LARGE ANTIPODAL FAMILIES

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Dedicated to Ted Bisztriczky on the occasion of his 60th birthday

Abstract. A family $\{A_i | i \in I\}$ of sets in $\mathbb{R}^d$ is *antipodal* if for any distinct $i, j \in I$ and any $p \in A_i$, $q \in A_j$, there is a linear functional $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi(p) \neq \varphi(q)$ and $\varphi(p) \leq \varphi(r) \leq \varphi(q)$ for all $r \in \bigcup_{i \in I} A_i$. We study the existence of antipodal families of large finite or infinite sets in $\mathbb{R}^3$.

1. Introduction

A set A of points in $\mathbb{R}^d$ is *antipodal* if for every pair of distinct points $x, y \in A$ there is a linear functional φ on $\mathbb{R}^d$ such that $\varphi(x) \neq \varphi(y)$ and $\varphi(x) \leq \varphi(z) \leq \varphi(y)$ for all $z \in A$. This notion was first introduced by Klee [7]. Danzer and Grünbaum [3] showed that an antipodal set in $\mathbb{R}^d$ has size at most 2^d.

A set A of points in $\mathbb{R}^d$ is *strictly antipodal* if for every pair of distinct points $x, y \in A$ there is a linear functional φ on $\mathbb{R}^d$ such that $\varphi(x) < \varphi(z) < \varphi(y)$ for all $z \in A \setminus \{x, y\}$. Grünbaum introduced this notion in [6], where he also formulated the statement that a strictly antipodal set in $\mathbb{R}^3$ has size at most 5. A complete proof of this fact follows from the classification of antipodal 3-polytopes given by T. Bisztricky and K. Böröczky [2]. These and related notions were subsequently studied by many authors. See for example the recent papers by Bisztriczky and others [1, 2] and the survey [10] for further details.

In this paper we consider antipodal and strictly antipodal families.

Definition 1. Let $\{A_i \mid i \in I\}$ be a family of subsets of $\mathbb{R}^d$. We say that this family is *antipodal* if for any $i, j \in I$, $i \neq j$, and any $p \in A_i$, $q \in A_j$, there is a linear functional $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi(p) \neq \varphi(q)$ and $\varphi(p) \leq \varphi(r) \leq \varphi(q)$ for any $r \in \bigcup_{i \in I} A_i$.

Definition 2. Let $\{A_i \mid i \in I\}$ be a family of subsets of $\mathbb{R}^d$. We say that this family is *strictly antipodal* if for any $i, j \in I$, $i \neq j$, and any $p \in A_i$, $q \in A_j$, there is a linear functional $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi(p) < \varphi(r) < \varphi(q)$ for any $r \in \bigcup_{i \in I} A_i \setminus \{p, q\}$.

Definition 3. Let $k(d)$ [resp. $k'(d)$] be the largest k such that for each m there exists an antipodal [resp. strictly antipodal] family of k sets in $\mathbb{R}^d$, each of size at least m.

Makai and Martini [8] considered the problem of determining $k(d)$ and $k'(d)$. They noted the following:

- $2^{d-1} \leq k(d) \leq 2^d - 1$, in particular, $4 \leq k(3) \leq 7$;

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- $k'(d) > c^d$ for some absolute constant $c > 1$;
- $k(2) = 2$ and $k'(2) = 1$;
- $3 \le k'(3) \le 5$.

They conjectured that $k(d) = 2^{d-1}$ (in particular, $k(3) = 4$) and $k'(3) = 3$.

Our results are the following:

(1) We classify antipodal families of 4 segments in $\mathbb{R}^3$ (Theorem 1).
(2) There does not exist a strictly antipodal collection of four $\mathcal{C}^1$ arcs in $\mathbb{R}^3$ (Theorem 2).
(3) There exists a strictly antipodal collection of 4 sets in $\mathbb{R}^3$, with three of them infinite and the fourth a singleton (Theorem 3).
(4) There does not exist an antipodal collection of six sets in $\mathbb{R}^3$ of arbitrary large size. In particular, $k(3) \le 5$ (Theorem 4).

Theorem 2 supports the conjecture of Makai and Martini that $k'(3) = 3$. On the other hand, Theorem 3 gives a near-counterexample.

