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A Note on Commodity Taxation: The Choice of Variable and the Slutsky, Hessian and Antonelli Matrices (SHAM)

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Policy problems require the specification of government choice variables, which may be prices or quantities, and the representation of preferences e.g. direct utility or distance functions. The purpose of this note is to provide an appropriate framework for switches between different descriptions of the optimization problem. We first assemble the relevant results on the matrices (Slutsky, Hessian and Antonelli) which arise in the different formulations. We then use the results to show the relationships between various analyses in the literature and finally point out how the results, whilst formally equivalent can lead to different emphases and interpretations (or misinterpretations).

1. THE PROBLEM

When we analyse a policy problem we have to select policy variables and a representation of household preferences. For example, in the context of optimum taxation we may regard either prices or quantities as variables of government choice and if we use quantities we may describe preferences using the direct utility function or the distance function. Corresponding to each choice there will be associated derivatives, which arise naturally in the analysis: in the examples given for quantities, the Hessian and Antonelli matrices respectively. Different problems and contexts may indicate different selections and authors vary in their predilections. It is important therefore in policy analysis to be able to switch conveniently from one representation to another. The purpose of this note is to set an appropriate framework for such switches in the problem of optimum commodity taxation. Accordingly, we (i) assemble (Section 2) the relevant results on Slutsky, Hessian and Antonelli matrices (hence the sub-title), (ii) employ (Section 3) these results to show the relationships between the approaches of Diamond and Mirrlees (1971), using prices as controls, Atkinson and Stiglitz (1972), who use quantities and the direct utility function, and Deaton (1979) who uses quantities and the distance function, (iii) point out (Section 4) how the results whilst formally equivalent (as we saw in Section 3) can lead to different emphases and interpretations (or misinterpretations).

2. THE FOUR MATRICES

There is no claim to originality concerning the results presented here on the Slutsky, Hessian and Antonelli matrices (SHAM); they are collected here for convenience in the comparison of approaches to the problem of optimum taxation. The standard problem in demand theory is the maximization of utility, expressed as a function of goods, subject to a budget constraint:

$$\text{Maximize}_q\, \nu(q) \tag{1}$$

$$\text{subject to } \textstyle\sum_i p_i q_i = x \tag{2}$$

where q is the n-vector of quantities, $\nu(\ )$ is the direct utility function, p is the vector of prices and x is total income or expenditure (we follow the notation of Deaton and Muellbauer (1980)). Where good i is supplied to the market, such as labour, then q_i is negative.

The first-order conditions are, where α is the Lagrange multiplier on the budget constraint (the marginal utility of income)

$$\frac{\partial \nu}{\partial q_i} = \alpha p_i \tag{3}$$

which, together with (2), provides $(n+1)$ equations in the $(n+1)$ unknowns q and α to give q and α as functions of p and x. We shall suppose in what follows that a demand function solving (1) and (2) does exist and that all the relevant functions possess whichever derivatives are needed for the statements at hand.

The classical analysis of demand responses then looks at shifts in the first-order conditions (2) and (3) as a result of changes in p and x, given by dp and dx (see e.g. Hicks (1936), and Samuelson (1947, Chapter V)). We have

$$\begin{bmatrix} \nu_{ij} & p_i \\ p_j & 0 \end{bmatrix} \begin{bmatrix} dq \\ -d\alpha \end{bmatrix} = \begin{bmatrix} \alpha dp \\ dx - qdp \end{bmatrix} \tag{4}$$

where $\{\nu_{ij}\}$ is the $(n \times n)$ Hessian of the direct utility function and the $(n+1)\times(n+1)$ matrix (which we call D) on the L.H.S. of (4) is the Hessian bordered by the price vector (with a zero in the $((n+1), (n+1))$ position).

Considering quantity as a function of price we have on multiplying through in (4) by D^{-1} (which we assume exists) and putting to zero in turn $dx - qdp$ (constant utility) and dp,

$$D^{-1} = \begin{bmatrix} (1/\alpha)s_{ij} & e_i \\ e_j & -\alpha_x \end{bmatrix} \tag{5}$$

where s_{ij} is the ij-th element of the Slutsky matrix, e_i is the income derivative $\partial q_i/\partial x$, α_x is the income derivative of the marginal utility of income and we adopt a similar notation for bordering matrices to that adopted for D. The Slutsky matrix is therefore given by the $(n \times n)$ first principal minor of the inverse, D^{-1}, of the bordered Hessian D.

