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Equilibrium transformations and the Revenue Equivalence Theorem

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Abstract

I develop a method that transforms an equilibrium strategy profile from one auction to another. The method is constructive and does not require complicated computation. This provides a new approach to revenue equivalence and extends the theorem to domains where it had not previously been known, in particular to simultaneous multiple object auctions with complete information and to auction environments having correlated private values and common values.

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1. Introduction

This paper explores an intuitive approach to the Revenue Equivalence Theorem [24], which is then generalized to establish revenue equivalence results in environments where these results cannot be easily established using standard techniques. This methodology will be used to analyze general models where the independent signals and private values assumptions are relaxed, and in simultaneous auction games with complete information where the bidders use mixed strategies.

The approach can be best illustrated using Vickrey's classical model, in which a seller is offering a single unit of a good for sale to n risk-neutral buyers. Each buyer knows his own valuation, and it is commonly known that all valuations are

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independently and identically distributed. The seller considers two mechanisms: a sealed-bid first-price auction and a sealed-bid second-price auction. Vickrey [24] showed that in the second-price auction, bidding one's value is a dominant strategy, hence "truth-telling" generates a symmetric equilibrium profile. In order to find a symmetric equilibrium profile in the first-price auction, one may proceed as follows. For each buyer and for each realization of his valuation, suppose that the buyer bids his expected payment (to the seller) conditional on winning in the second-price auction. To see that this is indeed an equilibrium in the first-price auction, notice that there is a natural one-to-one correspondence between *deviations* in the two games. If a player with a certain valuation bids as if he had a different valuation in one of the auctions then his expected payoff is exactly the same as it would be in the other auction. This is so because the probabilities of winning are the same (as the strategies are symmetric and increasing), and the expected payments are the same (by the definition of the proposed strategy in the first-price auction). The seller's expected revenues from the first- and second-price auctions are the same as well, because in both auctions, the buyer with the highest valuation wins and the winner's expected payment conditional on winning is the same. The present paper explores and generalizes this relationship between the correspondence of equilibrium strategies and the revenue equivalence of different auction formats.

Interestingly, while this idea can be traced back to Vickrey's seminal paper, subsequent studies of revenue-equivalent auctions (and those of optimal auctions, that is, auctions maximizing the seller's expected revenue) were based on insights that appear to be quite different. These papers, including [1–3,6,10,13,16,17], surveyed in [5], rely on the Revelation Principle. Using the principle, one can assume that the buyers simply report their true valuations to an auctioneer who then determines the winner and the payments of each buyer such that the relevant incentive and participation constraints are satisfied. It is shown that the allocation rule pins down (up to integration constants) the expected transfers that make the allocation rule incentive compatible. This result, which relies heavily on the assumptions of independent signals and risk-neutral buyers, can be used to extend Vickrey's work to other mechanisms that give the same revenues to the seller. For example, if the seller's reserve price is the same in two different auctions, then any monotonic equilibria in the respective auctions generate the same allocation, hence the seller's expected revenue will be the same in the two auctions as well. This approach, based on the Revelation Principle, has become standard in establishing revenue equivalence results. However, it hides the direct relationship between the equilibrium profiles of the first- and second-price auctions in Vickrey's model.

What is known about the seller's revenue when some of the assumptions of the classical model do not hold? Milgrom and Weber [12] develop a general auction model in which they drop two of the assumptions of Vickrey's model. First, the signal of one buyer may affect the expected valuation of the other buyers. Second, the signals do not have to be distributed independently. Instead, they may be affiliated. Milgrom and Weber show that (under regularity conditions) the classical auction formats can be ranked based on the seller's expected revenue. They show that the second-price auction produces no less revenue than the first-price auction.

One of the contributions of this paper is to show under which circumstances this ranking is weak; that is, when the first- and second-price auctions generate the same expected revenue to the seller in this model. (The method developed in this paper is also used to give a simple and intitive proof for Milgrom and Weber's revenue ranking result.)

In this paper, versions of the Revenue Equivalence Theorem are obtained for two different auction models. First, the independence and the private-value assumptions of Vickrey's classical model are relaxed (but symmetry and risk-neutrality are retained), in a general model similar to the one in [12]. Second, simultaneous multiple auction games with complete information are analyzed, where bidders use mixed strategies.

In both models new situations are identified where different sealed-bid auctions generate the same revenue for the seller. I approach revenue equivalence through a natural correspondence between strategy profiles in different auctions. The idea is simple: Assume that the buyers can condition their bids on certain information sets. If there are two mechanisms and a symmetric equilibrium profile is known for one of them, then one can construct a symmetric (not necessarily equilibrium) profile for the second mechanism such that for every player the expected "cost" of bidding (i.e., payment to the seller), conditional on the buyer's information, remains exactly the same as in the original equilibrium.[1] Call the corresponding strategy profile the *cost-equivalent* profile. I first show that the cost-equivalent profile in the second mechanism generates the same revenue for the seller as the original profile in the first mechanism; therefore, if the second profile is also an equilibrium profile then revenue equivalence holds. I then identify environments where the new strategy profile is indeed an equilibrium profile in the second mechanism. These include many first- and second-price auction environments where revenue equivalence has not been previously known to hold.

In addition to its enabling generalizations of the Revenue Equivalence Theorem, the major advantage of this approach is that it is constructive. If a symmetric equilibrium profile is given in one mechanism, then usually without complicated computations the method delivers a symmetric equilibrium profile in the other mechanism. In many situations, one of the two equilibria is indeed available.

The rest of the paper is organized as follows. In Section 2, I describe the underlying single-unit auction model and define cost equivalence in this model. In Section 3, a necessary and sufficient condition is given for a strategy profile that is cost-equivalent to an equilibrium profile in either a first- or a second-price auction to be an equilibrium profile in the other auction, which implies revenue equivalence. In Section 4 the applicability of the theorems from Section 3 is illustrated by showing how easy it is to transform an equilibrium from one auction to an equilibrium in another in several examples. In Section 5, a general model of a simultaneous auction with complete information is introduced. I redefine cost equivalence for these games

[1] In simultaneous auction models I focus on mixed strategy equilibria. In this case, the outcomes of the bidders' randomization devices—what they use for generating their mixed strategies—are interpreted as the information on which the bidders condition their bids.

and show how to transform an equilibrium from one auction to an equilibrium in another, which again generates revenue equivalence. In Section 6, these results are applied to specific simultaneous auction games. Section 7 concludes.

2. The single-unit model

There are n buyers (indexed by $1, \ldots, n$) and a single object to be sold in a sealed-bid auction game. The game is symmetric (ex ante) with respect to the players. The buyers are risk neutral. Each buyer's utility is the difference between the value of the object if it is won and any monetary payment. Formally, a buyer's ex post utility is: $v - m$, where v is the value of the object if she wins (0 if she does not win) and m is her payment to the auctioneer.

Each buyer observes a real-valued signal about the value of the object and can condition her bid on this signal. The signals are affiliated, and the expected value of the object is a weakly increasing non-negative function of each signal and is strictly increasing in the buyer's own signal; in particular, the signals of the other buyers can also affect the valuation of a buyer. The signals are distributed on the compact interval $[0,1]$, for simplicity, and the cumulative distribution function F_s of the highest signal among buyers $2, \ldots, n$ conditional on Buyer 1 having observed signal s has a density f_s. Assume, again for simplicity, that f_s is continuous and strictly positive on $[0,s]$ and that both F_s and f_s are differentiable with respect to s.[2]

A pure strategy in such a game is a function from $[0,1]$ to the set of real numbers $\mathbb{R}$. Throughout Sections 2–4 restrict attention to symmetric pure equilibria.

A buyer's monetary payment is called her *cost* and the payoff plus cost her *benefit*.

For the sake of convenience two additional pieces of notation are introduced. First, if Buyer 1 observes signal s, then $v(s,x)$ denotes the expected value of the object to 1 given that the highest signal among Buyers $2, \ldots, n$ was x. Assume $v(s,x)$ is continuous in x. Second, for a fixed symmetric strategy profile $\mathbf{b} = (b, \ldots, b)$, if buyer 1 observes signal s and bids $b(z)$—i.e. the bid that b assigns to the signal z—his expected benefit is only a function of s and z, and can therefore be written as $R(s,z)$. Observe that if b is strictly increasing in the signal and the highest bid wins, then

$$R(s,z) = \int_0^z v(s,x)\, dF_s(x).$$

Since $v(s,x)$ is continuous in x, $R(s,z)$ is differentiable in z (see for example [19, Theorem 10, p. 107]).

Suppose that there are two different high-bid-wins auctions and suppose that strategy profiles for each have the property that for each signal vector the profiles produce identical bid-orderings in the two games. Then the two mechanisms generate the same expected benefit as functions of the signals, and they differ only in the costs. The main idea of this paper takes advantage of this observation. Once an

[2] Notice that F_s and the uniform distribution on $[0,s]$ are absolutely continuous with respect to each other, hence the meaning of "almost everywhere" is the same according to both measures.

equilibrium for a mechanism with a certain cost structure is known, one can compute a strategy profile for another mechanism having a different cost structure such that if all buyers bid according to this strategy profile, each buyer's expected cost for each observed signal is the same as in the original game if all buyers bid there according to the equilibrium strategy profile. The problem then will be to find conditions such that the second profile is an equilibrium in the second mechanism. This is explained more formally below.