2. Classification of antipodal families of 4 segments in $\mathbb{R}^3$

Examples of antipodal families of four segments in $\mathbb{R}^3$

(a) Consider the cube Q with vertices $A_1 = (0,0,0)$, $A_2 = (1,0,0)$, $A_3 = (1,1,0)$, $A_4 = (0,1,0)$, $A_5 = (0,0,1)$, $A_6 = (1,0,1)$, $A_7 = (1,1,1)$, $A_8 = (0,1,1)$. Choosing subsegments of the 4 parallel sides $[A_1, A_5]$, $[A_2, A_6]$, $[A_3, A_7]$, $[A_4, A_8]$, we obtain an antipodal family of 4 segments.

(b) Taking subsegments of the sides $[A_1, A_5]$, $[A_2, A_6]$, $[A_3, A_4]$, $[A_7, A_8]$, we also obtain an antipodal family of 4 segments.

Since antipodality is an affine invariant property, affine images of the above examples are also examples of weakly antipodal collections.

Theorem 1. *Any antipodal family of four segments in $\mathbb{R}^3$ is affinely equivalent to one of the examples described in (a) and (b) above.*

Proof. Denote the segments by I_1, I_2, I_3 and I_4.

Lemma 1. *If two of the segments are coplanar then they are parallel but not collinear.*

Proof. Indeed, suppose that, say I_1 and I_2 are coplanar. Choose inner points $p \in I_1$ and $q \in I_2$ and a linear function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that $\varphi(p) < \varphi(q)$ and $\varphi(p) \le \varphi(r) \le \varphi(q)$ for all $r \in \bigcup_{i=1}^4 I_i$. Then φ is constant both on I_1 and on I_2. Thus, I_1 and I_2 are contained in the distinct parallel planes $\Pi_1 = \{\, x \in \mathbb{R}^3 \mid \varphi(x) = \varphi(p)\}$ and $\Pi_2 = \{\, x \in \mathbb{R}^3 \mid \varphi(x) = \varphi(q)\}$ respectively. Since I_1 and I_2 are coplanar and contained in two distinct parallel planes, they must be parallel and noncollinear. □

We shall proceed by case separation. If the segments were coplanar, then by Lemma 1 they would be parallel and noncollinear. However, in that case two of the segments would lie strictly in between the straight lines of the other two segments and the antipodaliy condition would obviously fail for those two segments. We may therefore suppose that the segments are not coplanar.

Consider the case when the segments are parallel to one another. Then intersecting the straight lines of the segments by a transversal plane, we obtain four noncollinear points in antipodal position in the plane. It is well known that four noncollinear points in the plane are in antipodal position if and only if they are vertices of a parallelogram [3]. This means that the four segments are on the four parallel sides of a parallelepiped. In this case, the family is affinely equivalent to an example of type (a).

By Lemma 1, it remains to study the case when two of the segments, say I_1 and I_2 are located on two skew lines. Denote by $\Sigma_1 \supset I_1$ and $\Sigma_2 \supset I_2$ the planes parallel to both I_1 and I_2. According to the antipodality assumption, the other two segments I_3 and I_4 are in the closed slab between Σ_1 and Σ_2.

Suppose first that there are no three mutually skew lines among the lines generated by the four segments. Then, by Lemma 1, both I_3 and I_4 are parallel either to I_1 or to I_2. If, for example, I_3 is parallel to I_1, then I_3 must be in the plane Σ_1, otherwise the antipodality condition would fail for any pair of inner points $p \in I_3$ and $q \in I_2$.

It is also easy to see, that I_3 and I_4 cannot be parallel to I_1 simultaneously, since in that case one of the coplanar parallel lines spanned by the segments I_1, I_3 and I_4, say the line of I_j would separate the other two and then the antipodality condition would fail for inner points of I_ℓ and I_j for $\ell \in \{1, 3, 4\} \setminus \{j\}$.

These observations yield that if there are no three mutually skew lines among the lines generated by the four segments, then I_1 and I_2 are parallel to exactly one of I_3 and I_4, and if we assume without loss of generality, that $I_1 \parallel I_3$ and $I_2 \parallel I_4$, then $I_3 \subset \Sigma_1$ and $I_4 \subset \Sigma_2$.