Multiplying out in $DD^{-1} = I$ we have, where H is the Hessian $\{\nu_{ij}\}$,

$$\frac{1}{\alpha} HS + pe' = I_n \tag{6a}$$

$$p'S = 0 \tag{6b}$$

$$He - p\alpha_x = 0 \tag{6c}$$

$$p'e = 1. \tag{6d}$$

Equation (6b) is well-known from the homogeneity degree zero of the Slutsky matrix, (6c) is simply the derivative with respect to income of (3), and (6d) the derivative with respect to income of (2) (i.e. the adding-up property).

Equations (6a) and (6c) give us the "inverse" relationship between the Hessian and Slutsky matrix. Multiplying (6a) by H^{-1} (we assume H is invertible) and using (6c) we have

$$S = \alpha H^{-1} - \hat{w}ee' \tag{7}$$

where $\hat{w}$ is the 'flexibility' of the marginal utility of income with respect to income x (see Houthakker (1960), Barten (1964) and Browning, Deaton and Irish (1985)). Note that (7) implies (see Houthakker (1960, p. 249), assuming $\alpha \neq 0$ or non-satiation) that necessary and sufficient conditions for additive separability are, $i \neq j$,

$$s_{ij} = -\hat{w}e_i e_j. \tag{8}$$

The Antonelli matrix A, defined below, is the generalized inverse of S and hence for our comparison between A and H we wish to see how H relates to this property (see e.g. Theil (1979, pp. 268–270), for a definition and discussion of the generalized inverse). Thus we pre-multiply (6a) by S and we have, using (6b),

$$\frac{1}{\alpha} SHS = S. \tag{9}$$

Post-multiplying (6a) by H we have, using (6c),

$$\frac{1}{\alpha} HSH + \alpha_x pp' = H. \tag{10}$$

For H and S to be true generalized inverses we would require, in addition to (9), that the second term on the l.h.s. vanishes. This cannot, of course, happen except in the case $\alpha_x = 0$ (when H would not be invertible—see (6c)).

The Antonelli matrix, like the Hessian, is a second derivative with respect to quantities, but it is a derivative of the distance function rather than the utility function. The distance function is defined as a function of the utility level u and quantities q by the identity

$$\nu(q/d) = u. \tag{11}$$

Thus $d(u, q) = 1$ if and only if $u = \nu(q)$. The Antonelli matrix $\{a_{ij}\}$ is defined by $a_{ij} = \partial^2 d/\partial q_i \partial q_j$.

One can show fairly easily (see Deaton (1979), or Deaton and Muellbauer (1980, pp. 57–58)) that

$$xSA = I_n - qr' \tag{12}$$

and

$$xAS = I_n - rq' \tag{13}$$

where $r \equiv p/x$. Multiplying (12) on the front by A and using the homogeneity degree zero of the Antonelli matrix in quantities we have

$$xASA = A \tag{14}$$

and similarly multiplying (12) on the back by S we get

$$xSAS = S \tag{15}$$

Equations (14) and (15) say that A and S are generalized inverses (putting x equal to one).

Given that H and A stand in a similar, but not identical, relationship to S it is natural to ask how they are related to each other. Differentiating (11) with respect to q_j (using the result that $a_i \equiv \partial d/\partial q_i = p_i/x$) we have

$$\frac{v_j}{a_j} = -\frac{d}{d_u}. \tag{16}$$

Cross-multiplying and differentiating w.r.t. q_i at constant u and rearranging we have

$$A + \hat{\alpha} a_{.u} r' + rr' = \frac{1}{\hat{\alpha}}(I - rq')H \tag{17}$$

where $a_{.u}$ is the vector with ith component a_{iu}, a_{iu} is $\partial^2 d/\partial q_i \partial u$, $\hat{\alpha}$ is $-1/d_u$ and we have put $d = 1$. Equation (17) expresses the relation between the Antonelli and Hessian matrices.