Assume that each buyer i bids b^i. Then any sealed-bid auction generates a cost function for Buyer 1, $c(b^1, \{b^i\}_{i=2}^n)$. (For example, if the mechanism is a first price auction,

$$c(b^1, \{b^i\}_{i=2}^n) = \begin{cases} b^1 & \text{if } b^1 \geqslant \max\limits_{i \in \{2,\dots,n\}} \{b^i\}, \\ 0 & \text{otherwise,} \end{cases}$$

assuming for simplicity that in case buyer 1 ties, she gets the object.) For a fixed symmetric strategy profile $\mathbf{b} = (b, \dots, b)$, the conditional expected cost of buyer 1 is

$$E_{s_2,\dots,s_n}(c(b(s_1), \{b(s_i)\}_{i=2}^n)|s_1)$$

that is, the expected cost conditioned on having observed s_1.

Definition 1. Suppose there are two mechanisms (α and β) generating different cost functions, c_α and c_β, respectively. Let $\mathbf{b}_\alpha = (b_\alpha, \dots, b_\alpha)$ be a symmetric strategy profile in the first mechanism, and $\mathbf{b}_\beta = (b_\beta, \dots, b_\beta)$ be a symmetric strategy profile in the second mechanism. They are called *cost-equivalent strategy profiles* if for each signal s_1:

$$E_{s_2,\dots,s_n}(c_\alpha(b_\alpha(s_1), \{b_\alpha(s_i)\}_{i=2}^n) \mid s_1) = E_{s_2,\dots,s_n}(c_\beta(b_\beta(s_1), \{b_\beta(s_i)\}_{i=2}^n) \mid s_1).$$

That is, if buyer 1 observes any signal and all buyers bid according to the corresponding cost-equivalent profiles in both mechanisms, then buyer 1's expected cost conditional on the observed signal is the same in both mechanisms.

Remark 1. In general neither existence nor uniqueness of a symmetric profile that is cost-equivalent to another is guaranteed. However, in cases of interest I will construct the unique cost-equivalent symmetric profile.

The following lemma claims that in high-bid-wins auctions where only the winner has to pay if the corresponding strategy profiles are generated by strictly increasing strategies, then revenue equivalence follows from cost equivalence. The proof is omitted because of its simplicity.

Lemma 1. *Let α and β be two high-bid-wins auctions where only the winner has to pay, and $\mathbf{b}_\alpha = (b_\alpha, \dots, b_\alpha)$ and $\mathbf{b}_\beta = (b_\beta, \dots, b_\beta)$ be cost-equivalent strategy profiles where b_α and b_β are strictly increasing functions. Then the expected revenue for the seller is the same in both auctions.*

Notice that if two profiles generate the same expected selling price they do not have to be cost-equivalent, that is, cost equivalence does not follow from revenue equivalence.

I now turn to: (1) identifying situations where whenever $\mathbf{b}_\alpha$ is a symmetric equilibrium strategy profile, the corresponding cost-equivalent strategy profile $\mathbf{b}_\beta$ is also an equilibrium, thereby establishing revenue equivalence for the seller; and (2) showing that in all of these situations the cost-equivalent symmetric profile can be constructed in an easy way.

3. Cost equivalence between first- and second-price auctions

From now on I restrict my attention to symmetric strategy profiles that are strictly increasing functions of the signals. Since transforming such an equilibrium in a third- or higher-price auction to a first-price equilibrium (and vice versa) is very similar to transforming a second-price equilibrium to a first-price equilibrium (and vice versa), for simplicity I also restrict my attention to the relationship between first- and second-price auctions.

The next step is to find conditions which guarantee that the cost-equivalent profile corresponding to an equilibrium profile is also an equilibrium profile. Recall that F_s denotes the cumulative distribution function of the highest signal among $s_2, \dots, s_n$ conditional on buyer 1 having observed s, and f_s denotes the corresponding density function. The key necessary and sufficient condition turns out to be

$$\text{If } x \leqslant s \text{ then } \left. \frac{\partial \frac{f_z(x)}{F_z(s)}}{\partial z} \right|_{z=s} = 0. \tag{B}$$

Prior to explaining the meaning of this condition I prove the following

Lemma 2. (B) *is equivalent to the following condition*:

$$\textit{If } x \leqslant s \leqslant z \textit{ then } \frac{f_s(x)}{F_s(s)} = \frac{f_z(x)}{F_z(s)}. \tag{A}$$

Proof. See the appendix.

The meaning of (A), and hence (B), is that if a player observes a signal z bigger than s, then the likelihood that the highest signal among the other players is x conditioned on it being smaller than s is the same as if the buyer observing z had observed s instead. Of course if the signals are distributed independently across buyers, this assumption is automatically satisfied. An example for joint distribution which satisfies (A) but not independence is the following. Suppose that the signals are uniformly and independently distributed on $[0, T]$, where T is a random variable. In this example $f_z(x)/F_z(s) = 1/s$, whenever $x \leqslant s \leqslant z$. (Notice that these signals are strictly affiliated.)

Since (A) and (B) are equivalent, from now on these conditions will be referred to as (A).

A natural question to ask is how easy it is to identify those joint cumulative distribution functions of the signals which satisfy (A). We can redefine the joint distribution in the following way: first define the distribution of the highest order statistic; then, conditional on the highest order statistic, define the distribution of the second highest order statistic, and so on. Once an ordered vector of signals is determined count for all its equally likely permutations. Formally, the joint distribution of n symmetrically distributed random variables can be defined by the cumulative distribution of the highest order statistic and the collection

$$\bigcup_{i=2}^{n} \{H_i^{x_1,\dots,x_{i-1}} \mid x_i \in [0,1], x_1 \geqslant \cdots \geqslant x_{i-1} \text{ and } H_i^{x_1,\dots,x_{i-1}} \text{ is a cdf on } [0, x_{i-1}]\},$$

where $H_i^{x_1,\dots,x_{i-1}}$ denotes the conditional cumulative distribution function of the ith highest signal given that x_j was the jth highest signal $(j = 1, \dots, i-1)$. I claim the following:

Proposition 1. *A symmetric joint distribution satisfies* (A) *if and only if* $H_2^z(x) = H(x)/H(z)$ *for some distribution function* H.

Proof. See the appendix.

Using Proposition 1, an interpretation of (A) is that the highest signal only determines the support of the second highest signal; the actual distribution is a restriction of a fixed measure (independent of the highest signal) to the support.

The main goal of this section is to show that (A) is both necessary and sufficient to guarantee that for first- and second-price auctions an increasing symmetric equilibrium in either of these can be converted into a revenue equivalent equilibrium of the other. To this end, let b_α be a strictly increasing strategy in the first-price auction and b_β in the second-price auction. If they generate symmetric cost-equivalent profiles they must satisfy the following equation for every s:

$$F_s(s)b_\alpha(s) = \int_0^s b_\beta(x)\, dF_s(x). \tag{1}$$

(The left-hand side is clearly the conditional expected cost function in the first-price auction, while the right-hand side is the conditional expected cost function in the second-price auction.)

Before continuing with the formal development, it is important to explain how to compute the cost-equivalent profile corresponding to a given symmetric strictly increasing profile and argue that if it exists it must be almost unique in the class of symmetric strictly increasing profiles. First let $\mathbf{b}_\beta$ be a strictly increasing symmetric profile in the second-price auction. From (1) if a strictly increasing strategy b_α generates a cost-equivalent profile in the first-price auction it must be defined uniquely by the following equation:

$$b_\alpha(s) = \frac{\int_0^s b_\beta(x)\, dF_s(x)}{F_s(s)}. \tag{2}$$

Observe that $b_\alpha(s)$ is the expected value of $b_\beta(x)$ conditional on x being smaller than s and s being observed. (In the proof of Theorem 2 it will be shown that this expected value is indeed strictly increasing in s.) Therefore if there exists a strictly increasing symmetric cost-equivalent profile corresponding to $\mathbf{b}_\beta$, it must be unique. Suppose now that $\mathbf{b}_\alpha$ is a strictly increasing symmetric profile in the first-price auction. First, rewrite (1) as

$$b_\alpha(s) = \int_0^s b_\beta(x) \frac{f_s(x)}{F_s(s)}\, dx.$$

Second, since (A) holds $f_s(x)/F_s(s) = f_1(x)/F_1(s)$; therefore (1) can be rewritten as

$$F_1(s) b_\alpha(s) = \int_0^s b_\beta(x)\, dF_1(x).$$

From the Fundamental Theorem of Calculus one can conclude that

$$b_\beta(x) = \frac{\partial(F_1(s) b_\alpha(s))}{\partial F_1(s)} \tag{3}$$

must hold almost everywhere. Since $b_\alpha(s)$ is strictly increasing and f_1 is positive, $\partial(F_1 b_\alpha)/\partial F_1$ indeed exists almost everywhere (see [19, Corollary 6, p. 104]). Therefore if there exists a strictly increasing cost-equivalent profile in the second-price auction corresponding to $\mathbf{b}_\alpha$, it is uniquely defined almost everywhere by (3).

3.1. The necessity of condition (A)

In this subsection it will be shown that (A) is indeed necessary to guarantee that a cost-equivalent strategy profile corresponding to an equilibrium profile is also an equilibrium profile. Three lemmas are needed to prepare for the proof of this result.

Lemma 3. *If $s' \geqslant s$ and $x' \geqslant x$ then*

(i) $$\frac{f_{s'}(x)}{F_{s'}(s)} \geqslant \frac{f_s(x)}{F_s(s)} \quad \textit{implies} \quad \frac{f_{s'}(x')}{F_{s'}(s)} \geqslant \frac{f_s(x')}{F_s(s)}$$

and

(ii) $$\frac{f_{s'}(x')}{F_{s'}(s)} \leqslant \frac{f_s(x')}{F_s(s)} \quad \textit{implies} \quad \frac{f_{s'}(x)}{F_{s'}(s)} \leqslant \frac{f_s(x)}{F_s(s)}.$$

Proof. See the appendix.

