Optimal Information Disclosure in Auctions and the Handicap Auction

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We analyse a situation where a monopolist is selling an indivisible good to risk-neutral buyers who only have an estimate of their private valuations. The seller can release, without observing, certain additional signals that affect the buyers' valuations. Our main result is that in the expected revenue-maximizing mechanism, the seller makes available all the information that she can, and her expected revenue is the same as it would be if she could observe the part of the information that is "new" to the buyers. We also show that this mechanism can be implemented by what we call a *handicap auction* in interesting applications. In the first round of this auction, each buyer picks a price premium from a menu offered by the seller (a smaller premium costs more). Then the seller releases the additional signals. In the second round, the buyers bid in a second-price auction where the winner pays the sum of his premium and the second highest non-negative bid. In the case of a single buyer, this mechanism simplifies to a menu of European call options.

1. INTRODUCTION

In many examples of the monopolist's selling problem (optimal auctions),[1] the seller has considerable control over the accuracy of the buyers' information concerning their own valuations. Often the seller can decide whether the buyers can access information that refines their valuations; however, she either cannot observe these signals or, at least, she is unaware of their significance to the buyers. For example, the seller of an oil field or a painting can determine the number and the nature of the tests the buyers can carry out privately (without the seller observing the results). Another example (due to Bergemann and Pesendorfer, 2002) is where the seller of a company has detailed information regarding the company's assets (*e.g.* its client list), but does not know how well these assets complement the assets of the potential buyers. Here, the seller can choose the extent to which she will disclose information about the firm's assets to the buyers. In other applications (*e.g.* selling broadcast rights for a future sports event), the buyers' valuations for the good become naturally more precise over time as the uncertainty resolves, and the seller can decide how long to wait with the sale.

When the buyers' information acquisition is controlled by the seller, that process can also be optimized by the mechanism designer. In the present paper we explore the revenue-maximizing mechanism for the sale of an indivisible good in a model where the buyers initially only have an estimate of their private valuations. The seller can costlessly release additional private signals to the buyers that affect their valuations. These signals may be correlated to the buyers' private

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[1] Early seminal contributions include Harris and Raviv (1981), Myerson (1981), Riley and Samuelson (1981), and Maskin and Riley (1984).

information, they are not observable to the seller, and the seller decides whether a buyer can observe them. This model captures the common theme of the motivating examples: the seller controls, but cannot learn certain private information that the buyers care about.[2]

Our main result is that in the optimal mechanism (and under certain conditions) the seller releases all the information she can, and her expected revenue is as high as it would be if she could observe the part of the information controlled by her that is "new" to the buyers. That is, in the optimal mechanism, the buyers do not enjoy additional informational rents from learning more about their *ex post* valuations when the access to additional signals is controlled by the seller.

We emphasize that the information disclosure policy of the seller does not have to be a 0–1 decision. That is, we could allow the seller to reveal her information only partially. For example, she could show an element of a partition where the new information belongs to or reveal a signal that is only imperfectly correlated to the new information. Our result still continues to hold: in the optimal mechanism the seller reveals all her information.

We also exhibit a simple mechanism, dubbed the *handicap auction*, which implements the revenue-maximizing outcome in some interesting applications of the general model. The first application is one where the buyers' original information pertains solely to the expected value of their valuation. (That is, each buyer's true valuation differs from his initial estimate by an independent noise.) In the second application, each buyer's *ex post* valuation is the realization of a normally distributed random variable. The buyers' initial estimates and the seller's signals are normally distributed, conditionally independent noisy observations of the buyers' true valuations. Note that in this "sampling" application a buyer's private information and the signal controlled by the seller are strictly affiliated.[3]

The handicap auction, which implements the optimal mechanism in the two applications above, consists of two rounds. In the first round, each buyer buys a price premium from a menu provided by the seller (a smaller premium costs more). Then, without observing, the seller releases as much information as she can. In the second round, the buyers bid in a second-price auction, where the winner is required to pay his premium over the second highest non-negative bid. We call the whole mechanism a handicap auction because buyers compete under unequal conditions in the second round, where a bidder with a smaller premium has an advantage.

When there is only a single buyer, the handicap auction simplifies to a mechanism that can be thought of as a menu of European call options offered by the seller. These types of options are widely used in financial forward markets. In our model, as in reality, an option with a higher strike price costs more to the buyer. Buyers with different initial estimates regarding the good's value sort themselves and choose different options.

Our model nests the classical (independent private values) auction design problem as a special case, where the additional signals are identically 0. In this case, the handicap auction implements the outcome of the optimal auction of Myerson (1981) and Riley and Samuelson (1981).

Information disclosure by the seller in an auction has been studied in the context of the winner's curse and in the linkage principle by Milgrom and Weber (1982). They investigate whether in traditional auctions the seller should commit to disclose public signals that are affiliated with the buyers' valuations. They find that the seller gains from committing to full disclosure because that reduces the buyers' fear of overbidding, thereby increasing their bids and hence the seller's revenue. Our problem differs from this classic one in many aspects. Most importantly, in our setting, the signals that the seller can release are private (not public) signals, in the sense that

each signal affects the valuation of a single buyer and can be disclosed to that buyer only. The seller will gain from the release of information not because of the linkage principle, but because the information can improve efficiency. As we show, some of the potential efficiency gain is appropriated by the seller.

Several papers have studied issues related to how buyers learn their valuations in auctions and what are the consequences that bear on the seller's revenue. One strand of the literature (Persico, 2000; Compte and Jehiel, 2001; and the references therein) focuses on the buyers' incentives to acquire information in different auction formats. Our approach is different in that we want to *design* a revenue-maximizing mechanism in which the seller has the opportunity to costlessly release (without observing) information to the buyers. In our model, it is the seller (not the buyers) who controls how much information the buyers acquire.

In Baron and Besanko (1984), Riordan and Sappington (1987), and Courty and Li (2000), a principal and an agent are contracting over two periods. Independently of the contract, the agent learns pay-off-relevant private information in both periods. These papers analyse the optimal two-stage revelation mechanism where the contract is signed in the first period, when the agent only knows his first-period type. In contrast, in our paper, we solve for the optimal mechanism in a multi-agent auction environment where the seller can *decide* whether the buyers receive additional private signals.

A related literature is that on optimal entry fees in auctions (*e.g.* French and McCormick, 1984; McAfee and McMillan, 1987; Levin and Smith, 1994). In these papers, in contrast to ours, entry is endogenous, and competition drives down the bidders' pay-offs to 0. As a result, the seller maximizes the total social surplus, and she levies (*ex ante*) entry fees in order to induce socially optimal entry. In the independent private values case, for example, socially optimal entry is achieved by setting entry fees to 0 and committing to an efficient allocation. Our approach is different in that we focus on the complete mechanism design problem with a fixed number of bidders, and transfers are only allowed at the stage when bidders already have private information. Consequently, our results are also markedly different from those in this literature. Interpreting the transfers in our model as (interim) entry fees, we find that it is optimal for the seller to set them different from 0 and to induce *ex post* inefficiency by sometimes not selling the good to the buyer with the highest true valuation.

Our motivation is closer to that of Bergemann and Pesendorfer (2002) and Ganuza (2004), where the seller decides how much private information the buyers may learn prior to participating in an auction. Ganuza (2004) focuses on the incentives of the auctioneer to release signals to the buyers that refine their private valuations before a second-price auction.[4] He finds that, in order to maximize the revenue, the seller should not reveal all her information. In our model, we find just the opposite: the seller releases everything she can. The reason for the discrepancy is that Ganuza (2004) fixes the selling mechanism as a second-price auction, and the seller must reveal her information before the bidders can take any action. In contrast, we allow the seller to design the fully optimal mechanism. A feature of this auction is that the seller first elicits the private information of the bidders, and only then does she reveal her own. The main difference between our paper and that of Bergemann and Pesendorfer (2002) is that in their paper, the seller first designs a disclosure policy and reveals information and only then proposes the selling mechanism. Moreover, the seller cannot commit between the two stages. In contrast, in our model the timing is simultaneous.[5] Under their assumptions, Bergemann and Pesendorfer (2002) show that the information structure that allows the seller to subsequently design the auction with the

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4. By allowing the buyers to have more private information, the seller faces a trade-off. More information increases efficiency and potentially the revenue, but it also increases the buyers' information rents in the second-price auction.

5. A less important difference is that in Bergemann and Pesendorfer (2002) the buyers do not possess private information at the beginning of the game.

largest expected revenue is necessarily imperfect. In this structure, buyers are only allowed to learn which element of a finite partition their valuation falls into.

Compared to these two papers, the difference of our approach is that we consider the design of the information structure and the transaction rules as one mechanism design problem, as a whole. This difference may first seem subtle, nevertheless, it is important. In our model, the seller can integrate the rules of information acquisition into the mechanism used for the sale itself. For example, the seller can charge the buyers for getting more and more accurate signals (perhaps in several rounds); the buyers may even be asked to bid for obtaining more information. In contrast to the two papers cited above, we show that the seller maximizes expected revenue by designing a mechanism in which she allows the buyers to learn their valuations as precisely as they can.[6] In other words, the trade-off between allowing the buyers to obtain more private information and generating more revenue disappears in the optimal auction.

The idea that "selling" the access to information may be profitable is not new and can be illustrated by an example. Suppose that there are two buyers who are unaware of their private valuations, which the seller can allow them to learn. Consider the following mechanism: the seller charges both buyers an entry fee. Then, she allows the buyers to observe their valuations and makes them play a second-price auction with zero reservation price. The second-price auction will be efficient. If the entry fees equal the buyers' *ex ante* expected profits then the seller appropriates the entire surplus.[7]

This simple solution—the seller committing to the efficient allocation, releasing the additional signals, and charging an entry fee equal to the expected efficiency gains—only works when the buyers do not have private information to start with. Otherwise (*e.g.* if the buyers privately observe signals, but their valuations also depend on other signals that they may see at the seller's discretion) the auctioneer, as we will show, does not want to commit to an efficient auction in the continuation, so the previously proposed mechanism does not work. One has to find a more sophisticated auction, and this is exactly what we will do in the remainder of the paper.

2. THE MODEL

2.1. *The environment*

There are n potential buyers for a single indivisible good sold by a seller (she). All parties are risk neutral. The seller's valuation for the good is normalized to 0; her objective is to maximize the expected revenue. Each buyer's pay-off is the negative of his payment to the seller, plus, in case he wins, the value of the object.

Buyer i's true valuation for the object is V_i. However, he only observes a noisy signal of it, v_i, which is his private information. The seller has the ability to disclose to buyer i, *without observing*, an additional noisy signal about V_i that is denoted by z_i.[8] Assume that $E[V_i \mid v_i, z_i]$ is strictly increasing in z_i. For example, z_i may be identical to V_i, that is, the seller may be able to reveal to buyer i his actual valuation for the good. We can also allow the seller to only *partially* reveal z_i to buyer i, for example, by disclosing a signal positively correlated to z_i. However, in

order to simplify the exposition (and without affecting the results), we will not formally model this possibility.[9]

Denote the distribution of buyer i's type, v_i, by F_i and the corresponding density by f_i. We assume that f_i is positive on the interval $[\underline{v}_i, \overline{v}_i]$ and satisfies the monotone hazard rate condition, that is, $f_i/(1 - F_i)$ is weakly increasing.

We emphasize that buyer i's initial signal and the seller's signal, v_i and z_i, *need not be independent* nor even conditionally independent given V_i. On the other hand, the pairs (v_i, z_i) are assumed to be independent across i. The independence of information *across buyers* is a standard assumption that rules out Crémer and McLean's (1988) full rent extracting mechanisms.

Since the buyers are risk neutral we can assume without loss of generality that $V_i = E[V_i \mid v_i, z_i]$, that is, after the seller reveals her signals, the buyers know their true valuations (*e.g.* $z_i \equiv V_i$ or $z_i \equiv V_i - v_i$). In other words, V_i is the buyer's posterior valuation given his and the seller's signals. Let H_{iv_i} denote the distribution of V_i conditional on v_i and h_{iv_i} the corresponding density.[10] We assume that if $v_i > \widehat{v}_i$ then H_{iv_i} first-order stochastically dominates $H_{i\widehat{v}_i}$, that is, $\partial H_{iv_i}/\partial v_i < 0$. We need the following two additional assumptions on the joint distribution of V_i, z_i, and v_i.