Applying the antipodality property for inner points of I_1 and I_3 and also for inner points of I_2 and I_4 we obtain two pairs of parallel supporting planes of the union of the four segments. These planes together with Σ_1 and Σ_2 bound a parallelepiped. The segments $I_1 \parallel I_3 \nparallel I_2 \parallel I_4$ are located on the opposite faces of this parallelepiped, so in this case the family is affinely equivalent to an example of type (b).

The final and less trivial case is when 3 of the segments, say I_1, I_2 and I_3 are on three mutually skew lines. We show that this case gives a contradiction. Applying the antipodality condition for each pair of I_1, I_2 and I_3 we obtain 3 slabs bounded by supporting planes of the union of the intervals. These slabs intersect in a parallelepiped with the property that each of I_1, I_2, I_3 is contained in an edge of the parallelepiped. Since parallelepipeds are affinely equivalent to the unit cube Q used in the construction of the examples, we can assume without loss of generality that $I_1 \subset [A_3, A_4]$, $I_2 \subset [A_5, A_8]$, $I_3 \subset [A_2, A_6]$.

We know that I_4 must be in the unit cube Q.

Let i, j, k be an arbitrary permutation of $1, 2, 3$. If I_4 were parallel to I_i, then the mutually skew segments I_4, I_j, I_k would lie on the edges of a rectangular box $Q' \subsetneq Q$. However, this would yield a contradiction, as such a box Q' cannot contain I_i. This means that I_1, I_2, I_3, I_4 must be mutually skew. In particular, I_4 cannot lie on any of the faces of the cube Q.

For $p, q \in \{1, 2, 3, 4\}$, $p \neq q$, denote by Σ_{pq} the plane containing I_p and parallel to I_q.

The straight line spanned by I_4 cuts the boundary of the cube Q at two points, call them S and T. We consider separate cases depending on the location of S and T.

If S and T are on opposite faces of Q, say $S \in \Sigma_{21}$ and $T \in \Sigma_{12}$, then the plane Σ_{42} cuts the cube in a rectangle , which has opposite sides through S and T parallel to I_2. Since this rectangle must be different from the face $\Sigma_{32} \cap Q$, I_3 and I_2 must be on opposite sides of Σ_{42}. This is a contradiction showing that S and T must lie on neighboring faces of Q.

The common edge e of the faces containing S and T either contains one of the segments I_1, I_2, I_3 (i.e., $e \in \{A_3A_4,\ A_5A_8,\ A_2A_6\}$) or contains none of them. The latter case can be split into two further subcases depending on whether the vertices of the common edge are covered by the union of the three straight lines spanned by I_1, I_2, I_3 (i.e., $e \in \{A_4A_8,\ A_5A_6,\ A_2A_3\}$) or not (i.e., $e \in \{A_1A_2,\ A_1A_4,\ A_1A_5,\ A_7A_3,\ A_7A_6,\ A_7A_8\}$).

Consider first the case, when S and T are on faces meeting along an edge containing one of the first three segments. We may suppose without loss of generality that $S \in \Sigma_{31}$ and $T \in \Sigma_{32}$. Then the plane Σ_{43} cuts off a triangular prism from the cube Q which contains the segment I_4 but does not contain the segments I_1 and I_2. This is a contradiction since Σ_{43} may not separate I_3 from I_1 and I_2.

Suppose now that the common edge e of the faces of Q containing S and T does not contain any of the segments I_1, I_2, I_3, but the endpoints of e are covered by the three straight lines spanned the segments. (See the left side of Figure 1.) Permuting the rôle of the indices if necessary, we may suppose that $S \in \Sigma_{31}$ and $T \in \Sigma_{21}$. The plane Σ_{42} cuts the face $Q \cap \Sigma_{21}$ in a segment $T'T''$, where $T \in T'T'' \parallel I_2$, $T' \in [A_5, A_6]$, $T'' \in [A_7, A_8]$. The intersection of Σ_{42} with the plane Σ_{31} must be the line $T'S$. Since I_2 and I_3 are on the same side of Σ_{42}, the line $T'S$ must intersect the boundary of the square $Q \cap \Sigma_{31}$ at a point $C_1 \in [A_2, A_6]$, for which $I_3 \subset [C_1, A_2]$. The plane Σ_{24} is parallel to Σ_{42} and the segments I_1 and I_3 are on the same side of it, therefore the line $\Sigma_{24} \cap \Sigma_{13}$ is parallel to $T'C_1$, goes through A_8 and intersects the segment $[A_3, A_4]$ at a point C_2 for which $I_1 \subset [A_3, C_2]$. The triangles $\triangle A_6C_1T'$ and $\triangle A_4A_8C_2$ are similar, thus $A_6T' : A_4C_2 = A_6C_1 : A_4A_8 < 1$, implying $A_6T' < A_4C_2$.