Lemma 4. *For all* $s \in [0,1]$ *there exist* $q(s) \in [0,s]$ *such that*

$$\left.\frac{\partial(f_z(x)/F_z(s))}{\partial z}\right]_{z=s} \leqslant 0 \quad \textit{if} \;\; x \in [0, q(s)]$$

and

$$\left.\frac{\partial(f_z(x)/F_z(s))}{\partial z}\right]_{z=s} \geqslant 0 \quad \textit{if} \;\; x \in [q(s), s].$$

Proof. See the appendix.

Lemma 5. *For all* $s \in [0,1]$

$$\int_0^s \left.\frac{\partial(f_z(x)/F_z(s))}{\partial z}\right]_{z=s} dx = 0.$$

Proof. See the appendix.

Finally I am ready the prove

Theorem 1. *If* $\mathbf{b}_\alpha = (b_\alpha, \dots, b_\alpha)$ *is an equilibrium profile in the first-price auction,* $\mathbf{b}_\beta = (b_\beta, \dots, b_\beta)$ *is an equilibrium profile in the second-price auction, and they are cost-equivalent, then* (A) *must hold.*

Proof. The payoff of a buyer in the second-price auction observing s and bidding $b_\beta(z)$ is

$$R(s,z) - \int_0^z b_\beta(x)\, dF_s(x). \tag{4}$$

In the first-price auction, bidding $b_\alpha(z)$ and using Eq. (2) it is

$$R(s,z) - F_s(z)\,\frac{\int_0^z b_\beta(x)\, dF_z(x)}{F_z(z)}. \tag{5}$$

Since $\mathbf{b}_\alpha$ and $\mathbf{b}_\beta$ are equilibrium profiles, (4) and (5) are both maximized at $z = s$. Since $F_s(x)$ is continuously differentiable with respect to x and $R(s,z)$ is differentiable with respect to z, both first-order conditions must be satisfied at $z = s$. Recall that R is identical in the two auctions. Therefore the derivatives of the respective second terms must be the same. The derivative of the second term of (4), after multiplying by $F_s(z)/F_s(z)$ and using the product rule for derivatives, is

$$f_s(z)\,\frac{\int_0^z b_\beta(x) f_s(x)\, dx}{F_s(z)} \\ + F_s(z)\left[b_\beta(z)\,\frac{f_s(z)}{F_s(z)} + \int_0^z b_\beta(x)\,\frac{\partial(f_s(x)/F_s(z))}{\partial z}\, dx\right]$$

and the derivative of the second term of (5) is

$$f_s(z)\frac{\int_0^z b_\beta(x)f_z(x)\,dx}{F_z(z)}$$
$$+\ F_s(z)\left[b_\beta(z)\frac{f_z(z)}{F_z(z)} + \int_0^z b_\beta(x)\frac{\partial(f_z(x)/F_z(z))}{\partial z}\,dx\right].$$

Observe that at $z = s$, the two expressions differ only in the respective second lines. Since

$$\frac{\partial(f_z(x)/F_z(z))}{\partial z}\bigg\rfloor_{z=s} = \frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s} + \frac{\partial(f_s(x)/F_s(z))}{\partial z}\bigg\rfloor_{z=s},$$

the only way to equalize them is to guarantee that:

$$\int_0^s b_\beta(x)\frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s}dx = 0.$$

However, using the notation of Lemma 4,

$$\begin{aligned}
&\int_0^s b_\beta(x)\frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s}dx \\
&= \int_0^{q(s)} b_\beta(x)\frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s}dx + \int_{q(s)}^s b_\beta(x)\frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s}dx \\
&\geqslant b_\beta(q(s))\int_0^{q(s)}\frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s}dx + b_\beta(q(s))\int_{q(s)}^s\frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s}dx \\
&= b_\beta(q(s))\int_0^s\frac{\partial(f_z(x)/F_z(s))}{\partial z}\bigg\rfloor_{z=s}dx = 0,
\end{aligned}$$

where the inequality follows from Lemma 4 and from b_β being strictly increasing and the last equality follows from Lemma 5. Observe that the inequality is strict whenever $\partial(f_z(x)/F_z(s))/\partial z\rfloor_{z=s}\neq 0$ for some $x(\leqslant s)$; therefore we can conclude that (A) is indeed necessary to guarantee that the cost-equivalent profile is an equilibrium profile. $\square$

The next subsection is devoted to the sufficiency of (A).

3.2. The sufficiency of condition (A)

Firstly, it will be shown that (A) is a sufficient condition to guarantee that the cost-equivalent strategy profile in the first-price auction corresponding to an equilibrium profile in the second-price auction is an equilibrium profile.

Theorem 2. *If* (A) *holds and* $\mathbf{b}_\beta = (b_\beta, \ldots, b_\beta)$ *is an equilibrium of the second-price auction, then the strategy profile* $\mathbf{b}_\alpha = (b_\alpha, \ldots, b_\alpha)$ *defined in* (2) *is an equilibrium profile in the first-price auction.*

Proof. Observe that since the signals are affiliated, the strategy b_α is indeed a strictly increasing function of the signal.[3] Therefore if all buyers adopt this strategy, once a certain signal is observed by a buyer the expected benefit of that buyer is the same as in the second-price auction.

It is clearly enough to consider deviations of the following form: after observing signal s a buyer bids $b_\alpha(z)$ instead of $b_\alpha(s)$. In that case her payoff would be (using (2)):

$$R(s,z) - F_s(z)b_\alpha(z) = R(s,z) - F_s(z)\frac{\int_0^z b_\beta(x)\,dF_z(x)}{F_z(z)}.$$

If $z \leqq s$ then by (A):

$$\frac{\int_0^z b_\beta(x)\,dF_z(x)}{F_z(z)} = \frac{\int_0^z b_\beta(x)\,dF_s(x)}{F_s(z)},$$

and if $z \geqq s$ then because the signals are affiliated[4]

$$\frac{\int_0^z b_\beta(x)\,dF_z(x)}{F_z(z)} \geqq \frac{\int_0^z b_\beta(x)\,dF_s(x)}{F_s(z)}.$$

Therefore, the payoff from this deviation can be weakly increased by replacing the expected cost with the smaller:

$$\int_0^z b_\beta(x)\,dF_s(x).$$

Hence the payoff from this deviation is weakly smaller than

$$R(s,z) - \int_0^z b_\beta(x)\,dF_s(x).$$

But this would have been the payoff to the buyer in the second-price auction if after observing signal s she had bid $b_\beta(z)$. Since b_β generated a symmetric equilibrium profile in the second-price auction this deviation cannot be profitable; therefore it is weakly smaller than

$$R(s,s) - \int_0^s b_\beta(x)\,dF_s(x).$$

But using (2) again this is just

$$R(s,s) - F_s(s)b_\alpha(s).$$

This is the payoff of the buyer in the first-price auction if he does not deviate. □

[3] $b_\alpha(s)$ is the expected value of the second-highest bid in the second-price auction, given that s was observed and it was the highest signal. Since b_β is a strictly increasing function of s and f_s is strictly positive on $[0,s]$, affiliation guarantees that the expectation above is also strictly increasing. For a formal proof see [12, Theorems 2 and 5].

[4] This inequality is again a straightforward consequence of [12, Theorems 2 and 5].

In order to show that (A) is also sufficient to guarantee that the cost-equivalent strategy profile in the second-price auction corresponding to an equilibrium profile in the first-price auction is an equilibrium profile the following is needed:

Lemma 6. *Let h and g be strictly increasing continuous functions on $[a,b]$. Suppose $h' = g$ whenever h' exists. Then h' must exist everywhere on $[a,b]$.*

Proof. See the appendix.

Now the following can be proven:

Theorem 3. *If* (A) *holds and $\mathbf{b}_\alpha = (b_\alpha, \dots, b_\alpha)$ is an equilibrium profile in the first-price auction, then the corresponding cost-equivalent profile $\mathbf{b}_\beta$ generated by b_β defined by* (3) *is an equilibrium profile in the second-price auction. Furthermore $b_\beta(s) = v(s,s)$.*

Proof. Since $\mathbf{b}_\alpha$ is a strictly increasing equilibrium profile in the first-price auction, b_α must be continuous. (If b_α is not continuous at s then there exists an $\varepsilon(>0)$ such that if a buyer observes a signal in $(s, s+\varepsilon)$ it is a profitable deviation to bid $\lim_{x\to s-} b_\alpha(x)$.) Observe that $b_\alpha(s)$ and $F_1(s)b_\alpha(s)$ are strictly increasing and therefore differentiable almost everywhere on $[0,1]$ with respect to the Lebesgue measure as well as to F (see [19, Corollary 6, p. 104] and footnote 2). Therefore $\partial(F_1 b_\alpha)/\partial F_1$ also exists almost everywhere.

Recall that $v(s,x)$ denotes the expected value of the object for a buyer if she has observed signal s and the highest signal among the other buyers is x. Since b_α generates a symmetric equilibrium profile, the following function of z must be maximized at $z = s$:

$$\int_0^z v(s,x)\, dF_s(x) - F_s(z) b_\alpha(z).$$

(This function is the expected payoff of the buyer who has observed s and bids $b_\alpha(z)$ given that the other buyers use the strategy b_α.) Suppose that $b'_\alpha(s)$ exists. Therefore the first-order condition for a maximum must be satisfied at $z = s$, that is

$$v(s,s) f_s(s) = f_s(s) b_\alpha(s) + F_s(s) b'_\alpha(s).$$

Equivalently

$$v(s,s) = b_\alpha(s) + \frac{F_s(s)}{f_s(s)} b'_\alpha(s).$$

Since the signals are affiliated and the value of the object is weakly increasing in each signal and strictly increasing in the buyer's own signal, $v(s,s)$ must be a strictly increasing function.[5] But

$$\frac{\partial(F_1(s) b_\alpha(s))}{\partial F_1(s)} = \frac{\partial(F_1(s) b_\alpha(s))}{\partial s} \frac{\partial s}{\partial F_1(s)} = b_\alpha(s) + \frac{F_1(s)}{f_1(s)} b'_\alpha(s).$$

[5] See footnote 3.

