u = -c^{-n}, \quad g(x, l) = x^{\delta}l^{\alpha}$

n	α	δ	γ
1	0·5	0·6	1·17
1	0·5	0·8	1·43
1	0·75	0·6	1·60
1	0·75	0·8	1·78
2	0·5	0·6	1·12
2	0·5	0·8	1·29
2	0·75	0·6	1·43
2	0·75	0·8	1·55

As is to be expected, the optimum γ is quite sensitive to variations in the parameters. Even for $n = 2$—which many would think not small—and with economies of scale that are very moderate—$\alpha+\delta = 1{\cdot}1$—the economy is supposed to wait until it can increase output by 12 per cent before investing again. To fill out the picture, consider an economy where initial output is 100 and the labour force is 100. If $g = \frac{1}{3}x^{0\cdot8}l^{0\cdot75}$, a plant with $x = 20$

and $l = 5$ has capital-output ratio 1·63 and output per man of 2·45. Then the optimal policy for $n = 2$ requires that, at the first step

$$t_0 = 3{\cdot}9$$
$$x_0 = 131$$
$$y_1 = 155.$$

No doubt these large investments and long periods are influenced by our somewhat peculiar production assumptions. But we suspect they and the high savings rates arise more from the basic feature of the model, that all growth is credited to investment. Once economies of scale are incorporated into a growth model, this is not such an absurd assumption; but we return to results with a flavour similar to Ramsey's, where for many plausible utility functions high investment is optimal. The associated desirability of very infrequent investment is even more striking.[1]

7. CONCLUDING REMARKS

We offer a few comments and speculations concerning the possible effects of complicating the model.

The effects of introducing discounting in this model are more significant than those in models without increasing returns. For example, we are no longer assured that we shall make an infinite sequence of investments on the optimum path—the proof of Lemma A.1 leans heavily on zero discounting. We suspect that, in our models, it is optimal to stop investing in finite time when utility is discounted. Also, consumption will no longer be constant between investments—it will fall at a rate sufficient to keep $u'(c)e^{-rt}$ constant. This means that the typical term in the series for the utility integral is no longer $tu(c)$, but $\phi(x, t)$, where ϕ is the utility from the optimum method of accumulating x in time t. The ϕ function can be derived fairly easily, at least for the iso-elastic utility case, but is messy, so that the first-order conditions, involving ϕ_x and ϕ_t, are difficult to work with.

The effects of allowing installed capacity to depreciate, in the absence of discounting, are less marked. It will still be optimum to have constant consumption between investments, and if utility integrals are to converge, we shall still need output to go to infinity and consequently an infinite sequence of investments. There is the possibility, also, that some part of inventories may evaporate before installation.

It would be interesting to introduce growing population into the model of Section 6, and we would hope to apply such a model to the development of new urban centres.

The model with homogeneous utility and production, and a constant rate of population growth v, has many features which are quite different from the modification of the Ramsey model introduced by Koopmans [3], Weizsäcker [8] and others. Economies of scale allow population growth to be amplified, and it is possible for consumption per head to grow at a constant rate $v(\alpha+\delta-1)/(1-\delta)$ for ever. In the Cobb-Douglas model with constant returns, consumption per head is ultimately constant. For this reason, no optimum policy exists unless utility (i.e. population times utility of *per capita* consumption) is sufficiently discounted. In our model, the optimum policy exists if

$$n > \frac{1-\delta}{\alpha+\delta-1},$$

[1] We have been asked about the possibilities of decentralized investment decisions in our models. They do not seem good. Of course, marginal cost should be equal to prices and marginal product to wage, but the plant manager has an incentive to attempt to build an arbitrarily large plant, and we think only the central planner can tell when to build. A modified form of decentralization with cost minimization subject to certain targets may be possible.

even without any discounting of utility. In that case one can show that t_i tends to a finite limit. It should not be difficult to identify the asymptotic behaviour of the optimum path, but we have not worked the case out in any detail.

A further modification that realism requires is to allow for a multiplicity of commodities, with investment of one kind or another taking place frequently, but investment in the production of any one type of commodity taking place only at fairly large intervals. The problem is to formulate an appealing but manageable model.

We do not think that any of these modifications would seriously affect the lessons of the models in this paper. If growth is presumed the result of capital accumulation with economies of scale, optimum savings rates are high for many plausible utility functions. If there were persistent increasing returns to capital alone, the economy could, and should, explode. If there are fixed investment costs, but otherwise constant returns, the optimal investment size is approximately the square root of output, and investment becomes almost continuous at high output levels. If, as seems realistic for economies, though not for the individual saver, there are economies of scale in production, but not when labour is fixed, the optimal size of plant may be very large, and the optimal period between investments very long. The value of waiting is high.

APPENDIX

The Existence of an Optimum

We take the model of Section 2, without labour. A similar argument should work for the model with labour. We shall use where stated:

Assumption A. u is increasing, strictly concave, satisfying $u(c)\to 0$ $(c\to\infty)$, and $u'(c)\to\infty$ $(c\to 0)$.

Assumption B. For any $y_0>0$, there exists a feasible path with convergent utility integral.

Assumption C. For all y_0 there exists $\delta>0$ such that all paths starting from y_0 and making the first investment at a time sooner than δ can be overtaken.

We note first that the problem can be simplified by considering only paths with constant consumption between investments. In fact, any path that does not have constant consumption between investments is inferior to one that does, because of the strict concavity of the utility function. We shall show that we may restrict our attention to paths that have an infinite number of periods of constant consumption.