Now we repeat the above arguments flipping the rôles of S and T and that of I_2 and I_3. Σ_{43} intersects the face $Q \cap \Sigma_{31}$ in a segment $[S', S'']$, where $S \in [S', S''] \parallel [A_6, A_2]$, $S' \in [A_5, A_6]$, $S'' \in [A_1, A_2]$. The intersection of Σ_{43} with the face $Q \cap \Sigma_{21}$ is a segment $[S', C_3]$ containing T and ending at a point $C_3 \in [A_5, A_8]$ for which $I_2 \subset [C_3, A_8]$. The plane $\Sigma_{34} \parallel \Sigma_{43}$ intersects the face $Q \cap \Sigma_{12}$ in a segment $[A_2, C_4] \parallel [S', C_3]$, the endpoint C_4 of which satisfies $C_4 \in [A_3, A_4]$ and $I_1 \subset [C_4, A_4]$. Using similarity of the triangles $\triangle A_5S'C_3$ and $\triangle A_3C_4A_2$ we obtain $A_3C_4 > S'A_5$.

I_1 must be in the intersection of the segments $[A_4, C_4]$ and $[A_3, C_2]$. On the other hand,

$$A_3C_2 + C_4A_4 = 2A_3A_4 - A_3C_4 - A_4C_2 < 2A_5A_6 - S'A_5 - A_6T' < A_3A_4,$$

which means that $[A_4, C_4]$ and $[A_3, C_2]$ are disjoint, a contradiction.

The last case that we should consider is when the common edge e of the faces of Q containing S and T has a vertex not covered by any of the straight lines spanned by the segments I_1, I_2, I_3. (See the right side of Figure 1.) By the similar rôle of I_1, I_2 and I_3 and that of A_1 and A_7, we may assume that $S \in \Sigma_{32}$, and $T \in \Sigma_{21}$. The plane Σ_{42} intersects the face $Q \cap \Sigma_{21}$ in a segment $[T', T'']$ parallel to I_2 and passing through T. Suppose T'' is the endpoint on $[A_7, A_8]$. The intersection of the plane Σ_{42} with the boundary of the cube Q is a rectangle $T'S'S''T''$, where the side $[S', S''] \parallel [T', T'']$ goes through S, and S' is located on the edge $[A_2A_6]$ in such a way that the segment I_3 is contained in the segment $[S', A_2]$. Similarly to the previous case, Σ_{24} cuts the cube Q in a rectangle $A_5C_1C_2A_8$,

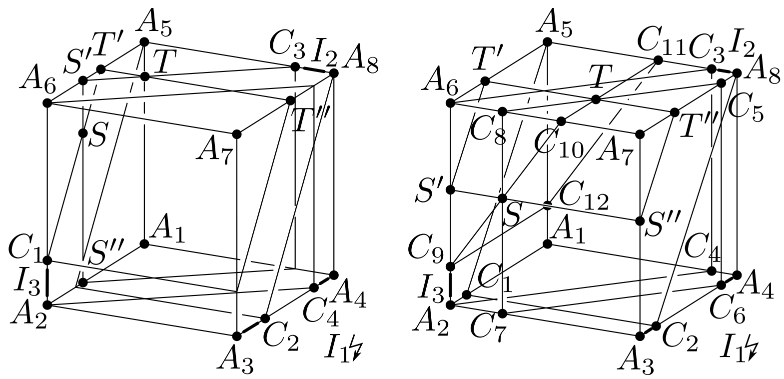

Figure 1.