We can complete the story of the matrices and in so doing bring out some important underlying symmetry by considering the fourth matrix, the Hessian (with respect to p), J, of the indirect utility function. One can view indirect and direct utility as dual concepts since the former is the level of the maximand in the solution to the problem of choosing quantities to maximize direct utility subject to the budget constraint and similarly the direct utility is the level of the minimand in the solution to the problem of choosing prices to minimize indirect utility subject to the budget constraint (Samuelson (1965)). Similarly the cost and distance functions are dual in the sense that the cost function is minimum expenditure $(p'q)$ subject to given utility $(d(u, q) = 1)$ where quantities are chosen and the distance function is minimum expenditure subject to given cost $(c(u, p) = 1)$ where prices are chosen (Deaton and Muellbauer (1980, pp. 54–59)).

We have studied the relation between S and H in (6) and (7) and between A and H in (17). Similarly one can study the relation between A and J, and S and J. Thus for the relation between A and J one can consider shifts in Roy's identity $\partial\psi/\partial p_i = -\alpha q_i$ where $\psi(p, x)$ is the indirect utility function in an analogous manner to the analysis of shifts in (3) which lead us to (6) and (7). Hence one can show, for example, that the Antonelli matrix is given by the $(n \times n)$ first principal minor of the $(n+1) \times (n+1)$ matrix which is the inverse of J bordered by the vector q. Using the normalized prices $r_i = p_i/x$ all the results of this section are valid with the roles of prices and quantities reserved.

3. APPLICATIONS TO COMMODITY TAXATION

The Ramsey-rule (1927) may be derived in the standard framework, where prices are chosen, from the first-order condition with respect to prices for the following problem, where t are indirect taxes,

$$\text{Maximize}_p \ \psi(p) \tag{18}$$

$$\text{subject to } t \cdot q \geqq \bar{R} \tag{19}$$

where $\psi(p)$ is the indirect utility function arising from the consumer maximization problem (1) and (2) *but with zero expenditure* x, and q and p are functionally related through the consumer's choice. The consumer's problem is homogeneous of degree zero in p, so we assume without loss of generality that one good, good 0, is untaxed and has producer and consumer price equal to unity.

It is straightforward to show (see Diamond and Mirrlees (1971)) that for all $n = m+1$ goods,

$$St = -\theta q \tag{20}$$

where

$$\theta = 1 - \frac{\alpha}{\lambda} - t'e \tag{21}$$

and λ is the Lagrange multiplier on the revenue constraint (19). This is often viewed as the Ramsey "rule" concerning the effects of taxes on quantities which is commonly stated as "the proportional reductions in compensated demands are the same for each good". Both Atkinson and Stiglitz (1972) and Deaton (1979) may be seen as seeking to "invert" the Slutsky matrix on the l.h.s. of (20) to find expressions for taxes. The former uses the Hessian and the latter the Antonelli matrix and both provide appendices to show how their results relate to the Ramsey rule. In the light of the results of Section 2 we can now carry out the two inversions in a parallel manner which brings out the relations between the three versions of the analysis.

The basic Atkinson-Stiglitz result is derived as follows. We multiply (20) on the front by H to give

$$HSt = -\theta Hq$$

and using (6a) and (21)

$$t = -\frac{\theta}{\alpha} Hq + \left(1 - \frac{\alpha}{\lambda} - \theta\right)p. \tag{22}$$

This is the first-order condition provided by Atkinson and Stiglitz, (1972) their equation (3.5), in the form with which they open the appendix to their paper.

Deaton (1979) begins with the Ramsey-rule in the form

$$St = -\theta q + \beta z^0 \tag{23}$$

where z^0 is the n-vector with 1 in the entry corresponding to the untaxed good and zeros elsewhere. The reason that it takes a slightly different form from (20) is that Deaton deals with *leisure* as the 0th consumption good rather than treating the demand for the 0th good as negative and thus a supply. The budget constraint is then $pq = T$ where T is the total time endowment (remembering that $p_0 = 1$), i.e. $T = x$. This imposes an asymmetry between goods which is not present in our analysis so far and this is discussed further below. Notice that the Ramsey 'rule' no longer holds for the 0-th good: it is the proportional reduction in (minus the) compensated labour supply which is equal to θ and *not* the proportional reduction in leisure demand. Thus we introduce βz^0 in (23) which is simply defined as the 0th component of $(St + \theta q)$.