Condition (A) however guarantees that $F_1(s)/f_1(s) = F_s(s)/f_s(s)$, and therefore we can conclude that

$$b_\beta(s) = \frac{\partial(F_1(s)b_\alpha(s))}{\partial F_1(s)} = v(s,s).$$

From Lemma 6 it follows that $\partial(F_1(s)b_\alpha(s))/\partial F_1(s)$ exists everywhere on $[0,1]$ and is always equal to $v(s,s)$. But $b_\beta(s) = v(s,s)$ indeed generates a symmetric equilibrium profile in the second-price auction (even if (A) does not hold); see [12, Theorem 6]. $\square$

Remark 2. The expected cost function generated by the first-price sealed-bid auction is automatically convex as a function of the probability of winning.

If a strictly increasing symmetric equilibrium is given in either the first- or the second-price auctions, then the corresponding cost-equivalent profile is unique in the class of strictly increasing profiles. This follows directly from the discussion before Section 3.1 and from the proof of Theorem 3. Notice that the cost-equivalent profile in the second-price auction corresponding to any symmetric increasing equilibrium in the first-price auction is always generated by the strategy $b_\beta(s) = v(s,s)$. Therefore the following holds:

Remark 3. The equilibrium in the class of symmetric strictly increasing strategy profiles is unique in both the first- and second-price auctions.

3.3. Revenue equivalence and ranking

In this subsection a simple and intuitive proof of the revenue ranking of the first- and second-price auctions (due to Milgrom and Weber in [12]) is provided and the most general form of the Revenue Equivalence Theorem is proven.

The argument of the proof of the revenue ranking is as follows. Firstly, it will be shown that if there is a symmetric strategy profile in the first-price auction such that no player has incentive to deviate upward, then any symmetric equilibrium profile must be below this one. (Given a strategy profile, there is no incentive to deviate upward, if no buyer can increase his payoff by bidding higher no matter what signal he had observed, given the other players follow the strategy profile.) Then it will be shown that the cost-equivalent profile in the first-price auction corresponding to any symmetric equilibrium profile in the second-price auction has the property that there is no incentive to deviate upward.

Lemma 7. *Let $\mathbf{b}_\gamma = (b_\gamma, \ldots, b_\gamma)$ be a symmetric increasing strategy profile in the first-price auction and $b_\gamma(0) = v(0,0)$. Suppose that $dF_s(z)b_\gamma(z)/dz\rfloor_{z=s}$ exists for all $s \in [0,1]$ and there is no incentive to deviate upward at any point. Then any equilibrium* *profile* $\mathbf{b}_\alpha = (b_\alpha, \ldots, b_\alpha)$ *is below* $\mathbf{b}_\gamma$, *that is,*

$$b_\gamma(s) \geqslant b_\alpha(s) \quad \forall s \in [0,1].$$

Proof. See the appendix.

Using this lemma one can prove that the second-price auction yields a higher revenue to the seller than the first-price one. This result is due to [12].

Proposition 2. *Any symmetric strictly increasing equilibrium profile in the second-price auction yields a* (*weakly*) *higher revenue to the auctioneer than any symmetric strictly increasing equilibrium profile in the first-price auction.*

Proof. Let $\mathbf{b}_\beta$ be a symmetric strictly increasing equilibrium profile in the second-price auction. Let $\mathbf{b}_\gamma$ be the cost-equivalent strategy profile in the first-price auction. That is,

$$b_\gamma(s) = \int_0^s \frac{b_\beta(x)}{F_s(s)}\, dF_s(x).$$

From the proof of Theorem 1 it is known that $dF_s(z)b_\gamma(z)\, dz\rfloor_{z=s}$ exists. In order to guarantee that the previous lemma can be applied, it has to be proved that at $\mathbf{b}_\gamma$ there is no incentive to deviate upward. Suppose that $z > s$. The payoff of a buyer in the first-price auction who observed signal s and bids $b_\gamma(z)$ given that all the other buyers follow b_γ is

$$R(s,z) - F_s(z)b_\gamma(z) = R(s,z) - F_s(z)\frac{\int_0^z b_\beta(x)\, dF_z(x)}{F_z(z)}.$$

Notice that, since the signals are affiliated

$$\frac{\int_0^z b_\beta(x)\, dF_z(x)}{F_z(z)} \geqq \frac{\int_0^z b_\beta(x)\, dF_s(x)}{F_s(z)}.$$

Hence the payoff from this deviation is weakly smaller than

$$R(s,z) - \int_0^z b_\beta(x)\, dF_s(x).$$

But this would have been the payoff to the buyer in the second-price auction if after observing signal s she had bid $b_\beta(z)$. Since b_β generated a symmetric equilibrium profile in the second-price auction this deviation cannot be profitable; therefore it is weakly smaller than

$$R(s,s) - \int_0^s b_\beta(x)\, dF_s(x) = R(s,s) - F_s(s)b_\gamma(s).$$

Therefore we can conclude that in $\mathbf{b}_\gamma$ there is no incentive to deviate upwards. From $b_\beta(0) = v(0,0)$ it follows that $b_\gamma(0) = v(0,0)$. The previous lemma can be applied and we can conclude that any symmetric strictly increasing equilibrium profile in the first-price auction is pointwise (weakly) smaller than any cost-equivalent profile corresponding to any symmetric strictly increasing equilibrium profile in second-price auction. From this the statement of the proposition follows. □

Finally, in light of Theorems 1–3, a general form of the Revenue Equivalence Theorem can be stated.

Revenue Equivalence Theorem. (1) *If* (A) *holds*, *then there exist symmetric equilibrium profiles* (*unique in the class of symmetric strictly increasing profiles*) *in both the first- and second-price auctions that generate the same expected revenue to the seller*.

(2) *If* (A) *does not hold*, *then there do not exist revenue-equivalent symmetric equilibrium profiles generated by strictly increasing strategies*.

Proof. If (A) holds then uniqueness follows from Remark 3. The two equilibria must be cost-equivalent (see Theorem 2 or Theorem 3), hence Revenue Equivalence follows from Lemma 1.

From the previous proposition it follows that the expected cost of a buyer as a function of her signal is weakly smaller in the first-price auction than in the second-price auction. If (A) does not hold, then the corresponding profiles cannot be cost-equivalent because of Theorem 1. Therefore the expected cost in the first-price auction as a function of the observed signal is for certain signals strictly smaller than in the second-price auction. By continuity and since f_s is strictly positive on $[0, s]$, it follows that the second-price auction generates a strictly higher revenue to the seller than does the first-price auction. □

4. Applications for single-unit auctions

In this section the goal is to demonstrate the constructiveness of the cost-equivalence concept by showing how easy it is to compute the equilibrium in a first-price auction once the equilibrium in the second-price one (and vice versa) is known. In each of the following examples the cost-equivalent profile corresponding to an equilibrium profile is also an equilibrium; therefore the Revenue Equivalence Theorem automatically holds.

It seems generally easier to find the symmetric equilibrium profile in the second-price auction than in the first-price auction. In a pure private-value model this is because truth-telling is a dominant strategy in the second-price auction. In a common-value model there is also a kind of truth-telling strategy, which is to bid the expected value of the object assuming that the signal a buyer has observed is the highest and that the second highest signal is the same. Therefore in the following examples first I construct the symmetric equilibrium profile in the second price auction, and then I transform it into a first-price equilibrium.

First consider an impure common-value model, where the buyers' signals are independently distributed.

Example 1 (Impure common-value model). Suppose there are two buyers and the signals are independently and uniformly distributed on the interval $[0, 1]$. The valuation of buyer 1 is $s_1 + \theta s_2$ where s_i is the signal observed by buyer i; that is, $v(s, x) = s + \theta x$. In this case from Theorem 3 the symmetric equilibrium profile in the second-price auction is generated by the strategy $b_\beta(s) = v(s, s) = (1 + \theta)s$. Using Theorem 2 the symmetric equilibrium profile in the first-price auction must be generated by

$$(1+\theta)\frac{\int_0^s x\,dx}{s} = (1+\theta)\frac{x}{2}.$$

Surprisingly, the equilibrium strategy in the first-price auction is also a linear function of the signal, but unlike in the second-price auction this requires the uniform distribution.

In the following example all the assumptions of Vickrey's model are kept, with the exception of independence.