Assumption 1. $(\partial H_{iv_i}(V_i)/\partial v_i)/h_{iv_i}(V_i)$ *is increasing in* V_i.

Assumption 2. $(\partial H_{iv_i}(V_i)/\partial v_i)/h_{iv_i}(V_i)$ *is increasing in* v_i.

We shall argue that these assumptions can be interpreted as a kind of *substitutability* in buyer i's posterior valuation between v_i and the part of z_i that is *new* to buyer i. In order to explain these assumptions and make it precise what we mean by *new* information, we perform an orthogonal decomposition of z_i in the next subsection.

2.2. *Decomposition of the seller's signal*

Imagine that buyer i's type is v_i and the seller can disclose to him, without observing, $s_i(z_i, v_i)$ instead of z_i, where s_i is strictly monotonic in z_i. First, notice that since s_i is monotonic, buyer i can recover z_i from $s_i(z_i, v_i)$ and execute the same Bayesian updating to compute his posterior valuation as before. Second, this change makes no difference for the seller either since she cannot observe z_i or s_i anyway. Hence, our model remains strategically the same. In particular, the seller's expected revenue is invariant to such a transformation of the signals.

In Lemma 1 we show that for each v_i there exists a particular transformation of z_i for which the random variable $s_i(z_i, v_i)$ is *independent* of v_i. Since s_i, which we call buyer i's *shock*, is orthogonal to the buyer's initial value estimate, it should be considered as the part of the seller-controlled information, z_i, that is *new* for the buyer.

Lemma 1. (i) *There exist functions u_i and s_i, such that $u_i(v_i, s_i(z_i, v_i)) \equiv V_i$, such that u_i is strictly increasing, s_i is strictly increasing in z_i, and $s_i(z_i, v_i)$ is independent of v_i.*
(ii) *All s_i's satisfying part (i) are positive monotonic transformations of each other.*

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9. We will show in Theorem 1 that by fully revealing her signals, the seller can do as well as in a specific "benchmark" situation that is not attainable by partial revelation. Therefore, restricting attention to the case where the seller can decide to either reveal z_i to buyer i or not reveal it at all does not affect the results.

10. The function H_{iv_i} is assumed to be twice continuously differentiable, strictly increasing on $\mathbb{R}$. This assumption is purely for convenience. As a result, any realization of the *ex post* valuation, V_i, is possible given any type v_i of buyer i.

Proof.   Define $s_i(z_i, v_i) \equiv H_{iv_i}(V_i)$, which is strictly increasing in z_i by assumption. Note that for any $y \in [0, 1]$, $\Pr(H_{iv_i}(V) \leq y) = \Pr(V \leq H_{iv_i}^{-1}(y)) = H_{iv_i}(H_{iv_i}^{-1}(y)) = y$. That is, s_i is uniform on $[0, 1]$ irrespective of the value of v_i, and hence it is independent of v_i. Let $u_i(v_i, s_i) = H_{iv_i}^{-1}(s_i)$, which is by definition identical to V_i. The monotonicity of u_i and part (ii) are proved in the Appendix.   ‖

In part (i) of the lemma, s_i is constructed as the percentile of the distribution of V_i conditional on v_i that the realization of the true valuation belongs to. The function u_i is the formula that buyer i can use to compute his posterior valuation given his type v_i and signal s_i. We will denote the c.d.f. of s_i by G_i.[11]

The transformation in the proof of Lemma 1 requires no assumption regarding the joint distribution of V_i, v_i, and z_i. However, assumptions made on this distribution translate into properties of the function u_i. For example, since $h_v > 0$ on $\mathbb{R}$ (see footnote 10), for all v_i, s_i, and v_i', there exists s_i' such that $u_i(v_i, s_i) = u_i(v_i', s_i')$.

Now we show what Assumptions 1 and 2 imply regarding the shape of u_i. Denote the partial derivatives of u_i by $u_{i1} = \partial u_i / \partial v_i$, $u_{i2} = \partial u_i / \partial s_i$, $u_{i11} = \partial^2 u_i / \partial v_i^2$, and $u_{i12} = \partial^2 u_i / \partial v_i \partial s_i$. The statement of the next lemma does not depend on the choice of s_i and u_i, as long as they satisfy part (i) of Lemma 1.

Lemma 2.   *Assumption* 1 *is equivalent to* $u_{i12} \leq 0$ *and Assumption* 2 *is equivalent to* $u_{i11}/u_{i1} \leq u_{i12}/u_{i2}$.

Proof.   See Appendix.   ‖

We are ready to interpret Assumptions 1 and 2. Assumption 1 says that the marginal impact of the s_i shock on buyer i's valuation is non-increasing in his type, v_i. Assumption 2 means that an increase in i's type, holding the *ex post* valuation constant, weakly decreases the marginal value of v_i.[12] The assumptions imply a certain monotonicity condition that is *sufficient* for our results. We will comment on the necessity of these assumptions in Section 4.3 after we present the results.

It is interesting to note that our assumptions are easier to interpret in terms of u_i, as decreasing-returns and substitutability conditions on v_i and s_i, than in terms of the correlations between V_i, v_i, and z_i. Intuitively, in order to determine the information rents that the seller has to leave with the buyers in the optimal mechanism, what matters is not how the "raw" signals are correlated (to each other and the buyers' actual values), instead, what matters is whether the buyers' original information, and the "new" part of the seller's signals are substitutes in the buyers' *ex post* valuations.

2.3. *Applications, examples*

In order to give a flavour of the type of situations that our model covers we now provide a few applications and examples where our assumptions hold.

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11. Notice that s_i constructed in part (i) of Lemma 1 is uniform on $[0, 1]$, therefore G_i could be assumed to be uniform on $[0, 1]$.

12. To see this interpretation of Assumption 2 note that the total differential of u_{i1} (the change in the marginal value of i's type) is $u_{i11}dv_i + u_{i12}ds_i$. Keeping u_i constant (moving along an "iso-value" curve) means $ds_i = -u_{i1}/u_{i2}dv_i$. Substituting this into the total differential of u_{i1} yields $(u_{i11} - u_{i12}u_{i1}/u_{i2})dv_i$. This expression is non-positive for $dv_i > 0$ if and only if $u_{i11}/u_{i1} \leq u_{i12}/u_{i2}$.

Example 1.   Assume that buyer i's true valuation for the good differs from his type by an additive and independent noise: $V_i = v_i + z_i$, where z_i is independent of v_i. Suppose that the seller can resolve the uncertainty in buyer i's valuation by disclosing, without observing, z_i. Since z_i is independent of v_i, the buyer's original private information pertains only to the *expected value* of the good. Among other things, v_i conveys no information regarding the precision of the buyer's estimate. Assume that all signals are independent across i's and that the distribution of v_i satisfies the monotone hazard rate condition. The transformed model obtains by setting $s_i \equiv z_i$ and $u_i(v_i, s_i) = v_i + s_i$. By the linearity of u_i, Assumptions 1 and 2 hold.

For a specific example, suppose that the object for sale is a car, and assume that the buyer knows its make, model, age, and mileage, but not its colour, which the seller can reveal. It seems reasonable to assume that a buyer's initial willingness to pay for the car (v_i) and his colour preference (z_i) are statistically independent.

Example 2a.   Consider the following familiar "normal sampling" problem. Suppose that buyer i's true valuation for the good, V_i, is drawn from a normal distribution with mean μ_i and precision (inverse variance) τ_{i0}. His signal, v_i, is normally distributed with mean V_i and precision τ_{iv}.[13] Suppose that the seller can allow buyer i to observe his true valuation, that is, $z_i \equiv V_i$. Clearly, V_i, z_i, and v_i are strictly affiliated. The distribution of v_i, which is normal, satisfies the hazard rate condition.

The c.d.f. of V_i conditional on v_i, H_{iv_i}, is normal with mean $(\tau_{i0}\mu_i + \tau_{iv}v_i)/(\tau_{i0} + \tau_{iv})$ and precision $\tau_{i0} + \tau_{iv}$. The realization of v_i simply "shifts" the conditional distribution of V_i to the right without altering its shape: for a unit increase in v_i, the mean (and only the mean) of i's true valuation increases by $\tau_{iv}/(\tau_{i0} + \tau_{iv})$.

In order to show that this example fits our model, we just need to verify Assumptions 1 and 2. The easiest way to do so is by transforming the seller's signal and showing that the conditions in Lemma 2 hold. Define $s_i \equiv H_{iv_i}(z_i)$, and let $u_i(v_i, s_i) = H_{iv_i}^{-1}(s_i) \equiv V_i$. It is immediate that u_i is strictly increasing in s_i. Since v_i shifts H_{iv_i} (as a function of z_i) to the right by $\tau_{iv}/(\tau_{i0} + \tau_{iv})$ for a unit increase in v_i, the derivative of $H_{iv_i}^{-1}$ with respect to v_i is constant, $\tau_{iv}/(\tau_{i0} + \tau_{iv})$. Therefore, u_i is linear and strictly increasing in v_i and Assumptions 1 and 2 are satisfied.

Example 2a generalizes to the case where z_i, just like v_i, is a noisy, normal signal of V_i (details available upon request). In Examples 1 and 2a, u_i is linear in v_i in the transformed model. We will show in Section 5 that the implementation of the optimal mechanism is particularly simple in this case. However, our model applies more generally.

Example 2b.   Suppose that the environment is the same as in Example 2a, with the only difference that the random variable V_i is not the monetary value of the good for buyer i; instead, it is the quantity of an input that the buyer obtains by buying the good. (Think of the auctioned good as a mineral field and V_i as the quantity of the raw material in the field.) Suppose that the buyer's monetary gain from owning the good is an increasing, continuously differentiable, and *concave* function of V_i, say, $r_i(V_i)$. The buyer is risk neutral towards monetary gains and losses.

Set, for example, $r_i(V_i) = V_i$ for $V_i \leq 0$ and $r_i(V_i) = 2(\sqrt{V_i + 1} - 1)$ for $V_i \geq 0$. Then, in the transformed model the signal s_i is the same as in the transformed version of Example 2a and the buyer's monetary valuation for the good is $\tilde{u}_i(v_i, s_i) \equiv r_i(u_i(v_i, s_i))$. This function is clearly non-linear, yet it also satisfies the conditions in Lemma 2 because u_i is linear in v_i and r_i is concave.

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13. The support of v_i is not a compact interval, but this does not cause a problem in the analysis.

In the following example, again, the buyer's signal is correlated to his true valuation, which the seller has the ability to reveal. We derive the transformed model where u_i is genuinely non-linear but our model still applies.

Example 3. Let v_i and $z_i \equiv V_i$ have joint p.d.f. $\phi(v_i, V_i) = 2e^{V_i}$ for $V_i \in [0, \ln(2)]$, $v_i \in [e^{V_i} - 1, 1]$. Then the densities of the marginals are $\phi_v(v_i) = \int_0^{\ln(v_i+1)} 2e^{V_i} dV_i = 2v_i$ for $v_i \in [0, 1]$ and $\phi_V(V_i) = \int_{e^{V_i}-1}^1 2e^{V_i} dv_i = (2 - e^{V_i})2e^{V_i}$ for $V_i \in [0, \ln(2)]$. The hazard rate condition holds for $f_i \equiv \phi_v$. The p.d.f. of V_i conditional on v_i is

$$h_{iv_i}(V_i) \equiv \phi_{V|v}(V_i \mid v_i) = \frac{\phi(v_i, V_i)}{\phi_v(v_i)} = \frac{e^{V_i}}{v_i}.$$

The corresponding cumulative distribution function is $H_{iv_i}(V_i) = (e^{V_i} - 1)/v_i$. Note that the domain of V_i conditional on v_i is $[0, \ln(v_i + 1)]$.

Perform the transformation by letting $s_i = H_{iv_i}(V_i)$. Then, $u_i(v_i, s_i) = \ln(v_i s_i + 1)$ ensures $u_i(v_i, s_i) \equiv V_i$. Now, $u_{i1} = u_{i2} = 1/(v_i s_i + 1)$, $u_{i11} = u_{i12} = -1/(v_i s_i + 1)$, therefore $u_{i12} < 0$ and $u_{i11}/u_{i1} = u_{i12}/u_{i2}$, so our assumptions hold.

