Abstract

This paper characterizes the unique symmetric mixed-strategy Nash equilibrium in the three-player version Lowest Unique Number Game. In this game, each player chooses a positive integer simultaneously and the player with the lowest unique number wins. The equilibrium mixing is shown to have full support and it is characterized by a constant hazard rate which is around .46.

1 Introduction

The literature on the Lowest Unique Number game typically consider this game with a finite action space (i.e., a player cannot choose a number larger than $N \in \mathbb{N}$) and provide a system of equations characterizing symmetric equilibria with full support, see for example Flitney (2008) and Baek and Bernhardsson (2010). This characterization is based on the observation that each player must be indifferent between choosing any of the numbers. If the largest element of the action space is $N$, there are $N - 1$ equations which describe such an indifference condition involving the mixing probabilities of the remaining players. For small number of players, these equations can be solved numerically. In a remarkable contribution, Östling et al. (2011) examines a version of this game where the number of players is Poisson-distributed. The authors prove several analytical results about symmetric equilibria and compute equilibria even when the expected number of players is large\footnote{The authors also argue that the equilibrium in their Poisson game describes the data on the Swedish lottery game Limbo fairly well.}.

In contrast to the literature, we do not impose any restriction on the action space: players can choose any positive integer. We first show that the support of every symmetric equilibrium mixing is full. We then argue that the Lowest Unique Number game is strategically equivalent to a dynamic exit-game in which players decide whether or not to exit in each period. In this game, a mixed strategy of a player can be described by specifying her exit probability in each period. We show that, in the unique symmetric equilibrium, the exit probability is constant in time and it is characterized as a root of a polynomial. The constant exit probability corresponds to the constant hazard rate of the equilibrium mixing of the Lowest Unique Number Game.
2 The Model and the Equilibrium

We consider the following symmetric normal-form game with three players. The action space of each player is \( \mathbb{N} \). The payoff of a player is one if she chooses the lowest unique number and zero otherwise. Formally, if Player \( i \) chooses \( n_i \) for \( i = 1, 2, 3 \), the Player 1’s payoff is

\[
u_1(n_1, n_2, n_3) = \begin{cases} 
1 & \text{if } n_1 < n_2, n_3, \\
1 & \text{if } n_1 > n_2 = n_3, \\
0 & \text{otherwise}.
\end{cases}
\]

We characterize the unique symmetric mixed-strategy Nash Equilibrium of this game.

Let \( \mathcal{P} \) denote the set of probability distributions over \( \mathbb{N} \). We first show that symmetric equilibrium strategies have full support.

**Lemma 1** Suppose that \( p^* \in \mathcal{P} \) constitutes a symmetric Nash Equilibrium. Then \( \text{supp } p^* = \mathbb{N} \).

**Proof.** We prove this statement by induction. Note that \( 1 \in \text{supp } p^* \) for otherwise Player 1 could choose 1 and surely wins if the other players use strategy \( p^* \). Suppose now that \( n \in \text{supp } p^* \). We are going to show that \( n + 1 \in \text{supp } p^* \). Suppose, by contradiction, that \( n + 1 \notin \text{supp } p^* \).

We argue that Player 1 is strictly better off by choosing \( n + 1 \) than by choosing \( n \). Note that \( n \) wins only if either the other two players tie at a number strictly smaller than \( n \) or because the other two players chose something larger than \( n \). In the former case, \( n + 1 \) also wins. In the latter case, the numbers chosen by the other two players are strictly larger than \( n \) because \( n + 1 \notin \text{supp } p^* \). Therefore, \( n + 1 \) would also win in this case. It remains to show that the probability that \( n + 1 \) wins but \( n \) does not is strictly positive. Observe that since \( n \in \text{supp } p^* \) the probability of the other two players tying at \( n \) is strictly positive. In this event, \( n + 1 \) wins but \( n \) loses. \(\blacksquare\)

To solve for an equilibrium of this game we consider the following dynamic exit-game in discrete time \( n = 1, 2, \ldots \) etc. At each time, players decide whether to exit or stay in simultaneously. Players do not observe other players’ actions. The payoff of a player is one if either she is the first to exit and none of the other player exited at the same time or the other two player exited before she exits. Otherwise, the payoff is zero. This game is obviously strategically equivalent to the Lowest Unique Positive Integer Game.

In this exit-game, one can represent a strategy by the set of exit probabilities at each time. That is, let \( \lambda_n (\in [0, 1]) \) denote the probability of exiting at \( n \) conditional on reaching \( n \). Then a strategy can be describe by \( \{\lambda_n\}_{n \in \mathbb{N}} \) where \( \lambda_n \) denotes the probability that the player chooses \( n \) in this dynamic problem conditional on not choosing \( 1, \ldots, n - 1 \). In other words, the distribution is characterized in terms of the hazards, note that \( \lambda_n = p_n / \left[ 1 - \sum_{i=1}^{n-1} p_i \right] \), where \( p_i \) denotes the (unconditional) probability of choosing \( i \).
Proposition 1 In the unique symmetric equilibrium of the Lowest Unique Positive Integer Game, 
\[ \lambda_n = \lambda^* \text{ for all } n \in \mathbb{N}, \text{ where } \lambda^* \text{ solves} \]
\[ (1 - \lambda)^2 = \lambda^2 + (1 - \lambda)^4. \] (1)

The value of \( \lambda^* \) defined in the statement of the proposition is around 0.46. The unique symmetric equilibrium strategy, \( p^* \in \mathcal{P} \), can be defined by \( p^*_n = (1 - \lambda^*)^{n-1} \lambda^* \) for all \( n \in \mathbb{N} \).