Example 2 (Correlated private-value model). Suppose there are n buyers and the signals are independently and uniformly distributed on the compact interval $[0, T]$, where T is a continuous random variable with a positive density on $[0, 1]$. The buyer's valuation for the object is the signal itself. The joint distribution of the signals clearly satisfies both affiliation and (A); therefore Theorem 2 can be applied. In the second-price auction, truth-telling is still a dominant strategy therefore it constitutes a symmetric equilibrium profile. If buyer 1 observes signal s, the others' signal distribution, conditional on s being the highest signal are independent and uniform on $[0, s]$. Therefore if $x \leqslant s$, $F_s(x) = x^{n-1}$. (Recall that F_s is the cumulative distribution function of the highest signal among buyers $2, \ldots, n$ given that buyer 1 has observed signal s.) The symmetric equilibrium in the first-price auction can be computed easily using Theorem 2:

$$b_\alpha(s) = \frac{\int_0^s x\,dx^{n-1}}{s^{n-1}} = \frac{n-1}{n}s.$$

Notice that this would have been the equilibrium strategy of the buyers in the first-price auction if the signals were distributed uniformly and independently on $[0, 1]$; that is, the distribution of T does not influence the equilibrium profile at all. This may seem surprising at first glance, but in the light of Theorem 2 this phenomenon becomes clear. I have shown that

$$b_\alpha(s) = \frac{\int_0^s x\,dF_1(x)}{F_1(x)};$$

therefore the cost-equivalent strategy in the second-price auction only depends on the distribution of the second-highest signal given that the highest signal is 1.

It is worth emphasizing that in the previous example as well as in the following two examples the signals are strictly affiliated. That is, the affiliation inequality is strict whenever the signals are not equal. Hence, one cannot use an argument to produce revenue equivalence based on pointing out that the ranking in [12] is weak whenever the signals are only weakly affiliated.

Next a pure common value model is analyzed. For further discussion of this problem see [4,11,12,14,15,18,25].

Example 3 (Mineral rights model). Suppose there are two buyers and the signals have the same joint distribution as in the previous example. The buyers' common valuation is T. Suppose that T is distributed according to $G(t)$ to be specified later. The joint density h of T and the signals is

$$h(t,x,y) = \begin{cases} \dfrac{g(t)}{t^2} & \text{if } x,y \leqslant t, \\ 0 & \text{otherwise.} \end{cases}$$

Let j denote the joint density function of the signals. Then $j(s,s)$ is clearly $\int_s^1 g(x)/x^2\,dx$, and the density function of T conditional on $s_1 = s_2 = s$ is defined as follows:

$$h(t \mid s_1 = s_2 = s) = \frac{\frac{g(t)}{t^2}}{\int_s^1 \frac{g(x)}{x^2}\,dx}.$$

From Theorem 3 and Remark 3, in the second-price auction the symmetric increasing equilibrium profile is generated by $v(s,s)$, which in this case is the expected value of T conditional on both buyers having observed signal s, that is

$$v(s,s) = \int_s^1 th(t \mid s_1 = s_2 = s)\,dt.$$

Suppose now that $G(t) = t^3$. Then $h(t \mid s_1 = s_2 = s)$ turns out to be $1/(1-s)$ and $v(s,s) = 1/2 + 1/2s$. Using Theorem 2 and that $f_s(x)/F_s(s) = 1/s$ if $x \leqslant s$

$$b_\alpha(s) = \frac{1}{2s}\int_0^s (1+x)\,dx = \frac{1}{2} + \frac{1}{4}s.$$

Next it is demonstrated that the signals do not have to be conditionally independent. Furthermore, the signal contains information about common as well as private value components. Such models were analyzed for example in [7–9].

Example 4. Suppose there are two buyers and the valuations have private as well as common components. Furthermore the two components are correlated. Buyer i's utility is

$$\alpha t + \beta s_i,$$

where t is the common component and s_i is the signal of buyer i. Further, assume that the joint distribution, h, of the signals and t is

$$h(t, s_1, s_2) = \begin{cases} \dfrac{2s_2}{s_1^2} & \text{if } t \geqslant s_1 \geqslant s_2, \\ \dfrac{2s_1}{s_2^2} & \text{if } t \geqslant s_2 \geqslant s_1. \end{cases}$$

This density is generated by the following process. First, t is distributed according to the cdf x^2 on $[0, 1]$. Conditional on the realization of t, the higher signal is distributed uniformly on $[0, t]$, and finally, conditional on the realization of the higher signal, the smaller one is distributed according to the cdf x^2/q^2, where q is the higher signal. Hence, from Proposition 1 the joint distribution clearly satisfies (A). The joint density of the signals, j, can be computed as follows:

$$\begin{aligned} j(s_1, s_2) &= \int_{\max\{s_1, s_2\}}^1 h(t, s_1, s_2)\, dt \\ &= \int_{\max\{s_1, s_2\}}^1 \frac{2\min\{s_1, s_2\}}{(\max\{s_1, s_2\})^2}\, dt = \frac{2(1 - \max\{s_1, s_2\})\min\{s_1, s_2\}}{(\max\{s_1, s_2\})^2}. \end{aligned}$$

To see that the signals are strictly affiliated observe that if $s_1 > s_2$

$$\begin{aligned} j(s_1, s_2)j(s_2, s_1) &= \left[\frac{(1 - s_1)2s_2}{s_1^2}\right]^2 < \left(\frac{(1 - s_1)2s_1}{s_1^2}\right)\left(\frac{(1 - s_2)2s_2}{s_2^2}\right) \\ &= j(s_1, s_1)j(s_2, s_2). \end{aligned}$$

Hence the density of t conditional on $s_1 = s_2 = s$ is

$$h(t \mid s_1 = s_2 = s) = \frac{h(t, s, s)}{j(s, s)} = \frac{1}{1 - s}.$$

Then $v(s, s)$ can be computed as follows:

$$v(s, s) = \alpha \int_s^1 t h(t \mid s_1 = s_2 = s)\, dt + \beta s = \alpha \frac{1 - s^2}{2(1 - s)} + \beta s = \alpha \frac{1 + s}{2} + \beta s.$$

Therefore, the equilibrium strategy in the second-price auction is

$$b_\beta(s) = \frac{\alpha}{2} + \frac{\alpha s}{2} + \beta s.$$

Now, using Theorem 2 and $f_s(x)/F_s(s) = 2x/s^2$ if $x \leqslant s$, the equilibrium strategy in the first-price auction turns out to be

$$b_\beta(s) = \int_0^s \left(\frac{\alpha}{2} + \frac{\alpha x}{2} + \beta x\right) \frac{2x}{s^2}\, dx = \frac{\alpha}{2} + \frac{\alpha s}{3} + \frac{2\beta s}{3}.$$

5. Simultaneous multiple object auctions with complete information

In this section, the method developed in the previous sections is applied to simultaneous multiple object auction models with complete information. First, the model is described. Then cost equivalence is defined and finally, two theorems are proven, which correspond to Theorems 2 and 3 showing how to transform a first-price equilibrium into a second-price one and vice versa in a simultaneous model.

There are n buyers competing in simultaneous sealed-bid high-bid-wins auctions for k not-necessarily-identical objects listed in some specified order. Each object is sold by the same mechanism. A pure strategy for a buyer is an element of $\mathbb{R}^k$, interpreted as an ordered list of bids. The highest of the n bids for each object wins that object at a cost which is a function of the bids on that object. The benefit of each buyer is $u(\boldsymbol{\omega})$, where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k) \in \{0,1\}^k$, and ω_i is one if the object is won by the buyer and zero otherwise. So $u : \{0,1\}^k \to \mathbb{R}$.[6] The buyer's payoff is his benefit less the sum of his costs. All buyers desire to maximize their respective expected payoffs. Since the focus here is on mixed-strategy equilibria where ties occur with probability zero, the tie-break rule need not be specified.

In many of the games described above there do not exist pure-strategy equilibrium, but there exist mixed-strategy equilibria. (See, for example, [20].) A mixed strategy is a probability measure on $\mathbb{R}^k$, and I restrict my attention to Borel measures only. Each Borel probability measure can be defined by its cumulative distribution function. In mixed-strategy equilibria, buyers use randomization devices, and their bids depend on the outcomes of these devices. Without loss of generality, we can assume the signal received from such a randomization device is a k-dimensional vector where the buyer bids the ith coordinate of the signal on object i. However, it is useful to have the buyers randomize in a slightly different way. First the buyer randomizes on $\mathbb{R}^k$, and once a k dimensional signal $\mathbf{x} = (x_1, \dots, x_k)$ is realized the buyer bids $p_i(x_i)$ on object i, where p_i is some strictly increasing function. Therefore a mixed strategy can be defined by a pair $(F, \mathbf{p})$, where F is a cumulative distribution function on $\mathbb{R}^k$ and $\mathbf{p} = (p_1, \dots, p_k)$ is an increasing mapping from $\mathbb{R}^k$ to $\mathbb{R}^k$. For instance, if a buyer is randomizing according to F and bids the signal itself, then with this notation he is using $(F, \mathbf{id})$, where $\mathbf{id} = (id, \dots, id)$ and id denotes the identity function. Notice that this way of randomizing is no more general than the previous one, where the buyers bid the signal itself, since using the strategy $(F, \mathbf{p})$ is equivalent to using the strategy $(F \circ \mathbf{p}^{-1}, \mathbf{id})$.[7] Notice that changing a buyer's strategy on a zero F-measure set does not effect the payoff of any buyer. Therefore $\mathbf{p}$ has to be defined F-almost-everywhere. (Uniqueness in the following discussion is also understood in the almost-everywhere sense.)

In what follows, a strong relationship between symmetric equilibria in first- and second-price auctions is established. It will be shown that if buyers use the mixed strategy $(F, \mathbf{p})$ in an equilibrium in one of these auctions, then there exists an

[6] The monocity of u is not required.

[7] $\mathbf{p}^{-1}$ is the inverse of $\mathbf{p}$.

equilibrium in the other auction in which buyers use the same F with an increasing mapping different from $\mathbf{p}$ to generate a mixed strategy. This means in particular that the supports of the corresponding equilibria are monotonic transformations of each other.

Cost equivalence between two symmetric mixed-strategy profiles is now defined as follows.