3. PREVIEW OF THE RESULTS FOR A SINGLE BUYER

In this section we preview our results for the case of a single buyer in the context of Example 2a. This special case allows us to outline the main results of the paper and some heuristic proofs. It also showcases the practical aspects of the optimal mechanism and its relation to mechanisms observed in reality, in particular, option contracts.[14]

In Example 2a the buyer's valuation for the object on sale, V, is normally distributed with mean μ and variance $1/\tau_0$. The buyer only observes the signal v, whose distribution conditional on V is normal with mean V and variance $1/\tau_v$. The unconditional distribution of v is normal with mean μ and variance $1/\tau_0 + 1/\tau_v$, hence the c.d.f. of v, F, satisfies the hazard rate condition. The conditional distribution of V given v, H_v, is also normal with mean $(\tau_0 \mu + \tau_v v)/(\tau_0 + \tau_v)$ and variance $1/(\tau_0 + \tau_v)$. The seller can disclose to the buyer, without observing, the realization of V. We want to find the optimal mechanism that incorporates the rules of information disclosure and sale.

The main difficulty of this mechanism design problem is that in this set-up, it seems impossible to characterize all feasible mechanisms. For example, it is clearly not without loss of generality to assume that the seller discloses V and asks the buyer to report it back. Our approach will be to analyse the optimal solution in a relaxed problem and prove that it is implementable in the original environment.

Suppose that the agents have access to a computer (black box) that works as follows: if the buyer privately inputs v and the seller reveals to the machine (without observing) V then it outputs $H_v(V)$. That is, the machine computes the percentile of the distribution of V given v that the realization of the true valuation belongs to. Call the output of the machine (which is a random variable) the signal s. In the proof of part (i) of Lemma 1 we established that $s \equiv H_v(V)$ is distributed uniformly on $[0, 1]$ for all realizations of v, hence it is independent of v.

It does not matter for the buyer whether he observes V or s; knowing v, he can compute $V = H_v^{-1}(s)$. However, the seller would be better off if she could commit to a mechanism contingent on s before observing s as she can always discard the information. The benchmark case is the hypothetical situation where the seller commits to a mechanism that may depend on s and *then*

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14. We thank the Editor, Juuso Välimäki, for suggesting that we include this section and pointing out the realistic features of the optimal mechanism.

observes s (but not V or v). The seller's expected revenue in the benchmark is an upper bound on that in the original problem.

The benchmark case can be analysed using standard tools of mechanism design. In particular, any feasible mechanism can be represented by a truthful direct mechanism, where the buyer is asked to reveal v, the seller observes s, trade takes place with probability $X(v, s)$, and the buyer pays an expected transfer of $T(v, s)$. If the buyer reports type $\hat{v}$ while his actual type is v, his pay-off in the mechanism is

$$\pi^*(v, \hat{v}) = \int_0^1 \left[H_v^{-1}(s) \, X(\hat{v}, s) - T(\hat{v}, s) \right] ds. \tag{1}$$

The mechanism is incentive compatible if $\pi^*(v, \hat{v})$ is maximized in $\hat{v}$ at $\hat{v} = v$. The first-order condition of maximization is $\partial \pi^*(v, \hat{v})/\partial \hat{v} = 0$ at $\hat{v} = v$ and the second-order condition boils down to $\int_0^1 X(v, s)ds$ being increasing in v (strictly increasing for sufficiency).

By the envelope theorem, $d\pi^*(v, v)/dv = \partial \pi^*(v, \hat{v})/\partial v$ at $\hat{v} = v$. In Example 2a we saw that $\partial H_v^{-1}(s)/\partial v = \tau_v/(\tau_0 + \tau_v)$, a constant that we will denote by λ. Therefore, $d\pi^*(v, v)/dv = \int_0^1 \lambda X(v, s)ds$. Integrating it in v and using (1) yields

$$\pi^*(v, v) = \underline{\pi} + \int_{\underline{v}}^v \int_0^1 \lambda X(y, s)ds\,dy, \tag{2}$$

where $\underline{v}$ is the infimum of types that receive the good with positive probability and $\underline{\pi}$ is this type's pay-off. The *ex ante* expectation of the buyer's surplus is

$$\int_{-\infty}^\infty \pi^*(v, v)dF(v) = \underline{\pi} + \int_{-\infty}^\infty \left[\int_0^1 \lambda X(v, s)ds \frac{1 - F(v)}{f(v)} \right] dF(v),$$

where we used integration by parts (or Fubini's theorem) in the second term.

The seller's expected pay-off is the difference between social surplus and the buyer's expected surplus, which by the previous expression can be written as

$$W = \int_{-\infty}^\infty \int_0^1 \left(H_v^{-1}(s) - \lambda \frac{1 - F(v)}{f(v)} \right) X(v, s)ds\,dF(v) - \underline{\pi}. \tag{3}$$

The seller maximizes W by choosing X and $\underline{\pi}$. The constraints to her problem are (i) $X \in [0, 1]$, (ii) $\int_0^1 X(v, s)ds$ weakly increasing in v, and (iii) $\underline{\pi} \geq 0$. The optimum is attained by setting $\underline{\pi} = 0$ and maximizing the integrand in (3) pointwise by letting

$$X(v, s) = \begin{cases} 1 & \text{if } H_v^{-1}(s) - \lambda(1 - F(v))/f(v) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The resulting X is between 0 and 1, it is strictly increasing in v as H_v^{-1} is increasing and $(1 - F)/f$ is decreasing in v. The constraints are satisfied and we have an optimum.

Equation (4) defines the allocation rule of the optimal mechanism in the benchmark case. According to it, the buyer receives the object exactly when his true valuation, $V \equiv H_v^{-1}(s)$,

exceeds the threshold $\lambda(1 - F(v))/f(v)$. By plugging (4) into (3), the seller's expected revenue in the optimal mechanism can be written as

$$W = \int\limits_{-\infty}^{\infty} \int\limits_{0}^{1} \max\left\{ H_v^{-1}(s) - \lambda \frac{1 - F(v)}{f(v)}, 0 \right\} ds\, dF(v). \tag{5}$$

The interpretation of the optimal mechanism in the benchmark is familiar from Bulow and Roberts (1989): the seller's marginal revenue from the buyer with type v and valuation V is $V - \lambda(1 - F(v))/f(v)$. The object is sold whenever this expression is non-negative, and the total revenue is the area under the marginal revenue curve.

Our main result is that the seller can implement the same outcome in the original model even *without observing s*. The intuition is that the seller's marginal revenue from buyer type v is $V - \lambda(1 - F(v))/f(v)$ in the original model as well, at least as long as the seller allows the buyer to learn V. However, this is not the entire argument: it turns out that the conditions under which the same allocation rule can be implemented without the seller observing s are stronger than they are in the benchmark model. In what follows, we provide a more formal analysis.

Consider mechanisms where the buyer first reports his type, v, and pays a fee $c(v)$. Then, the seller allows him to observe V. The buyer makes the decision whether or not to buy; if he buys, he pays an *additional* premium $p(v)$. (The functions c and p are announced by the seller before v is reported.) We argue that such a mechanism can implement the outcome of the benchmark, (4) and (5).

The buyer with type v who initially announces $\hat{v}$ buys the good in the end if and only if $V \geq p(\hat{v})$ because $c(\hat{v})$, which is paid no matter whether or not he buys, is sunk. If the seller sets $p(v) = \lambda(1 - F(v))/f(v)$ then the buyer who reports v truthfully gets the good if and only if $V \geq \lambda(1 - F(v))/f(v)$, which is exactly the allocation rule in the benchmark according to equation (4). Now we define a fee schedule $c(v)$ that, together with $p(v)$ defined above, elicits a truthful type announcement. By reporting $\hat{v}$, the buyer with signal v gets a pay-off of

$$\pi(v, \hat{v}) = \int\limits_{0}^{1} \max\left\{ H_v^{-1}(s) - p(\hat{v}), 0 \right\} ds - c(\hat{v}). \tag{6}$$

The first-order condition of incentive compatibility is $\partial \pi(v, \hat{v})/\partial \hat{v} = 0$ at $\hat{v} = v$, that is,

$$c'(v) = -p'(v)\left[1 - H_v(p(v))\right]. \tag{7}$$

The necessary second-order condition of the maximization is $p' \leq 0$, which holds strictly (implying sufficiency) for $p = \lambda(1 - F)/f$ by the hazard rate condition.[15] Equation (7) defines c up to a constant, which is set so that for an arbitrary v, $\pi(v, v) = \pi^*(v, v)$.

This mechanism implements the allocation rule of the benchmark. We only need to prove that the seller gets the same expected pay-off as well. By the definition of c, $\pi(v, v) = \pi^*(v, v)$ for some v. By differentiating (2) and using (4),

$$\frac{d}{dv}\pi^*(v, v) = \int\limits_{0}^{1} X(v, s)ds = \lambda\left[1 - H_v(\lambda(1 - F(v))/f(v))\right].$$

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15. Recall that in the benchmark case, the corresponding second-order condition requires that $\int_0^1 X(v, s)ds$ be increasing in v, which holds because $H_v^{-1}(s) - \lambda(1 - F(v))/f(v)$ is increasing in v. In order to implement the same allocation rule in the original model, the local second-order condition is stronger: it requires that $p = \lambda(1 - F)/f$ be decreasing in v.

Using (7), the total derivative of $\pi(v, v)$ is

$$\frac{d}{dv}\pi(v, v) = \int_{H_v(p(v))}^{1} (\lambda - p'(v))ds - c'(v) = \lambda[1 - H_v(p(v))].$$

Since $p = \lambda(1 - F)/f$, we have $d\pi(v, v)/dv = d\pi^*(v, v)/dv$ for all v. We conclude that the buyer's pay-off is the same under the proposed mechanism and the benchmark: $\pi(v, v) = \pi^*(v, v)$ for all v. The seller's revenue is the difference between the social surplus and the buyer's pay-off. In the proposed mechanism, the buyer's and the social surpluses are the same as their respective counterparts in the benchmark. Therefore, the seller attains the same expected revenue in both cases.

The mechanism consisting of the pair of functions (c, p) is remarkably simple and can be thought of as a *menu of European call options* offered by the seller. These types of options are widely used in financial forward markets and in labour and other production contracts as well. (Prominent examples include professional athletes' labour contracts and movie producers' option contracts regarding sequels.) In our model, it is the monopolist seller, not "time", that reveals new information to the buyer of the asset. However, this distinction makes no difference because the seller reveals all her information anyway. In the optimal mechanism, the buyer picks a fee, $c(v)$, depending on his prior, from a list provided by the seller. After having learnt his posterior, the buyer has to pay a corresponding additional strike price, $p(v)$, in case he decides to buy the good. Since $p = (1 - F)/f$ is decreasing in v, higher buyer types pick options with lower strike prices but larger upfront fees. (In financial markets, a European call option on the same future asset also costs more if the strike price is lower.) In our model, the reason why different buyers may choose options with different strike prices is that they have heterogeneous initial estimates regarding the asset's future value.

4. THE OPTIMAL MECHANISM IN THE GENERAL CASE

We now turn to the characterization of the expected revenue-maximizing mechanism in the general model. Recall that we showed in Section 2.2 that the seller's signal can be transformed so that the resulting random variable, s_i (buyer i's shock), is independent of v_i, and for an increasing function u_i, $V_i \equiv u_i(v_i, s_i)$. In Section 4.1 we characterize the optimal mechanism under the assumption that the seller, after having committed to a mechanism, is able to observe the realizations of the shocks. This provides an upper bound on her revenue in the original model as the seller can commit to ignore the shocks. Then, in Section 4.2, we show that the same outcome can be implemented even if the seller cannot observe the s_i's, but controls their release. While the buyers still enjoy information rents from their types, all their rents from observing the seller-controlled signals are appropriated by the seller. This is the main result of the paper.

4.1. *Benchmark: the seller can observe the shocks*

Suppose first, for benchmarking purposes only, that the seller can observe the s_i's after having committed to a selling mechanism.[16] The revelation principle applies, hence we can restrict attention to mechanisms where the buyers report their types and the seller determines the allocation and the transfers as a function of the reported types and the realization of the

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16. Since the seller observes the shocks only after having committed to a mechanism, "informed principal" type problems do not arise in the benchmark. It does not matter whether the buyers can observe the shocks as long as the mechanism is verifiable (*i.e.* the seller cannot lie about s_i).

shocks.[17] We will analyse truthful equilibria of direct mechanisms that consist of an allocation rule, $x_i(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i})$ for all i and an (expected) transfer scheme, $t_i(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i})$ for all i. Here, $x_i(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i})$ is the probability that buyer i receives the good and $t_i(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i})$ is the transfer that he expects to pay, given the reported types and the realization of the shocks.