where C_2 is a point of the segment $[A_3, A_4]$ for which $I_1 \subset [A_3, C_2]$. The plane Σ_{34} cuts the cube Q in a rectangle $A_2A_6C_3C_4$, where C_3 is a point of the segment $[A_5, A_8]$, for which $I_2 \subset [C_3, A_8]$. Similarly, the plane Σ_{43} cuts Q in a rectangle $C_5C_6C_7C_8$, where $T \in [C_5, C_8] \parallel [A_6, C_3]$, $S \in [C_7, C_8] \parallel I_3$, and C_6 is located on the segment $[A_3, A_4]$ in such a way that $I_1 \subset [C_6, A_4]$. Finally, the intersection of the cube with the plane Σ_{41} is a rectangle $C_9C_{10}C_{11}C_{12}$, where $C_{10} \in [A_6, A_7]$, $T \in [C_{10}, C_{11}] \parallel I_1$, $C_9 \in [A_2, A_6]$, $S \in [C_9, C_{10}]$ and $I_3 \subset [A_2, C_9]$. As before, I_1 must be in the intersection of the segments $[A_3, C_2]$ and $[C_6, A_4]$. Therefore, to obtain a contradiction, it is enough to show that these two segments are disjoint by proving $C_2A_4 > C_6A_4$.

As the triangles $\triangle A_5A_6C_3$, $\triangle A_7C_5C_8$, and $\triangle C_{10}TC_8$ are similar and $A_5A_6 > A_5C_3$ we have $A_7C_5 > A_7C_8$, $A_7T'' = TC_{10} > C_8C_{10}$ and

$$C_8A_6 > C_5A_8 = C_6A_4. \tag{1}$$

Comparing the right triangles $\triangle A_7T''S''$ and $\triangle C_8C_{10}S$ we obtain

$$\angle S'SC_9 = \angle C_8C_{10}S > \angle A_7T''S'' = \angle A_4C_2A_8. \tag{2}$$

Comparison of the right triangles $\triangle SS'C_9$ and $\triangle C_2A_4A_8$ using (2) and $A_4A_8 > S'C_9$ yields

$$C_2A_4 > S'S = C_8A_6, \tag{3}$$

which together with (1) gives $C_2A_4 > C_6A_4$.

□

3. Non-existence of strictly antipodal collections of four $\mathcal{C}^1$ arcs

Definition 4. We call a subset Γ of $\mathbb{R}^d$ a $\mathcal{C}^1$ arc if there is an injective continuously differentiable map $\gamma : [0,1] \to \mathbb{R}^n$ with nowhere $\mathbf{0}$ derivative the image of which equals Γ.

Theorem 2. *There are no strictly antipodal collections consisting of four $\mathcal{C}^1$ arcs in $\mathbb{R}^3$.*

Proof. Suppose that there is a strictly antipodal family of four arcs $\Gamma_1, \dots, \Gamma_4$ parametrized by the injective regular maps $\gamma_i : [0,1] \to \mathbb{R}^3$, $i = 1, \dots, 4$. Choose four arbitrary parameters $t_1, \dots, t_4 \in (0,1)$. The strict antipodality condition for $\gamma_i(t_i) \in \Gamma_i$ and $\gamma_j(t_j) \in \Gamma_j$ gives two parallel supporting planes of the union $\bigcup_{l=1}^{4} \Gamma_l$

passing through $\gamma_i(t_i)$ and $\gamma_j(t_j)$ respectively. The tangent of Γ_i at $\gamma_i(t_i)$ and the tangent of Γ_j at $\gamma_j(t_j)$ must be parallel to these supporting planes, otherwise Γ_i or Γ_j would cross one of the supporting planes. This means, that taking the tangent of Γ_i at $\gamma_i(t_i)$ and a sufficiently small segment on it around $\gamma_i(t_i)$ for $i = 1, \dots, 4$, we obtain four segments in antipodal position. By Theorem 1, in any antipodal collection of four segments, every segment is parallel to another one. This means that $\gamma_1'(t_1)$ is parallel to one of the vectors $\gamma_2'(t_2), \gamma_3'(t_3), \gamma_4'(t_4)$. Keeping t_2, t_3, t_4 fixed and letting t_1 run over the interval $(0, 1)$ we see that the direction of $\gamma_1'(t)$ can take only 3 different values. Since it changes continuously as well, the direction of γ_1' must be constant, so Γ_1 must be a segment. A similar argument shows that all the curves must be straight line segments, but it is obvious that straight line segments cannot be in strict antipodal position. □

4. Construction of a strictly antipodal collection of three $\mathcal{C}^1$ arcs and a singleton

Theorem 3. *There exists a strictly antipodal family of four sets consisting of three $\mathcal{C}^1$ arcs and a single point.*