Definition 2. Suppose there are two auctions, α and β, generating different cost functions, c_α and c_β, respectively. Let $(F, \mathbf{p}^\alpha)$ generate a symmetric mixed-strategy profile in the game where each object is auctioned by α, and let $(F, \mathbf{p}^\beta)$ generate a symmetric mixed-strategy profile in the game where each object is auctioned by β. The two profiles are called *cost-equivalent strategy profiles* if for each real number x and for each $j \in \{1, \dots, k\}$:

$$E_{x_2,\dots,x_n}(c_\alpha(p_j^\alpha(x), \{p_j^\alpha(x_i)\}_{i=2}^n)) = E_{x_2,\dots,x_n}(c_\beta(p_j^\beta(x), \{p_j^\beta(x_i)\}_{i=2}^n)).$$

Definition 2 not only requires that the expected overall cost of a buyer is the same in the two auctions after observing the same signal, but also that the expected cost of bidding on each individual object must be the same. Notice that the difference between Definitions 1 and 2 is that while the signal is exogenous in the first case, it is endogenous in the second one.

Restrict attention again to first- and second-price auctions and symmetric equilibrium profiles with respect to the players (but not to the objects). Let F be a cumulative distribution function on $\mathbb{R}^k$ and let F_i denote its ith marginal distribution. In what follows assume that F_i is continuous (in order to avoid any tie-break complications). Suppose that buyers $2, \dots, n$ use the mixed strategy $(F, \mathbf{p})$ in the first-price auction. Then the expected monetary payment in the first-price auction if a buyer bids $\mathbf{p}(\mathbf{x}) = (p_1(x_1), \dots, p_k(x_k))$ is

$$\sum_{i=1}^k F_i^{n-1}(x_i)p_i(x_i).$$

Similarly, if buyers $2, \dots, n$ use the mixed strategy $(F, \mathbf{q})$ the cost of bidding $\mathbf{q}(\mathbf{x}) = (q_1(x_1), \dots, q_k(x_k))$ in the second-price auction is

$$\sum_{i=1}^k \int_0^{x_i} q_i(y)\, dF_i^{n-1}(y).$$

The symmetric profile in the first-price auction generated by $(F, \mathbf{p})$ and the symmetric profile in the second-price auction generated by $(F, \mathbf{q})$ are cost-equivalent if and only if for all $i \in \{1, \dots, k\}$ and $x_i \in$ support F_i:

$$\int_0^{x_i} q_i(y)\, dF_i^{n-1}(y) = F_i^{n-1}(x_i)p_i(x_i). \tag{6}$$

Corresponding to Theorem 2 is:

Theorem 4. *If* $(F, \mathbf{id})$ *generates a symmetric equilibrium profile in the second-price auction, then the unique mixed strategy which generates the cost-equivalent symmetric profile in the first-price auction* $(F, \mathbf{p})$, *constitutes an equilibrium in the first-price auction, where* $\mathbf{p} : \mathbb{R}^k \to \mathbb{R}^k$, $\mathbf{p}(\mathbf{x}) = (p_1(x_1), \ldots, p_k(x_k))$, *and*

$$p_i(x_i) = \frac{\int_0^{x_i} y \, dF_i^{n-1}(y)}{F_i^{n-1}(x_i)}.$$

Proof. For each $i \in \{1, \ldots, k\}$, p_i is obviously strictly increasing wherever F_i is strictly increasing. The uniqueness of the corresponding cost-equivalent profile in the first-price auction is trivial from (6).

The idea of the proof is to show that bidding $\mathbf{x}$ ($\in$ support F) in the second-price auction is a profitable deviation if and only if bidding $\mathbf{p}(\mathbf{x})$ is a profitable deviation in the first-price auction. In fact I show that the payoff from bidding $\mathbf{x}$ in the second-price auction is the same as from bidding $\mathbf{p}(\mathbf{x})$ in the first-price auction.

Since for each $i \in \{1, \ldots, k\}$, p_i is strictly increasing, the expected benefit of a buyer who bids $\mathbf{x}$ in the second-price auction is the same as the expected benefit of a buyer who bids $\mathbf{p}(\mathbf{x})$ in the first-price auction. On the other hand, the two profiles are clearly cost-equivalent by (6); hence the cost of bidding $\mathbf{x}$ in the second-price auction is also the same as bidding $\mathbf{p}(\mathbf{x})$ in the first-price auction. $\square$

Corresponding to Theorem 3:

Theorem 5. *If* $(F, \mathbf{id})$ *generates a symmetric equilibrium profile in the first-price auction and* $q_i(x) = \partial(xF_i^{n-1}(x))/\partial F_i^{n-1}(x)$ *is a strictly increasing function on the support of* F_i *for all* $i \in \{1, \ldots, k\}$, *then the unique mixed strategy* $(F, \mathbf{q})$ *that generates the cost-equivalent symmetric profile in the second-price auction constitutes an equilibrium in the second-price auction, where* $\mathbf{q}(\mathbf{x}) = (q_1(x_1), \ldots, q_k(x_k))$.

Proof. First, observe that F_i must be strictly increasing on its support for each $i \in \{1, \ldots, k\}$. (If F_i is constant on (a, b) where $a < b$, then there exists an ε (>0) such that if a buyer observes a signal x with $x_i \in (b, b+\varepsilon)$, it is a profitable deviation to bid a instead of x_i on object i.) Therefore F_i^{n-1} is differentiable almost everywhere on its support. The uniqueness and existence of the corresponding cost-equivalent profile follows from (6) and Lemma 6.

The proof is now similar to the proof of the previous theorem. It is necessary only to show that the payoff from bidding $\mathbf{x}$ ($\in$ support F) in the first-price auction is the same as the payoff from bidding $\mathbf{q}(\mathbf{x})$ in the second-price auction. Since $\mathbf{q}$ is strictly increasing in each of its coordinates it must only be shown that the two profiles are cost-equivalent, that is, the expected cost of bidding $\mathbf{x}$ in the first-price auction is the same as the expected cost of bidding $\mathbf{q}(\mathbf{x})$ in the second-price auction. By the definition of q_i and the Fundamental Theorem of Calculus

$$\int_0^{x_i} q_i(y) \, dF_i^{n-1}(y) = \int_0^{x_i} \frac{\partial(yF_i^{n-1}(y))}{\partial F_i^{n-1}(y)} dF_i^{n-1}(y) = x_i F_i^{n-1}(x_i).$$

Therefore the cost of bidding $\mathbf{x}$ in the first-price auction is indeed the same as the cost of bidding $\mathbf{q}(\mathbf{x})$ in the second-price auction. $\square$

The expected selling price of an object is the expected cost of this object for buyer 1 conditional on buyer 1 having bid the highest on this object. Therefore the following can be concluded:

Revenue Equivalence Theorem. *For any symmetric equilibrium profile in the second-price auction generated by a mixed strategy with atomless marginal distributions, there exists a symmetric equilibrium profile in the first-price auction such that the expected selling price of each object is the same in both mechanisms. For any symmetric equilibrium profile in the first-price auction generated by a mixed strategy with an atomless marginal distribution, if the cost of bidding on each object is a strictly convex function of the probability of winning that object, there exists a symmetric equilibrium profile in the second-price auction such that the expected selling price of each object is the same in both auctions.*

Since the buyers are independently randomizing, the bids on each object are independently distributed. Hence, in this model a condition corresponding to (A) is automatically satisfied.

Notice that in a simultaneous auction model, while it is always possible to transform a second-price equilibrium into a first-price equilibrium, an additional condition is needed to guarantee that the first-price equilibrium can be transformed into a second-price one; namely $\partial(xF_i^{n-1}(x))/\partial F_i^{n-1}(x)$ must be a strictly increasing function on the support of F_i for all $i \in \{1, \dots, k\}$. This condition means that the expected cost of bidding on any of the objects is a strictly convex function of the probability of winning that object. Recall that in the case of single unit auction games the expected cost of a buyer in a first-price auction is always a strictly convex function of the probability of winning (see Remark 2). Therefore in a single-unit model one could transform a first-price equilibrium into a second-price equilibrium exactly when one could transform a second-price equilibrium into a first-price equilibrium. A natural question to ask is whether the nature of the equilibrium in simultaneous auction games automatically guarantees that this condition holds or whether there are examples of where it does not. The following example shows an equilibrium in a first-price auction for which this condition does not hold. Therefore we can conclude that the equilibrium structures of first-price auctions in some of these games are richer than the equilibrium structures of second-price auctions.