We shall prove that the revenue-maximizing allocation is in a way a generalization of the one in Myerson (1981). The object is rewarded to the bidder with largest non-negative "shock-adjusted virtual valuation", $W_i(v_i, s_i)$, where

$$W_i(v_i, s_i) = u_i(v_i, s_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} u_{i1}(v_i, s_i). \tag{8}$$

Ignoring ties, the optimal allocation rule x_i^* is defined as follows:

$$x_i^*(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i}) = 1 \text{ if } i = \arg\max_j \{W_j(v_j, s_j), 0\}. \tag{9}$$

Using the tools of Bayesian mechanism design, we obtain the following solution to the benchmark problem.

Proposition 1. *In the revenue-maximizing mechanism of the benchmark case (when the seller can observe s_i's after having committed to a selling mechanism), the allocation rule is defined by* (9). *The profit of buyer i with type v_i is*

$$\Pi_i^*(v_i) = \int_{\underline{v}_i}^{v_i} \int u_{i1}(y, s_i) X_i^*(y, s_i) dG_i(s_i) dy, \tag{10}$$

where $X_i^*(v_i, s_i) = \iint x_i^*(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i}) dF_{-i}(\boldsymbol{v}_{-i}) dG_{-i}(\boldsymbol{s}_{-i})$. The seller's revenue is

$$R^* = \iint \max_i \left\{ u_i(v_i, s_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} u_{i1}(v_i, s_i), 0 \right\} dF(\boldsymbol{v}) dG(\boldsymbol{s}). \tag{11}$$

Proof. See Appendix. ‖

It is useful, for use in subsequent steps of the analysis, to further describe some properties of the allocation rule.

Corollary 1. (i) *X_i^* is continuous in both of its arguments.*
(ii) *X_i^* is weakly increasing in both of its arguments.*
(iii) *If $v_i > \hat{v}_i$, $s_i < \hat{s}_i$ and $u_i(v_i, s_i) = u_i(\hat{v}_i, \hat{s}_i)$ then $X_i^*(v_i, s_i) \geq X_i^*(\hat{v}_i, \hat{s}_i)$.*

Proof. See Appendix. ‖

4.2. *The solution to the seller's problem*

We now show that the allocation rule and seller's revenue characterized in Proposition 1 can be implemented even if the seller cannot observe the shocks, as long as she can allow the buyers to observe them. In order to implement the allocation rule (9), she has to reveal her signals to the buyers (otherwise nobody can compute W_i). Therefore, in order to find a mechanism that implements the benchmark allocation rule, we can restrict attention to two-stage, incentive-compatible

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17. Using standard shorthand notation, $\boldsymbol{v}_{-i}$ and $\boldsymbol{s}_{-i}$ denote the vector of types and shocks of buyers other than i, while $\boldsymbol{v}$ and $\boldsymbol{s}$ denote the entire profiles.

direct mechanisms. In the first stage buyers report their v_i's; in the second stage each buyer observes his own s_i and reports it back. For all reporting profiles (v, s) and for all i, the seller allocates the good to buyer i with probability $x_i(v_i, v_{-i}, s_i, s_{-i})$ and buyer i pays $t_i(v_i, v_{-i}, s_i, s_{-i})$; these functions are set so that truth-telling is incentive compatible in both reporting stages.

Given a two-stage mechanism $\{x_i, t_i\}_{i=1}^n$, define

$$X_i(v_i, s_i) = \iint x_i(v_i, v_{-i}, s_i, s_{-i}) \, dF_{-i}(v_{-i}) dG_{-i}(s_{-i}),$$

$$T_i(v_i, s_i) \equiv \iint t_i(v_i, v_{-i}, s_i, s_{-i}) \, dF_{-i}(v_{-i}) dG_{-i}(s_{-i}),$$

as buyer i's expected probability of winning and expected transfers, respectively.

We will now analyse the consequences of incentive compatibility going backwards, starting in the second stage of the mechanism. In Lemma 3 we show how the allocation rule pins down the buyers' second-round profit functions given truthful revelation of types in the first round. Then, in Lemma 4 we describe what happens off the equilibrium path—after buyer i reports his type untruthfully and observes his shock. It turns out that the deviator will report an untruthful value for his shock in such a way that the two lies "cancel" each other and his true valuation, $u_i(v_i, s_i)$, is correctly inferred from his reports. Using these results, in Lemmas 5 and 6 (the final steps before proving Theorem 1), we derive the indirect profit function of buyer i.

In the second reporting stage, after truthful first round, buyer i with type v_i who observes s_i and reports $\hat{s}_i$ gets

$$\tilde{\pi}_i(s_i, \hat{s}_i; v_i) = u_i(v_i, s_i) X_i(v_i, \hat{s}_i) - T_i(v_i, \hat{s}_i). \tag{12}$$

Incentive compatibility in the second reporting stage is equivalent to

$$\tilde{\pi}_i(s_i, \hat{s}_i; v_i) \leq \tilde{\pi}_i(s_i, s_i; v_i) \text{ for all } i, v_i, s_i, \text{ and } \hat{s}_i. \tag{13}$$

The following lemma provides the conditions for a mechanism to be incentive compatible in the second reporting stage after a truthful first round.

Lemma 3. *If a two-stage mechanism is incentive compatible and $X_i(v_i, s_i)$ induced by the allocation rule is continuous in s_i then for all $s_i > \hat{s}_i$,*

$$\tilde{\pi}_i(s_i, s_i; v_i) - \tilde{\pi}_i(\hat{s}_i, \hat{s}_i; v_i) = \int_{\hat{s}_i}^{s_i} u_{i2}(v_i, z) X_i(v_i, z) dz. \tag{14}$$

Moreover, if (14) holds and X_i is weakly increasing in s_i then the two-stage mechanism is incentive compatible in the second round after truthful revelation in the first round.

Proof. See Appendix.   ‖

In order to complete the analysis of the second round of the mechanism, we also need to know what buyer i will do if he misreports his type in the first round. The following lemma claims that he will "correct his lie".[18]

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18. This correction is possible because for all v_i, $\hat{v}_i$, and s_i there exists $\hat{s}_i$ such that $u_i(v_i, s_i) = u_i(\hat{v}_i, \hat{s}_i)$. If this assumption did not hold then the seller could detect deviations more easily.

Lemma 4. *In the second round of an incentive-compatible two-stage mechanism, buyer i with type v_i who reported $\hat{v}_i$ in the first round and has observed s_i will report $\hat{s}_i = \sigma_i(v_i, \hat{v}_i, s_i)$ such that*

$$u_i(v_i, s_i) \equiv u_i(\hat{v}_i, \sigma_i(v_i, \hat{v}_i, s_i)). \tag{15}$$

Proof. Had buyer i indeed have type $\hat{v}_i$ (as reported) and observed $\hat{s}_i$, incentive compatibility in the second round would require

$$u_i(\hat{v}_i, \hat{s}_i)X_i(\hat{v}_i, \hat{s}_i) - T_i(\hat{v}_i, \hat{s}_i) \geq u_i(\hat{v}_i, \hat{s}_i)X_i(\hat{v}_i, s_i') - T_i(\hat{v}_i, s_i'),$$

for all s_i'. By (15), that is, $u_i(v_i, s_i) = u_i(\hat{v}_i, \hat{s}_i)$, this is equivalent to

$$u_i(v_i, s_i)X_i(\hat{v}_i, \hat{s}_i) - T_i(\hat{v}_i, \hat{s}_i) \geq u_i(v_i, s_i)X_i(\hat{v}_i, s_i') - T_i(\hat{v}_i, s_i'),$$

which means that type v_i that reported $\hat{v}_i$ in the first round and then observed s_i is indeed best off by reporting $\hat{s}_i$.   ‖

Now we move back to the first round and examine the consequences of incentive compatibility there. Our goal is to derive the buyers' equilibrium profit functions, and we proceed as follows. Using the result of Lemma 4 regarding continuation play after a first-round deviation (buyer i misreporting his type) we first derive the deviating buyer's profit function (see Lemma 5). In Lemma 6 we use these formulae for the deviator's pay-off to derive the buyers' equilibrium (or indirect) profit functions.

Lemma 5. *In an incentive-compatible two-stage mechanism, if type v_i of buyer i reports $\hat{v}_i$ in the first round then his pay-off is*

$$\pi_i(v_i, \hat{v}_i) = \pi_i(\hat{v}_i, \hat{v}_i) + \int \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i)X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i))dy\, dG_i(s_i), \tag{16}$$

where $\int_a^b$ denotes $-\int_b^a$ for $a < b$.

Proof. See Appendix.   ‖

Incentive compatibility in the first round is equivalent to, for all $v_i > \hat{v}_i$,

$$\pi_i(v_i, \hat{v}_i) \leq \pi_i(v_i, v_i) \text{ and } \pi_i(\hat{v}_i, v_i) \leq \pi_i(\hat{v}_i, \hat{v}_i). \tag{17}$$

Equation (16) is used in the following lemma to characterize the buyers' indirect profit functions in an incentive-compatible two-stage mechanism.

Lemma 6. *If a two-stage mechanism is incentive compatible and $X_i(v_i, s_i)$ induced by the allocation rule is continuous then buyer i's indirect profit (as a function of his type) can be written as*

$$\Pi_i(v_i) = \Pi_i(\underline{v}) + \int_{\underline{v}_i}^{v_i} \int u_{i1}(y, s_i)X_i(y, s_i)dG_i(s_i)dy. \tag{18}$$

Proof. See Appendix.   ‖

The significance of equation (18) is that it closely resembles equation (10), the profit function of buyer i under the benchmark case. By comparing the two formulae we see that if X_i^* can be implemented with $\Pi_i(\underline{v}) = 0$, then $\Pi_i(v_i) = \Pi_i^*(v_i)$ and the seller's revenue is the same as in the benchmark. The following result—which is the main result of the paper—shows that this is indeed the case.

Theorem 1. *When the seller can disclose without observing the buyers' shocks, the same outcome can be implemented as in the benchmark case described in Proposition 1.*

Proof. Set $X_i = X_i^*$ and suppose that all buyers except i report their types truthfully. Consider buyer i with type v_i contemplating to misreport to $\hat{v}_i < v_i$. Note that his deviation pay-off is

$$\pi_i(v_i, \hat{v}_i) - \pi_i(v_i, v_i) = \left[\pi_i(v_i, \hat{v}_i) - \pi_i(\hat{v}_i, \hat{v}_i)\right] - \left[\pi_i(v_i, v_i) - \pi_i(\hat{v}_i, \hat{v}_i)\right].$$

By (16) and (18), the difference of the two bracketed expressions can be written as

$$\int\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i^*(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy \, dG_i(s_i) - \int\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i^*(y, s_i) dy \, dG_i(s_i). \quad (19)$$

But since for all $y \in [\hat{v}_i, v_i]$, by property (iii) of X_i^* (Corollary 1),

$$X_i^*(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) \leq X_i^*(y, s_i),$$

the difference in (19) and hence $\pi_i(v_i, \hat{v}_i) - \pi_i(v_i, v_i)$ is non-positive. A similar argument can be used to rule out deviation to $\hat{v}_i > v_i$.

Buyer i's participation constraint holds in the first stage (when he only knows v_i) because $\Pi_i^* \geq 0$. His second-stage participation constraint fails if for some (v_i, s_i), $T_i^*(v_i, s_i) > X_i^*(v_i, s_i)u_i(v_i, s_i)$. This is a consequence of the simplification that transfers are made at the end. By asking buyer i to pay $\gamma_i(v_i) = \sup_{s_i}\{X_i^*(v_i, s_i)u_i(v_i, s_i) - T_i^*(v_i, s_i)\}$ in the first round and then $T_i^*(v_i, s_i) - \gamma_i(v_i)$ in the second round, the same allocation is implemented, and all interim participation constraints hold.

Finally, we point out that the necessary and sufficient condition that guarantees truthful reports in the first stage is that (19) is non-positive.   ‖

A key feature of the optimal allocation rule that makes the proof work is that for all $v_i, \hat{v}_i \in [\underline{v}, \bar{v}]$ and $s_i, \hat{s}_i \in \mathbb{R}$ such that $v_i > \hat{v}_i$ and $u_i(v_i, s_i) = u_i(\hat{v}_i, \hat{s}_i)$, the allocation rule "favours" the pair (v_i, s_i), that is, $X_i^*(\hat{v}_i, \hat{s}_i) \leq X_i^*(v_i, s_i)$, as seen in property (iii) in Corollary 1. In words, buyer i with type v_i and a given *ex post* valuation wins the object more often than he does with a lower type $\hat{v}_i$, but the same *ex post* valuation.