Proof. Consider the curves

$$\gamma_1(t) = (1 + t, at^2 - a, 2 + a - at^2),$$

$$\gamma_2(t) = (2 + a - at^2, 1 + t, at^2 - a),$$

$$\gamma_3(t) = (at^2 - a, 2 + a - at^2, 1 + t)$$

where $a = 1/100$ and $t \in [-1/10000, 1/10000]$. We prove that $O = (0, 0, 0)$, $\Gamma_1 = \operatorname{im}\gamma_1$, $\Gamma_2 = \operatorname{im}\gamma_2$ and $\Gamma_3 = \operatorname{im}\gamma_3$ form a strictly antipodal collection. Because of the rotational symmetry, it is enough to show the following two claims:

(1) *If $P = \gamma_1(t_0)$, then there are two parallel planes $\mathcal{S}_O$ and $\mathcal{S}_P$ such that $(\Gamma_1 \setminus \{P\}) \cup \Gamma_2 \cup \Gamma_3$ is contained in the open slab bounded by $\mathcal{S}_O$ and $\mathcal{S}_P$.*

(2) *If $P = \gamma_1(t_0)$ and $Q = \gamma_2(s_0)$, then there are two parallel planes $\mathcal{S}_P$ and $\mathcal{S}_Q$ such that $\{O\} \cup (\Gamma_1 \setminus \{P\}) \cup (\Gamma_2 \setminus \{Q\}) \cup \Gamma_3$ is contained in the open slab bounded by $\mathcal{S}_P$ and $\mathcal{S}_R$.*

(1) The tangent vector to γ_1 at P is $\mathbf{e} = (1, 2at_0, -2at_0)$. Let $\mathbf{f} = (0, 4, -1)$, and let $\mathcal{S}_O$ and $\mathcal{S}_P$ be the parallel planes having normal vector $\mathbf{n} = \mathbf{e} \times \mathbf{f} = (6at_0, 1, 4)$ and passing through O and P, respectively. The open slab bounded by $\mathcal{S}_O$ and $\mathcal{S}_P$ contains $\Gamma_1 \setminus \{P\}$, because $\mathbf{e}$ is a tangent to the parabola Γ_1 at P. Elementary calculation gives that

$$\mathbf{n} \cdot \overrightarrow{OP} = 8 + 3a + 6at_0 + 3at_0^2 > 8.$$

If $Q_s = \gamma_2(s) \in \Gamma_2$ and $R_s = \gamma_3(s) \in \Gamma_3$ are arbitrary points, then

$$\mathbf{n} \cdot \overrightarrow{OQ_s} = 6at_0(2 + a - as^2) + (1 + s) + 4(as^2 - a) = 1 + c_2,$$

$$\mathbf{n} \cdot \overrightarrow{OR_s} = 6at_0(as^2 - a) + (2 + a - as^2) + 4(1 + s) = 6 + c_3,$$

where $|c_2| < 0.02$ and $|c_3| < 0.02$, so

$$\mathbf{n} \cdot \overrightarrow{OO} = 0 < \mathbf{n} \cdot \overrightarrow{OQ_s} < \mathbf{n} \cdot \overrightarrow{OR_s} < 8 < \mathbf{n} \cdot \overrightarrow{OP},$$

hence the open slab bounded by $\mathcal{S}_O$ and $\mathcal{S}_P$ contains $\Gamma_2 \cup \Gamma_3$.

(2) The tangent vector to γ_2 at Q is $\mathbf{g} = (-2as_0, 1, 2as_0)$. Let now $\mathcal{S}_P$ and $\mathcal{S}_Q$ be the parallel planes having normal vector

$$\mathbf{n} = \mathbf{e} \times \mathbf{g} = (2at_0 + 4a^2 s_0 t_0, -2as_0 + 4a^2 s_0 t_0, 1 + 4a^2 s_0 t_0),$$

and passing through P and Q, respectively. The open slab bounded by $\mathcal{S}_P$ and $\mathcal{S}_Q$ contains $\Gamma_1 \setminus \{P\}$, because $\mathbf{e}$ is a tangent to the parabolic arc Γ_1 at P, and it also contains $\Gamma_2 \setminus \{Q\}$, because $\mathbf{g}$ is a tangent to the parabolic arc Γ_2 at Q. Elementary calculation gives that