Example 4. Suppose that there are two buyers and two identical objects. The benefit of a buyer is 1 if he has at least one object and zero otherwise; that is, the benefit function u is defined as

$$u(\omega_1, \omega_2) = \begin{cases} 0 & \text{if } \omega_1 + \omega_2 = 0, \\ 1 & \text{if } \omega_1 + \omega_2 \geqslant 1. \end{cases}$$

In [23], I show that the following strategy generates a symmetric mixed-strategy equilibrium in the first-price auction. The support of the strategy is two continuous strictly decreasing curves g_1 and g_2. g_1 is defined on $[0, a_1]$ and g_2 on $[0, a_2]$ where $a_1 \in (1/2, 1)$ and $a_2 \in (0, 1/2)$. They are defined implicitly by the following equations:

$$G^1(g_1(x)) = a_1 - G^1(x),$$

$$G^2(g_2(x)) = a_2 - G^2(x),$$

where $G^2(x) = x(1 - a_1)/a_2$ and

$$G^1(x) = \begin{cases} \dfrac{(1-a_1)x^2}{a_2(1-x)} & \text{if } x \in [0, a_2], \\ (1-a_1)\dfrac{x}{1-x} & \text{if } x \in [a_2, a_1]. \end{cases}$$

The measures on the curves are defined as follows:

$$\mu_1(\{(y, g_1(y)) \mid y \in [0, x)\}) = G^1(x),$$

$$\mu_2(\{(y, g_2(y)) \mid y \in [0, x)\}) = G^2(x).$$

It can be shown that $g_1 > g_2$, $g_i(a_i) = 0 = a_i - g_i(0)$, $g_i(g_i(x)) = x$ for $i = 1, 2$, and $\mu_1 + \mu_2$ is a probability measure on $\mathbb{R}^2$. Let F denote the cumulative distribution function corresponding to this strategy. The marginal distributions of F (F_1 and F_2) are $F_1(x) = F_2(x) = G^1(x) + G^2(x)$ if $x \in [0, a_2]$ and $F_1(x) = F_2(x) = G^1(x) + 1 - a_1$ if $x \in [a_2, a_1]$. It can be shown that if $x \in [a_2, a_1]$

$$\frac{\partial(xF_1(x))}{\partial F_1(x)} = 1.$$

Therefore Theorem 5 cannot be applied, and there does not exist a cost-equivalent mixed-strategy equilibrium in the second-price auction. Since $\partial(xF_1(x))/\partial F_1(x)$ is constant on $[a_2, a_1]$, it is possible that with an appropriate tie-break rule this problem can be solved, and the cost-equivalent equilibrium can be defined in the second-price auction. I do not know whether there exists a first-price auction with an equilibrium in which $\partial(xF_i^{n-1}(x))/\partial F_i^{n-1}(x)$ is actually decreasing.

6. Applications for simultaneous multiple object auctions

As we have seen, in single-unit auctions with incomplete information, it is generally easier to find the symmetric equilibrium profile in the second-price auction than in the first-price auction. Since the valuation of each buyer is common knowledge and the signals have nothing to do with the valuations, here there is no strategy corresponding to truth-telling. On the other hand, while in the first-price auction the expected cost of a bid on a certain object is just the product of the bid and the winning probability, in the second-price auction the expected cost is an integral. Hence, it seems to be easier here to search for equilibrium in the first-price auction than in the second-price auction. In each of the following examples, the equilibrium from a first-price auction is transformed into an equilibrium in the second-price auction.

Assume that the objects are identical. Therefore the buyers' benefit function u only depends on q, the number of objects won. First I exhibit another equilibrium in the game described in Example 4. This can be transformed into a second-price equilibrium.

Example 5. Recall there are two buyers and two objects. The benefit of a buyer is 1 if he has at least one object and zero otherwise; that is,

$$u(q) = \begin{cases} 0 & \text{if } q = 0, \\ 1 & \text{if } q = 1, 2. \end{cases}$$

In [23], I show the strategy defined in the same way as in Example 4, but with $a_1 = a_2 = 1/2$, constitutes a symmetric equilibrium in the first-price auction. The marginal distributions of F (F_1 and F_2) are $F_1(x) = F_2(x) = G^1(x) + G^2(x)$. The cost of bidding x on one of the objects is $x(G^1(x) + G^2(x)) = G^1(x)$. It can be shown that

$$\frac{\partial(xF_1(x))}{\partial F_1(x)} = \frac{\partial G^1(x)}{\partial(G^1(x) + G^2(x))} = 2x - x^2,$$

which is strictly increasing on $[0, \frac{1}{2}]$. Therefore Theorem 5 can be applied and one can conclude that $(F, \mathbf{q})$ generates a symmetric equilibrium profile in the second-price auction, where $q_i(x) = 2x - x^2$ for $i = 1, 2$. The expected revenue for the seller is 1. Furthermore, if the tie-breaking rule is defined such that in case of ties on both objects each buyer gets one of the objects, the pure strategy of bidding $(0, 0)$ clearly constitutes a symmetric equilibrium in both the first- and second-price auctions. They are clearly revenue-equivalent too. So, unlike in the single-unit model, in simultaneous auction games with complete information to get revenue-equivalent equilibrium profiles, neither uniqueness of the symmetric equilibria nor the fact that all symmetric equilibria must guarantee the same revenue for the seller is required.

The next example is especially interesting because of its unusual and beautiful equilibrium.

Example 6 (The Chopstick Auction, see Szentes and Rosenthal [21]). There are two buyers and three objects, the benefit of having only one object is zero, and the benefit of having either two or three objects is one. (That is, the marginal value of the first and third objects are zero.) In this game it can be shown that if the mechanism is a first price auction, then the uniform distribution on the surface of the regular tetrahedron spanned by the four points $(0, 0, 0)$, $(1/2, 1/2, 0)$, $(1/2, 0, 1/2)$, and $(0, 1/2, 1/2)$ constitutes a symmetric equilibrium profile. Denote this strategy by F. The marginal distributions of the bids are uniform on $[0, 1/2]$, that is, $F_i(x_i) = 2x_i$ for $i = 1, 2, 3$. Therefore the cost of bidding x on any of the objects in the first-price auction is $2x^2$. Notice that

$$\frac{\partial(x_i F_i(x_i))}{\partial F_i(x_i)} = \frac{\partial 2x_i^2}{\partial 2x_i} = 2x_i.$$

Therefore Theorem 5 can be applied and we can conclude that the following strategy constitutes a symmetric equilibrium profile in the second-price auction. The buyer randomizes in the same set as before but bids twice as much as in the first-price auction; that is, the support of the new equilibrium is the regular tetrahedron spanned by the points $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ and the distribution is again the uniform on its surface.

The next example is in some sense a generalization of the previous one, but with a very different equilibrium.

Example 7 (Majority auction games, see Szentes and Rosenthal [22]). There are n buyers competing for k objects. The benefit of a buyer as a function of the number of objects she wins is defined as follows:

$$u(q) = \begin{cases} 0 \text{ if } q<m, \\ m \text{ if } q\geqslant m, \end{cases}$$

where $m>k/2$. It can be shown that for each pair of (m,k) if n is large enough the following strategy (denoted by F) generates a symmetric equilibrium profile in the first-price auction. First the buyer selects m of the k objects randomly and bids x on those m objects and zero on the remaining $k-m$, where x is chosen according to the cdf

$$G(x) = \frac{\frac{k-m}{k} x^{\frac{1}{n-1}}}{1 - \frac{m}{k} x^{\frac{1}{m}}}.$$

It can be shown that the marginal distribution of F is $((k-m)/k) + (m/k)G$, and also that the cost of bidding x on any object in the first-price auction is

$$xF_i^{n-1}(x) = G^{n-1}(x).$$

Applying Theorem 5, the following strategy generates a symmetric equilibrium profile in the second-price auction. The buyers use the same randomization as in the first-price auction, but instead of bidding $\mathbf{x}$ they bid $\mathbf{q}(\mathbf{x})$ where

$$q_i(x) = \frac{\partial G^{n-1}(x)}{\partial F_i^{n-1}(x)} = \frac{k}{m}\left(\frac{G(x)}{\frac{k-m}{k} + \frac{m}{k} G(x)}\right)^{n-2}.$$

Observe that $q_i(x)$ is strictly increasing; therefore Theorem 5 indeed can be applied.

7. Discussion

Through cost equivalence I have identified a new class of auction environments where the Revenue Equivalence Theorem holds. This class is much larger than environments where Revenue Equivalence was previously known. The cost-equivalence approach also casts Revenue Equivalence in a new light and helps to resolve why it sometimes holds and sometimes does not.

Another point of interest is that the relationship between a first-price auction symmetric equilibrium strategy and a second-price auction symmetric equilibrium strategy is similar to the relationship between a function and its derivative whenever the equilibrium profiles are cost-equivalent. This relationship seems to be otherwise unexploited in the literature.

I has been shown that in a general single-unit auction model, (A) is necessary and sufficient to guarantee that if a strategy profile is an equilibrium in a first- or second-price auction then the corresponding cost-equivalent profile is an equilibrium in the other auction. It is fairly easy to recognize those distributions which satisfy (A) (see Proposition 1). However it seems to be hard to characterize those which also satisfy affiliation.

Mixed strategies in simultaneous multiple object auctions with perfect information have not been extensively analyzed before, and Revenue Equivalence Theorem had not been established for these games (See however, [21–23]). The Revenue Equivalence Theorem here is in a different sense than the usual one. Neither uniqueness of equilibrium nor the fact that all equilibria must guarantee the same revenue for the seller is required (see Example 5). However for each equilibrium in the second-price auction, there exists a corresponding equilibrium in the first-price auction that gives the same revenue to the seller. If the cost of a buyer is a convex function of the probability of winning in a first-price equilibrium, then there exists an equilibrium in the second-price auction which generates the same expected revenue to the seller. This convexity condition is not satisfied in every equilibrium (see Example 4). Therefore, unlike in single-unit auction models, the equilibrium structures of first-price auctions are generally richer than those of second-price auctions. Furthermore, the supports of the two equilibria have the same topological structure. (In fact, one is a monotone transformation of the other.)

Entry fees and reservation prices also can be incorporated into the model without any difficulty. In the simultaneous auction models, cost-equivalence analysis can be extended to any pair of sealed-bid auctions. In the single-unit model one must conjecture the sufficient and necessary condition to guarantee that the cost-equivalent profile corresponding to an equilibrium profile is also an equilibrium. However, for each pair of mechanisms this condition is different from (A).

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Appendix

Proof of Lemma 2. We can rewrite (A) as

$$\text{If } x \leqslant s \text{ then } \left.\frac{\partial \frac{f_z(x)}{F_z(s)}}{\partial z}\right]_{z \geqq s} = 0.$$

Therefore it is necessary only to show that (B) implies (A).