4.3. *Necessary and sufficient conditions*

The assumptions made in Section 2 (in particular, Assumptions 1 and 2) are not necessary for our main result to hold. Below we discuss what are the necessary and sufficient conditions for the allocation X_i^*, defined by (9), to be implementable by an incentive-compatible mechanism in both the benchmark and the original models.

The benchmark case.   At the end of the proof of Proposition 1, we have concluded that X_i^* can be implemented in the benchmark case if and only if for all $i, \widehat{v}_i, v_i \in [0, 1]$,

$$\int \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i^*(y, s_i) dy \, dG_i(s_i) \geq \int (u_i(v_i, s_i) - u_i(\hat{v}_i, s_i)) X_i^*(\hat{v}_i, s_i) dG_i(s_i), \qquad (20)$$

where $\int_{\hat{v}_i}^{v_i}$ denotes $-\int_{v_i}^{\widehat{v}_i}$ for $v_i < \hat{v}_i$. Notice that if there were no shocks (s_i was 0 with probability 1), this condition would boil down to the necessary condition identified in the standard optimal auction literature (see Myerson, 1981), which requires a buyer with a higher type to win with a higher probability. That is, X_i^* has to be increasing in v_i. Because of the shocks, this condition only has to be satisfied in expectation in the benchmark. Our sufficient conditions imply that X_i^* is increasing in v_i, not only in expectation, but for all s_i. This is because Assumptions 1 and 2 guarantee that buyer i's shock-adjusted virtual valuation, W_i, is increasing in v_i and therefore X_i^* is increasing, too. As a matter of fact the monotonicity of W_i in v_i follows from the concavity of u_i in v_i, which is a much weaker requirement than Assumptions 1 and 2.

The original model.   Recall that the allocation rule X_i^* is implemented in a two-stage mechanism. In the first stage, buyers report their types. In the second stage, they learn their shocks and report them. Hence, there will be two conditions, one for each stage, that are necessary and sufficient for incentive compatibility.

The first condition guarantees that at the second stage buyer i reports s_i truthfully, given truthful reports in the first stage. At the end of the proof of Lemma 3 we pointed out that this condition holds, if and only if, for all $i, \widehat{s}_i, s_i$, and v_i such that $\widehat{s}_i < s_i$,

$$X_i^*(v_i, s_i) \geq X_i^*(v_i, \hat{s}_i). \qquad (21)$$

By the second stage the seller knows the buyers' types. Hence the problem of inducing them to report their new information is essentially the same as in a standard optimal auction. That is why we get the usual condition: a buyer with a larger shock must win the object with higher probability. Assumption 1 implies that W_i is increasing in s_i and hence X_i^* is increasing too. Notice that in the benchmark case it does not matter whether X_i^* is monotonic in s_i. The seller observes the shocks and does not have to elicit this information from the buyers. As a result, if (20) holds but (21) does not, X_i^* can be implemented in the benchmark but not in the original model.

The second condition guarantees that buyer i reports her type truthfully at the first stage. At the end of the proof of Theorem 1 we showed that this holds if and only if for all $i, \widehat{v}_i, v_i \in [0, 1]$,

$$\int \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i^*(y, s_i) dy dG_i(s_i) \geq \int \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i^*(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy dG_i(s_i). \qquad (22)$$

This condition is probably the hardest to interpret. Recall that the definition of σ_i is that $u_i(\hat{v}_i, \sigma_i(v_i, \hat{v}_i, s_i)) = u_i(v_i, s_i)$. If one replaces $\sigma_i(y, \hat{v}_i, s_i)$ by s_i in the R.H.S. of (22), then this condition would coincide with the necessary and sufficient condition in the benchmark, (20). Unlike in the benchmark, if a buyer misreports his type in the first stage, he can also misreport his shock in the second stage. As we have proved in Lemma 4, he would indeed do so, and instead of s_i, he would report $\sigma_i(v_i, \hat{v}_i, s_i)$. That is, he would make a report in the second stage such that the *ex post* valuation computed from his reports was his true valuation. This explains the discrepancy between (20) and (22). It is not enough that X_i^* is increasing in v_i in expectation, it must

be increasing even under the second-stage deviation strategy of the buyers. Again, Assumption 2 implies that (22) does not only hold in expectation but also pointwise. This is because X_i^* satisfies property (iii) in Corollary 1.

To summarize, condition (21) is irrelevant for the benchmark, since the seller observes the shocks and does not have to induce the buyers to report them. Condition (22) is stronger than condition (20) because a deviation in the first stage is linked to a deviation in the second stage, which is not possible in the benchmark. Furthermore, Assumption 1 is sufficient for (21), and Assumption 2 is sufficient for (22). The concavity of u_i in v_i implies (20), which is a weak consequence of Assumptions 1 and 2.

We want to point out that the conditions (20)–(22) can be stated in terms of the joint distribution of V_i, v_i, and z_i without introducing the variable s_i. These conditions involve u_i, u_{i1}, and X_i^*, which can all be expressed without s_i. Recall from the proof of Lemma 2 that $s_i \equiv H_{v_i}(V_i)$, $u_i(v_i, s_i) = V_i$, and $u_{i1}(v_i, s_i) = (\partial H_{iv_i}(V_i)/\partial v_i)/h_{iv_i}(V_i)$. Therefore, buyer i's shock-adjusted virtual valuation, W_i, is

$$V_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \frac{\partial H_{iv_i}(V_i)/\partial v_i}{h_{iv_i}(V_i)}.$$

Since according to the optimal allocation rule the object goes to the buyer with the largest shock-adjusted virtual valuation, X_i^* can also be defined without the introduction of s_i. However, as we have seen, X_i^* and the necessary–sufficient monotonicity conditions can be interpreted more easily with the help of the transformed signal, s_i.

Below we show, by examples, that sometimes the allocation rule X_i^* is only implementable in the benchmark case, but not in the original model. These examples also illustrate that Assumptions 1 and 2 are not necessary.

Example 4. Suppose that there is a single buyer, and in the transformed model $u(v, s) = \alpha(v + s) + \beta vs$. In order to guarantee that u is increasing in both arguments, assume $(\alpha, \beta) \in A \equiv \{(a, b) : a \geq \max\{0, -b\}\}$ and $v, s \in [0, 1]$. Let $F(v) = 1 - (1 - v)^k$ with $k \geq 1$ and assume s is uniform. We shall characterize the subsets of A such that X^* is implementable in the benchmark case and in the original model, respectively.

The shock-adjusted virtual valuation of the buyer is $W(v, s) = \alpha(v + s) + \beta vs - (1 - v)(1 + \beta s)/k$. Recall that $X^*(v, s) = 1$ if $W(v, s) \geq 0$ and 0 otherwise. Since $\partial W/\partial v = (\alpha + \beta s)(k + 1)/k$, W is increasing in v when $\alpha \geq \max\{0, -\beta\}$. Therefore, condition (20) holds for all $(\alpha, \beta) \in A$ and X^* is implementable on the entire set A.

Condition (21) is satisfied if and only if W is increasing in s. Since $\partial W/\partial s = \alpha + \beta v - (1 - v)\beta/k$, this requires $\alpha \geq \max\{\beta/k, -\beta\}$.[19] Hence, $\alpha \geq \max\{\beta/k, -\beta\}$ is necessary and sufficient for (21) to hold.

Finally, (22) is satisfied if and only if $X^*(v, \sigma(v + \delta, v, s)) \leq X^*(v + \delta, s)$ whenever $\delta \geq 0$. This holds if and only if $W(v, \sigma(v + \delta, v, s)) \leq W(v + \delta, s)$. We show that this condition holds for all $(\alpha, \beta) \in A$. In this example

$$\sigma(v + \delta, v, s) = \frac{\alpha(\delta + s) + \beta(v + \delta)s}{\alpha + \beta v}.$$

Therefore, $W(v + \delta, s) - W(v, \sigma(v + \delta, v, s))$ is

$$\frac{1 - v}{k}\left(\alpha + \beta \frac{\alpha(\delta + s) + \beta(v + \delta)s}{\alpha + \beta v}\right) - \frac{1 - v - \delta}{k}(\alpha + \beta s) = \frac{\delta(\alpha + \beta s)}{k}\frac{\alpha + \beta}{\alpha + \beta v} > 0.$$

19. If $\alpha < \max\{\beta/k, -\beta\}$, then there exists an interval on which W is decreasing in s, which implies that (21) does not hold.

We conclude that while X^* is implementable in the benchmark case for all $(\alpha, \beta) \in A$, in the original model it is only implementable for $\alpha \geq \max\{\beta/k, -\beta\}$.

This example also shows that Assumptions 1 and 2 are not necessary for our results. Notice that Assumption 1 holds if and only if $\beta \leq 0$ and Assumption 2 holds if and only if $\beta \geq 0$. However, X^* is implementable whenever $\alpha \geq \max\{\beta/k, -\beta\}$.

In the following two examples the orthogonal transformation yields special cases of the parametric model of Example 4.

Example 5. Suppose that there is a single buyer whose true valuation is distributed according to the c.d.f. V^2 on $[0, 1]$. His signal, v, is a uniform draw from $[0, V]$. Assume that the seller can disclose, without observing, the realization of V to the buyer.

Notice that the joint distribution of (V, v) is uniform on the triangle $0 \leq v \leq V \leq 1$. As a result, the conditional distribution of V given v is also uniform on the interval $[v, 1]$ with c.d.f. $H_v(V) = (V - v)/(1 - v)$. The unconditional c.d.f. of v is $F(v) = 1 - (1 - v)^2$. To perform the transformation, define $s \equiv H_v(V)$. Then, $u(v, s) = H_v^{-1}(s) = v + s - vs$.

By direct calculation, $u_1 = 1 - s$, $u_{11} = 0$, $u_2 = 1 - v$, and $u_{12} = -1$. Therefore, Assumption 1 holds ($u_{12} \leq 0$), but Assumption 2 does not ($u_{11}/u_1 > u_{12}/u_2$). However, the model is equivalent to Example 4 with $k = 2$, $\alpha = 1$, and $\beta = -1$. Since $\alpha \geq \max\{\beta/k, -\beta\}$, X^* can be implemented in the original model. This illustrates that Assumptions 1 and 2 are not necessary for Theorem 1.

It is interesting to note that the shock-adjusted virtual valuation simplifies to $W = [3u(v, s) - 1]/2$. Recall that in the optimal allocation the buyer gets the good whenever $W \geq 0$, that is, $u(v, s) \equiv V \geq 1/3$. The allocation rule only depends on the buyer's true valuation and not on his original signal, v. In this example (and its extension to multiple symmetric buyers), there is no distortion in the allocation of the good conditional on sale.

Example 6. Suppose that the joint p.d.f. of v and V is $\phi(v, V) = 1/v$ for $0 \leq V \leq v \leq 1$, and the seller can disclose V to the buyer. The p.d.f. of v is $f(v) \equiv \phi_v(v) = \int_0^v 1/v \, dV = 1$, that is, uniform on $[0, 1]$. The distribution of V conditional on v is $H_v(V) = V/v$, that is, uniform on $[0, v]$. Define $s \equiv H_v(V) = V/v$. Then, $u(v, s) = H_v^{-1}(s) = vs$.

This model is equivalent to Example 4 with $k = 1$, $\alpha = 0$, and $\beta = 1$. Since $\max\{\beta/k, -\beta\} > \alpha \geq \max\{0, -\beta\}$, X^* cannot be implemented in the original model.

4.4. *Discussion of the optimal mechanism*

As we have pointed out, in the optimal mechanism, two *ex ante* symmetric buyers with the same *ex post* valuation may not have the same probability of winning. According to Corollary 1, an increase in a buyer's type, holding his true valuation constant, weakly *increases* the probability that he wins the object.[20] This is so because $W_i(v_i, s_i)$ in equation (8) is weakly increasing in v_i even if $u_i(v_i, s_i)$ is compensated. (Example 5 is a knife-edge case where W_i remains constant for such a compensated change.) The economic reason for this property is that the seller uses discrimination—giving preference to someone with a higher type—in order to screen the buyers.