**Proof.** By Lemma 1, each player must be indifferent between exiting at \( n \) and at \( n + 1 \) for all \( n \in \mathbb{N} \). Observe that, conditional on a player not exiting before \( n \), her payoffs generated by exiting at \( n \) and \( n + 1 \) are the same if at least one other player exited before \( n \). Indeed, if both players exited at the same time, \( n \) and \( n + 1 \) generate a payoff of one. Otherwise, both of them generate a payoff of zero. Therefore, to compare the payoffs generated by \( n \) and \( n + 1 \), it is enough to condition on the event that both other players are still in at \( n \). Then, if the player exists at \( n \), she wins if neither of the other players exits at \( n \) which happens with probability \( (1 - \lambda_n)^2 \). If she exits at \( n + 1 \), she wins if the other two exited at \( n \), which happens with probability \( \lambda_n^2 \), and if none of the other players exits at \( n \) and \( n + 1 \), which happens with probability \( (1 - \lambda_n)^2 (1 - \lambda_{n+1})^2 \).

So, the payoffs generated by \( n \) and \( n + 1 \) are the same if, and only if,
\[ (1 - \lambda_n)^2 = \lambda_n^2 + (1 - \lambda_n)^2 (1 - \lambda_{n+1})^2. \] (2)

Observe that when \( \lambda_n = \lambda_{n+1} \), the previous equation becomes equation (1) which implies that \( \lambda^* \) indeed defines an equilibrium.

It remains to show that there does not exist any other symmetric equilibrium. In other words, we need to argue that if \((\lambda_n)_{n \in \mathbb{N}}\), \( \lambda_n \in [0, 1] \), satisfy (2) for each \( n \) then \( \lambda_n \equiv \lambda^* \) or \((\lambda_n)\) does not constitute an equilibrium. First, note that (2) can be rewritten as
\[ \lambda_{n+1} = 1 - \sqrt{1 - 2\lambda_n}. \]
The right-hand side is increasing in \( \lambda_n \), zero at zero, and one at 1/2. Furthermore, it’s value is strictly smaller than \( \lambda_n \) on \([0, \lambda^*)\) and it is strictly larger than \( \lambda^* \) on \((\lambda^*, 1/2]\). We consider two cases depending on whether \( \lambda_1 \) is larger or smaller than \( \lambda^* \).

**Case 1:** \( \lambda_1 < \lambda^* \). In this case, \( \lambda_n \) converges to zero. This implies that for a large enough \( n \), conditional no player exiting before \( n \), the probability that any of the players exits at \( n \) is arbitrarily close to zero. Hence, exiting at \( n \) generates a payoff of close to one conditional on no player exiting before. But that is impossible because that payoff cannot exceed one third in a symmetric equilibrium².

**Case 2:** If \( \lambda_1 > \lambda^* \) then (2) implies that \( \lambda_{n+1} > \lambda_n \) for all \( \lambda_n \in (\lambda^*, 1/2] \). Since \( \lambda^* \) is the unique solution of (1), \( \lambda_n \) does not converge and there must exist an \( m \in \mathbb{N} \) such that \( \lambda_m > 1/2 \), implying that \( \lambda_{m+1} \) is not defined. ■

²Intuitively, if \( \lambda_1 < \lambda^* \) then the exit probabilities (\( \lambda_n \)) imply that a player never exits with a positive probability, which is clearly suboptimal.
3 Discussion

What happens if the players’ action spaces are restricted to be a finite set \( \{1, \ldots, N\} \). Of course, Lemma 1 continues to hold and players must deploy mixing with full support in any symmetric equilibrium. Equation (2) can again be used to define equilibrium strategies with the boundary condition \( \lambda_N = 1 \).

What if there are more than three players? Again, the arguments of the proof of Lemma 1 are applicable and players still randomize on full support. However, considering exit games to characterize equilibria is no longer useful. The reason is that, in the three-person version of the game, once a player exists, the game effectively ends. Therefore, the computation describing the indifference between two consecutive numbers could be conditioned on the event that no player exited before. This observation led to equation (2) which only depends on the exit probabilities at the two consecutive number. When there are more players, some players may exit simultaneously and lose, and hence, the game continues. So, the indifference condition at two numbers depends on the distribution of the number of players still in the game. In turn, this distribution depends on each exit probability at an earlier time. Consequently, the equation corresponding to equation (2) is no longer helpful.

References