By the product rule for derivatives, rewrite (B) as

$$\left.\frac{\partial f_z(x)/\partial z}{F_z(s)}\right]_{z=s} - \left.\frac{f_z(x)}{F_z^2(s)}\frac{\partial F_z(s)}{\partial z}\right]_{z=s} = 0 \quad \forall x \leqslant s.$$

Rearranging terms

$$\left.\frac{\partial f_z(x)/\partial z}{f_z(x)}\right]_{z=s} = \left.\frac{\partial F_z(s)/\partial z}{F_z(s)}\right]_{z=s} \quad \forall x \leqslant s. \tag{A.1}$$

Observe that if h and g are integrable functions and $h(x)/g(x) = c$ on the interval $[a, d]$ then

$$\frac{\int_a^d h(x)\,dx}{\int_a^d g(x)\,dx} = c.$$

Applying this argument to the left-hand side of (A.1) and integrating from 0 to y ($\leqq s$):[8]

$$\left.\frac{\int_0^y \partial f_z(x)/\partial z\,dx}{\int_0^y f_z(x)\,dx}\right]_{z=s} = \left.\frac{\partial \int_0^y f_z(x)\,dx/\partial z}{\int_0^y f_z(x)\,dx}\right]_{z=s} = \left.\frac{\partial F_z(y)/\partial z}{F_z(y)}\right]_{z=s}.$$

Hence

$$\left.\frac{\partial f_z(x)/\partial z}{f_z(x)}\right]_{z=s} = \left.\frac{\partial F_z(y)/\partial z}{F_z(y)}\right]_{z=s},$$

where $x \leqslant y \leqslant s$. This last equation is equivalent to

$$\left.\frac{\partial f_z(x)/\partial z}{F_z(y)}\right]_{z=s} - \left.\frac{f_z(x)}{F_z^2(y)}\frac{\partial F_z(y)}{\partial z}\right]_{z=s} = 0.$$

[8] Recall that 0 is the bottom of the support of the signal.

But this is just

$$\left.\frac{\partial(f_z(x)/F_z(y))}{\partial z}\right]_{z=s} = 0.$$

Since $y \leqq s$, this is just (A). □

Proof of Proposition 1. Let h_2^z denote the density function corresponding to H_2^z. If $x \leqslant z$ then

$$f_z(x) = F_z(z)h_2^z(x), \tag{A.2}$$

that is, f_z is the product of the probability that z is the highest signal given that z is observed and the density of the second highest signal given that z was the highest signal. Furthermore by integrating f_z from 0 to $s(\leqslant z)$ we get that

$$F_z(s) = F_z(z)H_2^z(s).$$

Assume there exists an H such that $H_2^z(x) = H(x)/H(z)$. Let h denote the density function corresponding to H. Then

$$F_z(s) = F_z(z)\frac{H(s)}{H(z)}.$$

Substituting the previous equation into (A.2) and using $h_2^z(x) = h(x)/H(z)$, I get

$$\frac{f_z(x)}{F_z(s)} = \frac{h(x)}{H(s)}.$$

Since this is true for all $z \geqslant s$, (A) is satisfied.

Assume now that (A) holds and apply it for $z = 1$. Then

$$\frac{f_s(x)}{F_s(s)} = \frac{f_1(x)}{F_1(s)}.$$

Substituting the previous equation into (A.2) we get: $h_2^z(x) = f_1(x)/F_1(z)$. Therefore with $H = F_1$, $H_2^x(y)$ can be written as $H(y)/H(x)$. □

Proof of Lemma 3. Since the proofs of the two statements are basically identical I prove only (i). Assume by contradiction that $f_{s'}(x)/F_{s'}(s) \geqslant f_s(x)/F_s(s)$ but $f_{s'}(x')/F_{s'}(s) < f_s(x')/F_s(s)$. Then

$$(f_{s'}(x)/F_{s'}(s))(f_s(x')/F_s(s)) > (f_s(x)/F_s(s))(f_{s'}(x')/F_{s'}(s))$$

would also be true. Multiplying each side by $F_{s'}(s)F_s(s)$ I get that

$$f_{s'}(x')f_s(x) < f_s(x')f_{s'}(x).$$

But the signal of Buyer 1 and the highest signal among Buyers 2, …, n are affiliated (see [12, Theorem 2]). Therefore the previous inequality cannot hold since it violates the affiliation inequality. □

Proof of Lemma 4. It is enough to show that if $x' \geqslant (\leqslant) x$ then

$$\left.\frac{\partial(f_z(x)/F_z(s))}{\partial z}\right]_{z=s} \geqslant (\leqslant) 0 \quad \text{implies} \quad \left.\frac{\partial(f_z(x')/F_z(s))}{\partial z}\right]_{z=s} \geqslant (\leqslant) 0.$$

Assume that for x $\partial(f_z(x)/F_z(s))/\partial z \geqslant 0$ at $z = s$. Therefore there exists a positive δ such that whenever $0 \leqslant \varepsilon \leqslant \delta$:

$$\frac{f_{s+\varepsilon}(x)}{F_{s+\varepsilon}(s)} - \frac{f_s(x)}{F_s(s)} \geqslant 0.$$

Using the previous lemma the same must also be true for any x' which is bigger than x; therefore we can conclude that for every $x' \geqslant x$

$$\left.\frac{\partial(f_z(x')/F_z(s))}{\partial z}\right]_{z=s} \geqslant 0.$$

Similarly if for any x: $\partial(f_z(x)/F_z(s))/\partial z \leqslant 0$ at $z = s$, the same must also be true for any $x' \leqslant x$. $\square$

Proof of Lemma 5. Observe that

$$\int_0^s \left.\frac{\partial(f_z(x)/F_z(s))}{\partial z}\right]_{z=s} dx = \left.\frac{\partial \int_0^s f_z(x)/F_z(s)\, dx}{\partial z}\right]_{z=s}.$$

But since $\int_0^s f_z(x)/F_z(s)\, dx = 1$ this derivative must be zero. $\square$

Proof of Lemma 6. Notice that since h is increasing, h' exists almost everywhere on $[a, b]$ (see for example [19, Corollary 6, p. 104]). Since h is continuous and h' is increasing whenever it exists, h must be convex. Since g is continuous $h'_-(x) < h'_+(x)$ is impossible. Therefore h must be differentiable everywhere on $[a, b]$. $\square$

Proof of Lemma 7. First, it is shown that $F_s(z)b_\alpha(z)$ is differentiable according to z at $z = s$. Since $\mathbf{b}_\alpha$ is an equilibrium profile for all $z > s$

$$R(z, z) - F_z(z)b_\alpha(z) \geqslant R(z, s) - F_z(s)b_\alpha(s),$$

therefore

$$\frac{R(z, z) - R(z, s)}{z - s} \geqslant \frac{F_z(z)b_\alpha(z) - F_z(s)b_\alpha(s)}{z - s} = \frac{F_s(z)b_\alpha(z) - F_s(s)b_\alpha(s)}{z - s} + b_\alpha(z)\frac{F_z(z) - F_s(z)}{z - s} - b_\alpha(s)\frac{F_z(s) - F_s(s)}{z - s}.$$

Similarly,

$$R(s, s) - F_s(s)b_\alpha(s) \geqslant R(s, z) - F_s(z)b_\alpha(z),$$

hence

$$\frac{F_s(z)b_\alpha(z) - F_s(s)b_\alpha(s)}{z - s} \geqslant \frac{R(s, z) - R(s, s)}{z - s}.$$

Therefore

$$\frac{R(s,z)-R(s,s)}{z-s} \leqslant \frac{F_s(z)b_\alpha(z)-F_s(s)b_\alpha(s)}{z-s} \leqslant \frac{R(z,z)-R(z,s)}{z-s} + b_\alpha(s)\frac{F_z(s)-F_s(s)}{z-s} - b_\alpha(z)\frac{F_z(z)-F_s(z)}{z-s}$$

Notice, the left-hand-side as well as the right-hand-side of the previous inequality converges to $R_2(s,s)$ as z converge to s. Hence, one can conclude that the right derivative of $F_s(z)b_\alpha(z)$ at $z=s$ exists and is $R_2(s,s)$. In a similar manner one can show the same for the left derivative. Hence $dF_s(z)b_\alpha(z)/dz\rfloor_{z=s}$ exists.

Suppose by contradiction, that for some s $b_\alpha(s) > b_\gamma(s)$, and hence $F_s(s)b_\alpha(s) > b_\gamma(s)F_s(s)$. Notice that b_α must be continuous and $b_\alpha(0)$ must be $v(0,0)$. Then there must exist an s^* such that $b_\alpha(s^*) \geqslant b_\gamma(s^*)$ and

$$\left.\frac{dF_{s^*}(z)b_\alpha(z)}{dz}\right\rfloor_{z=s^*} > \left.\frac{dF_{s^*}(z)b_\gamma(z)}{dz}\right\rfloor_{z=s^*}.$$

Notice that since in $\mathbf{b}_\gamma$ there was no incentive to deviate upward

$$\left.\frac{dR(s^*,z)}{dz}\right\rfloor_{z=s^*} \leqslant \left.\frac{dF_{s^*}(z)b_\gamma(z)}{dz}\right\rfloor_{z=s^*}.$$

Hence,

$$\left.\frac{dR(s^*,z)}{dz} - \frac{dF_{s^*}(z)b_\alpha(z)}{dz}\right\rfloor_{z=s^*} < 0$$

which contradicts the first-order condition that guarantees that $\mathbf{b}_\alpha$ is an equilibrium profile. □

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