This property implies that the auction is not efficient, even under *ex ante* symmetry of the bidders and conditional on the object being sold. In contrast, in the classical symmetric set-up with deterministic valuations, the optimal auction of Myerson (1981) and Riley and Samuelson (1981) is efficient conditional on sale.

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20. In the previous section, we showed that this is a *necessary condition* for implementation when the seller cannot observe her signals. However, it is not necessary in the benchmark, when the seller can observe the shocks.

In order to better explain Theorem 1, consider a set-up where the buyers are *ex ante* symmetric (the v_i's are identically distributed) and the shocks are additive, mean zero random variables. Let us compare the optimal allocation rule in our model with that of the revenue-maximizing auction when *nobody* (neither the seller nor the buyers) can observe them. In the latter case, the seller should allocate the good to the buyer with the largest non-negative virtual value estimate, $v_i - (1 - F(v_i))/f(v_i)$. In our model, in the optimal mechanism, the good will be allocated more efficiently, as the winner will now be the buyer with the highest non-negative shock-adjusted virtual valuation, $v_i + s_i - (1 - F(v_i))/f(v_i)$.[21] According to Theorem 1, the seller, by controlling the release of the shocks and without actually observing them, can implement this rule and appropriate the increase in efficiency.[22]

5. THE HANDICAP AUCTION

In this section we show how to implement the optimal mechanism when in the transformed model u_{i1} is constant, like in Examples 1 and 2a. We call the mechanism *handicap auction*.

A handicap auction consists of two rounds. In the first round, each buyer i, knowing his type, chooses a price premium p_i for a fee $C_i(p_i)$, where C_i is a fee schedule published by the seller. The buyers do not observe the premia chosen by others. The second round is a traditional auction, but the winner is required to pay his premium over the price. Between the two rounds, the seller may send messages to the buyers. In the optimal handicap auction the seller discloses her signals to the buyers, and the second round is a second-price (or English) auction with a zero reservation price. We call this mechanism a handicap auction because in the second round, the buyers compete under unequal conditions: a bidder with a smaller premium has an advantage. The bidders *buy* their premium in the initial round, which is a form of price discrimination.[23] If there is only a single buyer, then, as we have seen it in Section 3, the handicap auction simplifies to a *menu of buy options*: p_i can be thought of as the strike price, and the upfront fee, $C_i(p_i)$, is the cost of the option.

It is a dominant strategy for buyer i who chose premium p_i in the first stage to bid $u_i(v_i, s_i) - p_i$ in the second stage of the handicap auction. Assuming that the buyers follow this weakly dominant strategy in the second round, the handicap auction can be represented by pairs of functions, $p_i : [\underline{v}, \bar{v}] \to \mathbb{R}$ and $c_i : [\underline{v}, \bar{v}] \to \mathbb{R}$, for $i = 1, \ldots, n$, where $p_i(v_i)$ is the price premium that in equilibrium type v_i picks for a fee of $c_i(v_i) \equiv C_i(p_i(v_i))$.

Proposition 2. *If u_{i1} is constant then the optimal mechanism of Theorem 1 can be implemented via a handicap auction $\{c_i, p_i\}_{i=1}^n$, where*

$$p_i(v_i) = \frac{1 - F_i(v_i)}{f_i(v_i)} u_{i1}, \tag{23}$$

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21. It is easy to see that if $v_i - (1 - F(v_i))/f(v_i) < v_j - (1 - F(v_j))/f(v_j)$, but, by adding the shocks to both sides the inequality is reversed, then $v_i + s_i > v_j + s_j$. Therefore, an allocation based on the shock-adjusted virtual valuations "pointwise" improves efficiency. (This may not be true if F_i's are not identical.)

22. If the buyers' *ex ante* type distributions are not identical then, as the seller gets to observe the signals, the efficiency of the optimal mechanism may only improve in *ex ante* expectation. Still, there will be some efficiency gain, which will be extracted by the seller even if she cannot observe the additional signals.

23. The handicap auction can also be implemented as a mechanism where, in the first round, each bidder buys a discount (larger discounts cost more) and then participates in a second-price auction with a positive reservation price, where the winner's discount is applied towards his payment.

and $c_i(v_i)$ is defined by

$$c_i(v_i) = \iint \left[\max\{W_i(v_i, s_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\} - \int_{\underline{v}}^{v_i} u_{i1} \mathbf{1}_{\{W_i(y, s_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} dy \right] dF_{-i} dG,$$

(24)

where $W_i(v_i, s_i) = u_i(v_i, s_i) - p_i(v_i)$ and $W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}) = \max_{j \neq i}\{W_j(v_j, s_j), 0\}$.

From the seller's perspective, the premium-fee schedule works as a device to discriminate among buyers with different value estimates. A buyer with a high type, expecting to win often, chooses a small price premium for a large fee in order not to pay much when he wins. Low types choose large price premia, which are cheaper, but make winning more expensive. In the optimal handicap auction, just like in the general optimal mechanism, two buyers with the same *ex post* valuation do not have the same probability of winning. The buyer with a larger v_i chooses a smaller price premium, bids higher in the second round, and ends up winning more often.

One may suggest that the way the seller can appropriate all rents from the additional information is that in the handicap auction she essentially charges the buyers a type-dependent upfront fee equal to the "value" of the information they are about to receive. This intuition may be appealing, but it overly simplifies the workings of the mechanism. The value of the additional information to the participants is not well defined because it depends on the rules of the selling mechanism. This value could be different if the seller chose a mechanism different from the handicap auction.

It may be useful to compute a numerical example not only for illustrative purposes but also to see how a seller may be able to calculate the parameters of the optimal handicap auction (the price premium-fee schedule) in a practical application.

Example 1 (Revisited).   Consider the set-up of Example 1, with the following additional assumptions: the buyers' types are distributed uniformly on $[0, 1]$ and the shocks are distributed according to a standard logistic distribution.[24]

First, suppose that there is a single buyer. Then, as seen also in Section 3, the handicap auction can be thought of as a menu of buy options, represented by $C_1(p_1)$, where p_1 is the strike price and $C_1(p_1)$ is the fee of the option. In the first round, the buyer chooses a price p_1 and pays $C_1(p_1)$; in the second round (after having observed s_1), he has the option to buy the good at price p_1. Again, we represent this menu as a pair of functions, $c_1(v_1)$ and $p_1(v_1)$, $v_1 \in [0, 1]$.

The expected revenue-maximizing strike price schedule is given by (23), $p_1(v_1) = 1 - v_1$. The fee schedule in (24) becomes

$$c_1(v_1) = \frac{1}{2}\ln(1+e) - 1 + v_1 + \frac{1}{2}\ln(1 + e^{1-2v_1}).$$

We can also express the cost of the option as a function of the strike price

$$c_1 = C_1(p_1) = \frac{1}{2}\ln[(1+e)(1 + e^{2p_1 - 1})] - p_1.$$

C_1 is a strictly decreasing function on $[0, \infty)$. The buyer with a higher estimate buys an option with a lower strike price at a higher cost. For example, if the buyer has the lowest estimate, $v_1 = 0$, then he buys the option of getting the good at $p_1 = 1$, which costs $c_1 = \ln[(1+e)/e] \approx 0.31$

24. The c.d.f. of the standard logistic distribution is $G_i(s_i) = e^{s_i}/(1 + e^{s_i})$, $s_i \in (-\infty, +\infty)$.

upfront, and yields zero net surplus. In contrast, the highest type, $v_1 = 1$, buys a call option with zero strike price at a cost of about 0.81.

As the number of buyers increases, the premium-fee schedule, $C_i(p_i)$, shifts down and flattens out. This is so because the premium as a function of the buyer's type, given by (23), remains the same, $p_i(v_i) = 1 - v_i$. As there are more buyers in the second round of the auction, competition increases and the bidders' gains from learning their valuations precisely decrease. As a result, they are willing to pay less upfront: the fee schedule, $c_i(v_i)$, computed using (24), decreases for all v_i as n increases.

6. CONCLUSIONS

We have analysed an auction model where the seller can reveal without observing signals to the buyers that, together with the buyers' original private information, affect their valuations. We have derived the expected revenue-maximizing mechanism comprising both the rules of information disclosure and those of the transaction.

In the optimal mechanism, the seller discloses all the information that she can and obtains the same expected revenue as if she could observe the "new" part of the additional signals that she can release without observing. The buyers do not enjoy any additional information rents from the signals whose disclosure is controlled by the seller. We have provided necessary and sufficient conditions for this result. The optimal mechanism can be implemented via a "handicap auction" in interesting applications.

There are selling mechanisms that have similar features to our optimal auction. For a single buyer, the optimal mechanism is a menu of "buy" options, where the cost of the option is negatively related to the strike price of the good. Similar contracts are used in financial, labour, and other production relationships. Sellers often demand that bids incorporate payments contingent on further exploration of the good's value, just like in our optimal mechanism.[25] Firms that solicit bids from underwriters before their initial public offering favour the most aggressive bidders in subsequent bidding for financial contracts.[26] These types of mechanisms, like our optimal auction, have the feature that *ex post* discrimination among bidders is based upon a prior contractual relationship.

The overall conclusion of our investigation is that under certain conditions, the seller who controls the "flow of information" in an auction appropriates the rents of that information.

APPENDIX: OMITTED PROOFS

Proof of Lemma 1 (Monotonicity).   Differentiating the identity $s_i \equiv H_{iv_i}(u_i(v_i, s_i))$ in v_i and s_i, respectively, rearranging, and using $u_i(v_i, s_i) = V_i$ yield

$$\frac{\partial u_i(v_i, s_i)}{\partial v_i} = -\frac{\partial H_{iv_i}(V_i)/\partial v_i}{h_{iv_i}(V_i)}, \tag{25}$$

$$\frac{\partial u_i(v_i, s_i)}{\partial s_i} = \frac{1}{h_{iv_i}(V_i)}. \tag{26}$$

The density h_{iv_i} is positive and by assumption $\partial H_{iv_i}(V_i)/\partial v_i < 0$, therefore u_i is strictly increasing in both arguments.

(ii) Suppose that s_i and $\tilde{s}_i$ are independent of v_i and there exist functions u_i and $\tilde{u}_i$, strictly increasing in their second argument, such that $u_i(v_i, s_i) \equiv \tilde{u}_i(v_i, \tilde{s}_i) \equiv V_i$.[27] We want to show that $\tilde{s}_i$ is a monotonic transformation of s_i,

25. See, for example, the privatization of the Antamina mine by the Peruvian government, as described in Moel and Tufano (1997).

26. See Krigman, Shaw and Womack (2001) and the references therein.

27. The latter condition means that for all $\xi \in \mathbb{R}$, $\Pr(u_i(v_i, s_i) \le \xi) = H_{iv_i}(v) = \Pr(\tilde{u}_i(v_i, \tilde{s}_i) \le \xi)$.

that is, there exists an increasing function λ such that for all v_i and σ,

$$\Pr(u_i(v_i, s_i) \leq u_i(v_i, \sigma)) = \Pr(\tilde{u}_i(v_i, \tilde{s}_i) \leq \tilde{u}_i(v_i, \lambda(\sigma))). \tag{27}$$

The issue is that λ must be the same function for all realizations of v_i.

Define λ so that for all σ, $\Pr(s_i \leq \sigma) = \Pr(\tilde{s}_i \leq \lambda(\sigma))$. (The function λ associates the same percentiles of s_i and $\tilde{s}_i$ with each other.) Note that this function is increasing and does not depend on the realization of v_i as both s_i and $\tilde{s}_i$ are independent of v_i. By the monotonicity of u_i in its second argument, the L.H.S. of (27) equals $\Pr(s_i \leq \sigma)$, which in turn equals $\Pr(\tilde{s}_i \leq \lambda(\sigma))$ by the definition of λ. By the monotonicity of $\tilde{u}_i$ in its second argument, $\Pr(\tilde{s}_i \leq \lambda(\sigma))$ equals $\Pr(\tilde{u}_i(v_i, \tilde{s}_i) \leq \tilde{u}_i(v_i, \lambda(\sigma)))$, which is the R.H.S. of (27), therefore the equality indeed holds. ‖

Proof of Lemma 2. In this proof we omit the reference to the identity of buyer i. Recall from Lemma 1 that the model can be transformed into one with orthogonally normalized v and s by letting $s \equiv H_v(V)$ and $u(v, s) = H_v^{-1}(s)$. First, we prove the lemma for this s and u and then argue that the choice of s and u does not matter.