Lemma A.1. *If there exists an $\bar{x}$ for which $g(\bar{x})>0$, and u is strictly increasing then for any y_0, the path $c(t) = y_0$ $(t \geqq 0)$ can be overtaken.*

Proof. Consider the feasible path with consumption $c(t) = c_0<y_0$ for $0 \leqq t \leqq T_0$, where T_0 is such that $g((y_0-c_0)T_0)>0$, and let $y_1 = y_0+g((y_0-c_0)T_0)$, $c(t) = y_1>y_0$ for $t>T_0$. Since $u(y_1)>u(y_0)$

$$\int_0^{T_0} u(c_0)dt+\int_{T_0}^{T} u(y_1)dt \quad \text{overtakes} \quad \int_0^{T} u(y_0)dt.$$

The lemma says that it is always worth making another investment. Note the role played by zero discounting.

We are now in a position to prove that an optimum exists. Define $V(y)$ as the supremum of all possible utility integrals along feasible paths given that one starts with output y and no accumulated savings. We know that the utility integral is bounded above by zero and Assumption A gives us a lower bound, so the definition of V is legitimate. We group together some of the assumptions already made for a formal statement of the theorem.

Theorem. *If g is increasing and continuous, and Assumptions A, B and C hold, then an optimum policy exists.*

Proof. The assumptions of the theorem are sufficient for Lemma A.1, and thus justify the recursive relation.

$$V(y_0) = \sup_{c,\, t} \{tu(c) + V[y_0 + g(t(y_0 - c))]\}, \qquad \ldots(1')$$

where the supremum is taken over the set

$$0 \leqq c \leqq y_0 \qquad \ldots(2')$$

$$\delta(y_0) \leqq t \leqq V(y_0)/u(y_0) + 1. \qquad \ldots(3')$$

The upper bound in (3′) is justified because, if it is not satisfied,

$$tu(c) \leqq tu(y_0) < V(y_0) + u(y_0) < V(y_0),$$

so that such values of t can be excluded from consideration.

An easy but long proof shows that V is a continuous function; we separate it into Lemma A.2 immediately following this theorem. Hence, using the continuity of g and u, the expression within the braces in (1′) is a continuous function of c, t. Inequalities (2′), (3′) define a compact set, so the upper bound is attained at c_0, t_0 (say), and

$$V(y_0) = t_0 u(c_0) + V[y_0 + g(t_0(y_0 - c_0))].$$

Define $y_1 = y_0 + g(t_0(y_0 - c_0))$. Choose c_1, t_1 in the corresponding way. Define y_i, c_i, t_i in a similar fashion for all $i \geqq 1$. Add all the equations

$$V(y_i) = t_i u(c_i) + V(y_{i+1}), \quad i = 0, 1, 2, \ldots$$

to obtain, since values of V are non-positive

$$V(y_0) \leqq \sum_{i=0}^{\infty} t_i u(c_i). \qquad \ldots(4')$$

Now consider the policy $c(t) = c_i$ for $\sum_0^{i-1} t_j \leqq t \leqq \sum_0^i t_j$, where we define $t_{-1} \equiv 0$. If $\sum_0^\infty t_j$ diverges, this is a well-defined feasible policy over the indefinite future, and

$$V(y_0) = \sum_{i=0}^{\infty} t_i u(c_i). \qquad \ldots(5')$$

If, on the other hand, $\sum_0^\infty t_i = T$, a finite number, then we must specify what happens beyond T. Note that we must have $y(t) \to \infty$ as $t \to T$ from below. For suppose not, then $y(t)$ being monotonic we must have $y(t) \to \bar{y}$, a finite limit. But then as $i \to \infty$, we have $t_i \to 0 < \delta(\bar{y})$, and this contradicts Assumption C.

With infinite output we can sustain infinite consumption and thus a zero utility integral from T on. For the sequence (c_i, t_i) followed by this phase of infinite consumption, the utility integral is $\geqq V(y_0)$ from (4′).

Thus we have shown that an optimum policy exists, and consists of the sequence (c_i, t_i) followed, if possible, by a phase of infinite consumption.

Lemma A.2. *With the assumptions of the theorem, the function V is continuous.*

Proof. As in the theorem

$$V(y) = \sup \{tu(c) + V[y + g(t(y - c))]\}.$$

Let $\delta > 0$ and let t^*, c^* be such that

$$t^* u(c^*) + V[y + g(t^*(y - c^*))] > V(y) - \delta/3.$$

Let $y' < y$ and $\gamma = c^*/y < 1$ (by Lemma A.1 and the remark preceding it). Consider t', c' satisfying $c' - \gamma y'$ and

$$y' + g(t'(y' - c')) = y + g(t^*(y - c^*)). \qquad \ldots(6')$$

These exist and are unique if $g(x)$ is unbounded over $[0, \infty]$ since we have assumed g continuous and increasing.

Starting from y' a policy t', c' achieves, if followed up suitably, a utility integral greater than

$$t'u(c') + \{V[y' + g(t'(y' - c'))] - \delta/3\}.$$

Thus, using earlier inequalities and (6′), we have

$$\begin{aligned} V(y') &> t'u(c') + V[y' + g(t'(y' - c'))] - \delta/3 \\ &= t'u(c') + V[y + g(t^*(y - c^*))] - \delta/3 \\ &> t'u(c') - t^*u(c^*) + V(y) - 2\delta/3. \end{aligned}$$

Now use (6′), and remember $c' = \gamma y'$, $c^* = \gamma y$. By continuity of g and u, by taking y' sufficiently closed to y we can make $t'u(c') - t^*u(c^*) > -\delta/3$, hence

$$V(y') > V(y) - \delta.$$

clearly $V(y') < V(y)$ for $y' < y$.

These two together give us continuity of V.

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