Deaton shows in the appendix to his paper how his expression for optimum taxes may be derived from (23). One multiplies on the front by the Antonelli matrix, uses (13) and writes $\bar{R} = \rho T$ to obtain

$$t = \beta T A z^0 + \rho p. \tag{24}$$

4. APPRAISAL AND INTERPRETATION

The derivation in a parallel manner of (22) from (21) as in Atkinson and Stiglitz (1972) and (24) from (23) as in Deaton (1979), using the Hessian and Antonelli matrices respectively, illustrates the similarities between the two approaches to the optimum tax problem using quantities as controls. There are a number of interesting differences. First an advantage of the Antonelli matrix is that it is homogeneous in q whereas H is not, so that on multiplying (23) through by A, Aq vanishes whereas we retain Hq on the r.h.s. of (22). This allows the elegance of (24) which, in contrast to (22), relates taxes to prices and price responses without quantities appearing.

A disadvantage of the use of the distance function and the Antonelli matrix is that it requires positive total expenditure x. The strict positivity of expenditure is used in showing the minimization in the fundamental duality relationships noted at the end of Section 2 (see Deaton (1979, p. 393)). Further, the derivative property of the distance function $\partial d/\partial q_i = p_i/x$ obviously breaks down at $x = 0$. The problem is clear using the definition of the distance function (11). With the consumption of all goods positive the line joining the origin, to the point representing q, will in general intersect the indifference curve u. Thus we have a well-defined distance $d(u, q)$. But if we adopt the conventional approach to factors in general equilibrium models and treat them simply as negative demands we have, for example, the problem that, with one good and one factor, there would, in general, be either zero or two intersections of the line (joining the origin to q) and the indifference curve. In the former case the distance d is undefined and in the latter one has to have a convention for choosing one of the points. The borderline case is zero expenditure when the line is tangent to the indifference curve. At such a point small increases in u or decreases in q would give $d(\ )$ undefined. But this case is precisely the central one when lump-sum incomes are absent—as one usually supposes for the Ramsey problem.

The way out is to define all consumptions to be non-negative, by replacing labour supply q_0 by leisure $(T - q_0)$ where T is large enough to ensure non-negativity of leisure. This is quite often the procedure adopted in labour supply analysis and is that followed by Deaton as we have seen. It has a number of unsatisfactory aspects. First, the total time availability T is hard to define let alone measure. Secondly, elasticities of leisure demand and elasticities with respect to leisure are difficult subjects for our intuition if we do not know how to define and measure leisure. Thirdly, the great convenience of factors as negative demands in general equilibrium theory is lost. Fourthly, the absence of lump-sum income is a central feature of the commodity tax problem and it is partially obscured by making expenditure strictly positive.

Fifthly, there is a temptation to elevate the innocent normalization (we choose good zero to be untaxed without loss of generality) into something of real substance. This temptation is illustrated by Deaton's (erroneous) suggestion (1981, p. 1256) that the reason complementarity and substitutability with leisure are important in the determination of which goods should be taxed most heavily is that government revenue has been specified in terms of labour as the numeraire. It can easily be checked that the ranking of goods by the optimum proportion of tax in price is *independent of numeraire or choice of untaxed good.* Further, with constant producer prices it does not matter whether one specifies a government revenue in terms of this good or that good (it is the revenue level at producer prices that counts—one can always transform one good into another at the constant producer prices). The crucial reason for the central role of complementarity with leisure in the results concerning the optimum proportion of tax in price is that there is an endowment of leisure which cannot be taxed in this second-best problem. In a sense one might interpret the attempt to exploit complementarities with leisure in the second-best solution as being in the spirit of the first-best which would be to take the relevant proportion of the endowment as a lump-sum tax.

None of the three methods should be regarded as unambiguously superior to the other. The choice of variables for optimization and the representations of preferences depend on the job at hand, the kind of results or information one has in mind, and the predilection of the practitioner. It is important, however, to establish routes for passing from one approach and set of results to the other so that one can use the various techniques to bring out different aspects of the solution, whilst avoiding any confusion which might arise from differences amongst the techniques themselves.

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