By partially differentiating $(\partial H_v(V)/\partial v)/h_v(V)$, Assumption 1 can be written as

$$\frac{\partial h_v/\partial v}{\partial H_v/\partial v} \leq \frac{h_v'}{h_v}, \tag{28}$$

and Assumption 2 can be expressed as

$$\frac{\partial^2 H_v(V)}{\partial v^2} \bigg/ \frac{\partial H_v(V)}{\partial v} \leq \frac{\partial h_v(V)}{\partial v} \bigg/ h_v(V). \tag{29}$$

From (26), using the chain rule we get

$$\frac{\partial^2 u(v, s)}{\partial s \partial v} = -\frac{1}{h_v^2(V)} \left(\frac{\partial h_v(V)}{\partial v} + h_v'(V) \frac{\partial H_v^{-1}(s)}{\partial v} \right) \bigg|_{V = H_v^{-1}(s)}$$

$$= \frac{1}{h_v^2(V)} \left(\frac{h_v'(V)}{h_v(V)} \frac{\partial H_v(V)}{\partial v} - \frac{\partial h_v(V)}{\partial v} \right) \bigg|_{V = H_v^{-1}(s)}, \tag{30}$$

where the second line follows from (25). Since the density h_v is positive and $\partial H_v/\partial v \leq 0$, $u_{12} \leq 0$ translates into (28).

Differentiate (25) in v to get

$$\frac{\partial^2 u(v, s)}{\partial v^2} = -\frac{\partial[\partial H_v(H_v^{-1}(s))/\partial v]/h_v(H_v^{-1}(s))}{\partial v}.$$

This can be rewritten as

$$\frac{\partial^2 u(v, s)}{\partial v^2} = -\frac{1}{h_v(V)} \left[\frac{\partial^2 H_v(V)}{\partial v^2} - \frac{\partial H_v(V)}{\partial v} \frac{\partial h_v(V)}{\partial v} \frac{1}{h_v(V)} \right]$$

$$+ \frac{1}{h_v^2(V)} \frac{\partial H_v(V)}{\partial v} \left[\frac{\partial h_v(V)}{\partial v} - \frac{h_v'(V)}{h_v(V)} \frac{\partial H_v(V)}{\partial v} \right]$$

$$= \frac{1}{h_v(V)} \left[\frac{2}{h_v(V)} \frac{\partial H_v(V)}{\partial v} \frac{\partial h_v(V)}{\partial v} - \frac{h_v'(V)}{h_v^2(V)} \left(\frac{\partial H_v(V)}{\partial v} \right)^2 - \frac{\partial^2 H_v(V)}{\partial v^2} \right],$$

where $V = H_v^{-1}(s)$. Hence, using this expression and (25),

$$\frac{\partial^2 u(v, s)}{\partial v^2} \bigg/ \frac{\partial u(v, s)}{\partial v} = \frac{-2}{h_v(V)} \frac{\partial h_v(V)}{\partial v} + \frac{h_v'(V)}{h_v^2(V)} \frac{\partial H_v(V)}{\partial v} + \frac{\partial^2 H_v(V)}{\partial v^2} \bigg/ \frac{\partial H_v(V)}{\partial v} \bigg|_{V = H_v^{-1}(s)}.$$

By (30) and (26),

$$\frac{\partial^2 u(v, s)}{\partial v \partial s} \bigg/ \frac{\partial u(v, s)}{\partial s} = \frac{1}{h_v(V)} \left(\frac{h_v'(V)}{h_v(V)} \frac{\partial H_v(V)}{\partial v} - \frac{\partial h_v(V)}{\partial v} \right) \bigg|_{V = H_v^{-1}(s)}.$$

Hence $u_{i11}/u_{i1} \le u_{i12}/u_{i2}$ is equivalent to

$$\frac{h'_v(V)}{h^2_v(V)}\frac{\partial H_v(V)}{\partial v} + \frac{\partial^2 H_v(V)}{\partial v^2} \bigg/ \frac{\partial H_v(V)}{\partial v} - \frac{2}{h_v(V)}\frac{\partial h_v(V)}{\partial v} \le \frac{h'_v(V)}{h^2_v(V)}\frac{\partial H_v(V)}{\partial v} - \frac{1}{h_v(V)}\frac{\partial h_v(V)}{\partial v}.$$

The first terms in each side cancel and we get exactly (29).

So far we assumed that $u(v, s) = H_v^{-1}(s)$. It remains to show that the choice of s_i and u_i does not affect the statement of this lemma as long as they satisfy part (i) in Lemma 1. To see this, consider a positive monotonic transformation of s_i, $\tilde{s}_i \equiv \lambda(s_i)$, and a corresponding $\tilde{u}_i$ with $\tilde{u}_i(v_i, \tilde{s}_i) \equiv u_i(v_i, \lambda(s_i))$. For simplicity assume λ is differentiable, hence $\lambda' > 0$. Since $\tilde{u}_{i1} = u_{i1}$, $\tilde{u}_{i2} = u_{i2}\lambda'$, $\tilde{u}_{i11} = u_{i11}$, and $\tilde{u}_{i12} = u_{i12}\lambda'$, the statement of this lemma is valid for $\tilde{u}_i$ whenever it is valid for u_i. ‖

Proof of Proposition 1.   Fix a direct mechanism and define

$$X_i(v_i, s_i) = \iint x_i(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i}) dF_{-i}(\boldsymbol{v}_{-i}) dG_{-i}(\boldsymbol{s}_{-i}). \tag{31}$$

If buyer i with type v_i reports type $\hat{v}_i$ then his profit is

$$\pi_i(v_i, \hat{v}_i) = \iint [x_i(\hat{v}_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i}) u_i(v_i, s_i) - t_i(\hat{v}_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i})] dF_{-i}(\boldsymbol{v}_{-i}) dG(\boldsymbol{s}). \tag{32}$$

The mechanism is incentive compatible if and only if $\pi_i(v_i, \hat{v}_i) \le \pi_i(v_i, v_i)$, for all i and all $v_i, \hat{v}_i \in [\underline{v}, \bar{v}]$. Define $\Pi_i(v_i) \equiv \pi_i(v_i, v_i)$ as the equilibrium (or indirect) profit function of buyer i. The seller's expected revenue can be written as the difference of the social surplus and the buyers' expected profits

$$\sum_{i=1}^n \iint [u_i(v_i, s_i) x_i(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i}) - \Pi_i(v_i)] dF(\boldsymbol{v}) dG(\boldsymbol{s}). \tag{33}$$

The benchmark problem is to maximize (33) subject to the buyers' incentive compatibility and participation constraints.

The incentive constraint guaranteeing that a bidder with type v_i does not report $\hat{v}_i$ (and vice versa) can be rewritten as

$$\int [u_i(v_i, s_i) - u_i(\hat{v}_i, s_i)] X_i(\hat{v}_i, s_i) dG_i(s_i) \le \pi_i(v_i, v_i) - \pi_i(\hat{v}_i, \hat{v}_i)$$

$$\le \int [u_i(v_i, s_i) - u_i(\hat{v}_i, s_i)] X_i(v_i, s_i) dG_i(s_i). \tag{34}$$

From this inequality chain, using the standard techniques of Bayesian mechanism design, one can conclude that

$$\Pi_i(v_i) = \Pi_i(\underline{v_i}) + \int\limits_{\underline{v_i}}^{v_i} \int\limits_{s_i} u_{i1}(y, s_i) X_i(y, s_i) dG_i(s_i) dy. \tag{35}$$

Given the profit function of buyer type v_i in (35), we can write the seller's expected revenue (after the application of Fubini's Theorem) as

$$\iint \sum_i x_i(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i}) \left[u_i(v_i, s_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} u_{i1}(v_i, s_i) \right] dF(\boldsymbol{v}) dG(\boldsymbol{s}). \tag{36}$$

The allocation rule $x_i^*(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i})$ maximizes the integrand in (36) pointwise.

We need to show that this rule is incentive compatible. The candidate optimal allocation rule, $x_i^*(v_i, \boldsymbol{v}_{-i}, s_i, \boldsymbol{s}_{-i})$, hence $X_i^*(v_i, s_i)$, is weakly increasing in v_i. This is because W_i is increasing in v_i since, by assumption, u_i is increasing in v_i, $(1 - F_i)/f_i$ is weakly decreasing in v_i, and finally, Assumptions 1 and 2 imply that u_i is concave in v_i. Suppose $v_i > \hat{v}_i$ and use (35) to rewrite

$$\Pi_i^*(v_i) - \Pi_i^*(\hat{v}_i) = \int\int\limits_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i^*(y, s_i) dy \, dG_i(s_i).$$

Since $X_i^*(y, s_i)$ is increasing in y,

$$\int_{\hat{v}_i}^{v_i} \int u_{i1}(y, s_i) X_i^*(y, s_i) dy \, dG_i(s_i) \geq \int_{\hat{v}_i}^{v_i} \int u_{i1}(y, s_i) dy \, X_i^*(\hat{v}_i, s_i) dG_i(s_i)$$

$$= \int (u_i(v_i, s_i) - u_i(\hat{v}_i, s_i)) X_i^*(\hat{v}_i, s_i) dG_i(s_i). \tag{37}$$

But this is exactly the incentive compatibility condition for buyer type v_i not to imitate $\hat{v}_i < v_i$, (34). A similar argument applies when $\hat{v}_i > v_i$.

We want to point out that the necessary and sufficient condition for X_i^* to be implementable is (37), where, by convention, if $v_i < \hat{v}_i$, $\int_{\hat{v}_i}^{v_i} = -\int_{v_i}^{\hat{v}_i}$. This condition may hold even if X_i^* is not increasing in its first argument. ∥

Proof of Corollary 1.   Continuity of X_i^* follows because the distributions are atomless. X_i^* is weakly increasing in v_i and s_i because (8) is strictly increasing in both variables, which in turn follows from the monotone hazard rate condition, $u_{i1} > 0$, $u_{i11} \leq 0$, $u_{i2} > 0$, and $u_{i12} \leq 0$. Finally, (iii) is another monotonicity property stating that X_i^* is weakly increasing in v_i even if s_i is adjusted to keep $u_i(v_i, s_i)$ constant. To see this, consider $v_i > \hat{v}_i$ and $s_i < \hat{s}_i$ with $u_i(v_i, s_i) = u_i(\hat{v}_i, \hat{s}_i)$ and compare (8) for (v_i, s_i) and $(\hat{v}_i, \hat{s}_i)$. The first term of (8) is the same by assumption; in the second term $(1 - F_i(v_i))/f_i(v_i) \leq (1 - F_i(\hat{v}_i))/f_i(\hat{v}_i)$ by the monotone hazard rate condition, $u_{i1}(v_i, s_i) \leq u_{i1}(\hat{v}_i, \hat{s}_i)$ because $u_{i11}/u_{i1} \leq u_{i12}/u_{i2}$ (see footnote 12).   ∥

Proof of Lemma 3.   Rewrite (13) as

$$\tilde{\pi}_i(s_i, \hat{s}_i; v_i) = \tilde{\pi}_i(\hat{s}_i, \hat{s}_i; v_i) + [u_i(v_i, s_i) - u_i(v_i, \hat{s}_i)] X_i(v_i, \hat{s}_i). \tag{38}$$

By reversing the roles of s_i and $\hat{s}_i$, we get

$$\tilde{\pi}_i(\hat{s}_i, s_i; v_i) = \tilde{\pi}_i(s_i, s_i; v_i) - [u_i(v_i, s_i) - u_i(v_i, \hat{s}_i)] X_i(v_i, s_i).$$

From these equalities standard techniques of Bayesian mechanism design yield (14).