$$\begin{aligned}
\mathbf{n} \cdot \overrightarrow{OP} &= (2at_0 + 4a^2 s_0 t_0)(1 + t_0) + (-2as_0 + 4a^2 s_0 t_0)(at_0^2 - a) + \\
&\quad + (1 + 4a^2 s_0 t_0)(2 + a - at_0^2) \\
&= 2 + d_1, \\
\mathbf{n} \cdot \overrightarrow{OQ} &= (2at_0 + 4a^2 s_0 t_0)(2 + a - as_0^2) + (-2as_0 + 4a^2 s_0 t_0)(1 + s_0) + \\
&\quad + (1 + 4a^2 s_0 t_0)(as_0^2 - a) \\
&= -a + d_2,
\end{aligned}$$

where $|d_1| < 0.02$ and $|d_2| < 0.07a$. If $R_u = \gamma_3(u) \in \Gamma_3$ is an arbitrary point, then

$$\begin{aligned}
\mathbf{n} \cdot \overrightarrow{OR_u} &= (2at_0 + 4a^2 s_0 t_0)(au^2 - a) + (-2as_0 + 4a^2 s_0 t_0)(2 + a - au^2) + \\
&\quad + (1 + 4a^2 s_0 t_0)(1 + u) \\
&= 1 + d_3,
\end{aligned}$$

where $|d_3| < 0.02$. So

$$\mathbf{n} \cdot \overrightarrow{OQ} < 0 = \mathbf{n} \cdot \overrightarrow{OO} < \mathbf{n} \cdot \overrightarrow{OR_u} < \mathbf{n} \cdot \overrightarrow{OP},$$

hence the open slab bounded by $\mathcal{S}_P$ and $\mathcal{S}_Q$ contains $\{O\} \cup \Gamma_3$, too. □

5. Non-existence of large antipodal families of six sets

Theorem 4. *For some $m \in \mathbb{N}$ there does not exist an antipodal family $\{A_i | i \in I\}$ with $\#I = 6$ and each $\#A_i \geq m$. In particular, $k(3) \leq 5$.*

Proof. We use the fact, independently proved in [11] and [2], that an antipodal set of 6 points in $\mathbb{R}^3$ can be partitioned into two parts of 3 points each, with the two parts contained in two parallel planes.

Let $\{A_1, \ldots, A_6\}$ be an antipodal family in $\mathbb{R}^3$ with each $\#A_i \geq m$. For each choice of points $a_i \in A_i$, $1 \leq i \leq 6$, the set $\{a_1, \ldots, a_6\}$ is an antipodal set. By the above fact, there is a partition $\{J, K\}$ of $\{1, \ldots, 6\}$ into two sets of size three, such that $\{a_i | i \in J\}$ and $\{a_i | i \in K\}$ are on parallel planes. Note that the parallel planes are uniquely determined by the partition, otherwise both sets of size 3 would be collinear. However, it follows from the definition of antipodal family that no three points from distinct sets can be collinear.

There are $\binom{6}{3}$ such partitions. By the multipartite analogue of Ramsey's theorem (due to Erdős [4]; see also Chapter 5, Theorem 4 of [5]) it follows that if m is sufficiently large, then there exist a fixed partition $\{J_0, K_0\}$ of $\{1, \ldots, 6\}$ and subsets $B_i \subseteq A_i$, $1 \leq i \leq 6$, with $\#B_i = 2$, such that $\{a_i | i \in J_0\}$ and $\{a_i | i \in K_0\}$ are on parallel planes for any choice of $a_i \in B_i$. Without loss of generality $J_0 = \{1, 2, 3\}$ and $K_0 = \{4, 5, 6\}$.

It is now easily seen that the pair of parallel planes is independent of the choice of $a_i \in B_i$. Indeed, fix $a_1 \in B_1, a_2 \in B_2, a_3 \in B_3$, and let Π_1 be the plane through a_1, a_2, a_3. Then for any $a_4 \in B_4$, the plane Π_2 through a_4 parallel to Π_1 must contain $B_5 \cup B_6$. Similarly, B_4 must also be contained in Π_2. It then follows that $\{B_4, B_5, B_6\}$ is an antipodal family in the plane Π_2. However, Martini and Makai [8] showed that there does not exist such a family. □

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