To see the second part of the claim, suppose buyer i reports $\hat{s}_i < s_i$ after seeing s_i. His deviation gain is

$$\tilde{\pi}_i(s_i, \hat{s}_i; v_i) - \tilde{\pi}_i(s_i, s_i; v_i) = [\tilde{\pi}_i(s_i, \hat{s}_i; v_i) - \tilde{\pi}_i(\hat{s}_i, \hat{s}_i; v_i)] - [\tilde{\pi}_i(s_i, s_i; v_i) - \tilde{\pi}_i(\hat{s}_i, s_i; v_i)].$$

Recall from (38) that the first bracketed difference is $[u_i(v_i, s_i) - u_i(v_i, \hat{s}_i)] X_i(v_i, \hat{s}_i)$, and by (14) the second one can be rewritten as

$$\int_{\hat{s}_i}^{s_i} u_{i2}(v_i, z) X_i(v_i, z) dz \geq \int_{\hat{s}_i}^{s_i} u_{i2}(v_i, z) X_i(v_i, \hat{s}_i) dz = [u_i(v_i, s_i) - u_i(v_i, \hat{s}_i)] X_i(v_i, \hat{s}_i). \tag{39}$$

The inequality follows from X_i being weakly increasing in s_i. Therefore, the gain from deviation is non-positive. A similar argument can be used to show that a deviation to $\hat{s}_i > s_i$ is not profitable either. We want to point out that (39) is the necessary and sufficient condition that guarantees that bidder i truthfully reports her shock, given truthful reports in the first stage. Also notice that if the allocation X_i is continuously differentiable in its second argument than X_i being weakly increasing in s_i is equivalent to (39). (This is because if X_i was not increasing in s_i, there would exists an interval $[\hat{s}_i, s_i]$, where X_i is increasing and hence the opposite inequality would hold.)   ∥

Proof of Lemma 5.   Buyer i's expected profit when his type is v_i but reports $\hat{v}_i$ in the first round is

$$\pi_i(v_i, \hat{v}_i) = \int [u_i(v_i, s_i) X_i(\hat{v}_i, \sigma_i(v_i, \hat{v}_i, s_i)) - T_i(\hat{v}_i, \sigma_i(v_i, \hat{v}_i, s_i))] dG_i(s_i).$$

Using (15) and Lemma 4 we have

$$\pi_i(v_i, \hat{v}_i) = \int \tilde{\pi}_i(\sigma_i(v_i, \hat{v}_i, s_i), \sigma_i(v_i, \hat{v}_i, s_i); \hat{v}_i) dG_i(s_i). \tag{40}$$

Suppose $v_i > \hat{v}_i$. Note that by (15) and the monotonicity of u_i, we have $\sigma_i(v_i, \hat{v}_i, s_i) > s_i > \sigma_i(\hat{v}_i, v_i, s_i)$. By (14), we can rewrite (40) as

$$\pi_i(v_i, \hat{v}_i) = \int \left[\tilde{\pi}_i(s_i, s_i; \hat{v}_i) + \int_{s_i}^{\sigma_i(v_i, \hat{v}_i, s_i)} u_{i2}(\hat{v}_i, z) X_i(\hat{v}_i, z) dz \right] dG_i(s_i).$$

This becomes, by $\pi_i(\hat{v}_i, \hat{v}_i) = \int \tilde{\pi}_i(s_i, s_i, \hat{v}_i) dG_i(s_i)$,

$$\pi_i(v_i, \hat{v}_i) = \pi_i(\hat{v}_i, \hat{v}_i) + \int \int_{s_i}^{\sigma_i(v_i, \hat{v}_i, s_i)} u_{i2}(\hat{v}_i, z) X_i(\hat{v}_i, z) dz \, dG_i(s_i). \tag{41}$$

Note that σ_i defined by (15) is continuous and monotonic. Hence the image of $\sigma_i(y, \hat{v}_i, s_i)$ on $y \in [\hat{v}_i, v_i]$ is $[s_i, \sigma_i(v_i, \hat{v}_i, s_i)]$, and thus we can change the variable of the inside integral in (41) from $z \in [s_i, \sigma_i(v_i, \hat{v}_i, s_i)]$ to $y \in [\hat{v}_i, v_i]$ to get

$$\pi_i(v_i, \hat{v}_i) = \pi_i(\hat{v}_i, \hat{v}_i) + \int \int_{\hat{v}_i}^{v_i} u_{i2}(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) \sigma_{i1}(y, \hat{v}_i, s_i) dy \, dG_i(s_i).$$

By differentiating (15) in v_i (using the implicit function theorem),

$$u_{i1}(v_i, s_i) = u_{i2}(\hat{v}_i, \sigma_i(v_i, \hat{v}_i, s_i)) \sigma_{i1}(v_i, \hat{v}_i, s_i). \tag{42}$$

The last two equations imply (16) for the case of $v_i > \hat{v}_i$. An identical argument yields the same for $v_i < \hat{v}_i$. ‖

Proof of Lemma 6.   Using (16), incentive compatibility in the first round (the inequality system (17)) is equivalent to, for all $v_i > \hat{v}_i$,

$$\int \frac{\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy}{v_i - \hat{v}_i} dG_i(s_i) \leq \frac{\pi_i(v_i, v_i) - \pi_i(\hat{v}_i, \hat{v}_i)}{v_i - \hat{v}_i}$$

$$\leq \int \frac{\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(v_i, \sigma_i(y, v_i, s_i)) dy}{v_i - \hat{v}_i} dG_i(s_i).$$

By $X_i \leq 1$ and the concavity of u_i in v_i,

$$\frac{\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy}{v_i - \hat{v}_i} \leq \frac{\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) dy}{v_i - \hat{v}_i} \leq u_{i1}(\hat{v}_i, s_i),$$

and by assumption $u_{i1}(\hat{v}_i, s_i)$ has a finite expectation with respect to s_i. Therefore, by the Lebesgue convergence theorem

$$\lim_{v_i \to \hat{v}_i} \int \frac{\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy}{v_i - \hat{v}_i} dG_i(s_i)$$

$$= \int \lim_{v_i \to \hat{v}_i} \frac{\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy}{v_i - \hat{v}_i} dG_i(s_i) = \int u_{i1}(\hat{v}_i, s_i) X_i(\hat{v}_i, s_i) dG_i(s_i).$$

By analogous reasoning

$$\lim_{\hat{v}_i \to v_i} \int \frac{\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(v_i, \sigma_i(y, v_i, s_i)) dy}{v_i - \hat{v}_i} dG_i(s_i) = \int u_{i1}(v_i, s_i) X_i(v_i, s_i) dG_i(s_i).$$

Therefore, if X_i is continuous in v_i then $\Pi_i(v_i) \equiv \pi_i(v_i, v_i)$ is differentiable everywhere, and

$$\Pi'_i(v_i) = \int u_{i1}(v_i, s_i) X_i(v_i, s_i) dG_i(s_i).$$

Since this derivative is finite for all v_i, Π_i is Lipschitz continuous and hence it can be recovered from its derivative, and we obtain (18).   ‖

Proof of Proposition 2.   If, for all $j = 1, \ldots, n$ and $v_j \in [\underline{v}, \overline{v}]$, type v_j of buyer j purchases a premium $p_j(v_j) = (1 - F_j(v_j))/f_j(v_j) u_{i1}$ then buyer i will win in the second round if and only if, for all j,

$$u_i(v_i, s_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} u_{i1} \geq \max \left\{ u_j(v_j, s_j) - \frac{1 - F_j(v_j)}{f_j(v_j)} u_{i1}, 0 \right\}.$$

This is so because in the second round, every buyer j bids $u_j(v_j, s_j) - p_j(v_j)$ by the assumption that the buyers follow weakly dominant strategies in the second round. Hence the allocation rule is indeed the same as in the optimal mechanism, provided that all buyers behave "truthfully", that is, every buyer j with type v_j chooses $p_j(v_j)$ for a fee $c_j(v_j)$ defined in (24).

We show that the handicap auction defined by (23) and (24) is incentive compatible. Let $\pi_i^h(v_i, \hat{v}_i)$ denote the pay-off of buyer i with type v_i if he "deviates" to $\hat{v}_i$ (chooses $c_i(\hat{v}_i)$ and $p_i(\hat{v}_i)$). First, we show that for buyer i there is no incentive to deviate downwards. Suppose $\hat{v}_i < v_i$. Then the deviator bids $u_i(v_i, s_i) - p_i(\hat{v}_i)$ in the second round and his pay-off is

$$\pi_i^h(v_i, \hat{v}_i) = -c_i(\hat{v}_i) + \iint \max\{u_i(v_i, s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\} \, dF_{-i} dG.$$

Using the definitions of c_i and W_i, this can be rewritten as

$$\iint \int_{\underline{v}}^{\hat{v}_i} u_{i1} \mathbf{1}_{\{W_i(y, s_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} \, dy - \max\{u_i(\hat{v}_i, s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\} \, dF_{-i} dG$$

$$- \iint \max\{u_i(v_i, s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\} \, dF_{-i} dG.$$

Notice, if $v_i = \hat{v}_i$ then the last two terms cancel each other; therefore the integral in the first line equals $\pi_i^h(\hat{v}_i, \hat{v}_i)$. This implies that

$$\pi_i^h(v_i, v_i) = \pi_i^h(\hat{v}_i, \hat{v}_i) + \iint \left[\int_{\hat{v}_i}^{v_i} u_{i1} \mathbf{1}_{\{W_i(y, s_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} \, dy \right] dF_{-i} dG \tag{43}$$

and

$$\pi_i^h(v_i, \hat{v}_i) = \pi_i^h(\hat{v}_i, \hat{v}_i) - \iint \max\{u_i(\hat{v}_i, s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\} \, dF_{-i} dG$$

$$+ \iint \max\{u_i(v_i, s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\} dF_{-i} dG. \tag{44}$$

Since $\partial u_i(v_i, s_i) / \partial v_i = u_{i1}$ is a constant, $u_i(v_i, s_i)$ can be written as $u_{i1} v_i + r_i(s_i)$. We claim that the difference between the integrands in the second and third terms can be rewritten as

$$\max\{u_{i1} v_i + r_i(s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\} - \max\{u_{i1} \hat{v}_i + r_i(s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}), 0\}$$

$$= \int_{\hat{v}_i}^{v_i} u_{i1} \mathbf{1}_{\{u_{i1} y + r_i(s_i) - p_i(\hat{v}_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} \, dy. \tag{45}$$

If $u_{i1} \hat{v}_i + r_i(s_i) - p_i(\hat{v}_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})$ then the difference is

$$u_{i1}(v_i - \hat{v}_i) = \int_{\hat{v}_i}^{v_i} u_{i1} dy = \int_{\hat{v}_i}^{v_i} u_{i1} \mathbf{1}_{\{u_{i1} y + r_i(s_i) - p_i(\hat{v}_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} dy.$$

Suppose $u_{i1} v_i + r_i(s_i) - p_i(\hat{v}_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}) \geq u_{i1} \hat{v}_i + r_i(s_i) - p_i(\hat{v}_i)$. Let $y^* \in \{\hat{v}_i, v_i\}$ be such that $W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}) = u_{i1} v_i + r_i(s_i) - p_i(\hat{v}_i)$. Then (45) is

$$u_{i1} v_i + r_i(s_i) - p_i(\hat{v}_i) - W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}) = u_{i1}(v_i - y^*) = \int_{y^*}^{v_i} u_{i1} dy$$

$$= \int_{\hat{v}_i}^{v_i} u_{i1} \mathbf{1}_{\{u_{i1} y + r_i(s_i) - p_i(\hat{v}_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} dy.$$

If $W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i}) \geq u_{i1}y + r_i(s_i) - p_i(\hat{v}_i)$ then the value of both sides of (45) is 0. Therefore, (45) indeed holds. From (44) and (45) it follows that

$$\pi_i^h(v_i, \hat{v}_i) = \pi_i^h(\hat{v}_i, \hat{v}_i) + \iint \left[\int_{\hat{v}_i}^{v_i} u_{i1} \mathbf{1}_{\{u_{i1}y + r_i(s_i) - p_i(\hat{v}_i) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} dy \right] dF_{-i} dG.$$

Notice that since p_i is decreasing, $u_{i1}y + r_i(s_i) - p_i(\hat{v}_i) \leq u_{i1}y + r_i(s_i) - p_i(y)$ whenever $y \in \{\hat{v}_i, v_i\}$. Hence

$$\pi_i^h(v_i, \hat{v}_i) \leq \pi_i^h(\hat{v}_i, \hat{v}_i) + \iint \left[\int_{\hat{v}_i}^{v_i} u_{i1} \mathbf{1}_{\{u_{i1}y + r_i(s_i) - p_i(y) \geq W_{-i}^0(\boldsymbol{v}_{-i}, \boldsymbol{s}_{-i})\}} dy \right] dF_{-i} dG = \pi_i^h(v_i, v_i),$$

where the equality follows from (43). This means that for buyer i there is no incentive to deviate downwards. An almost identical argument shows that there is no incentive to deviate upwards either. ‖

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