London Taught Course Centre

2020 examination

Graph Theory

Instructions to candidates

This open-book exam has 3 questions. Some parts are harder than others. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed. You are also allowed to use any theorems you find, provided they are properly referenced.

1 Let G_0 be the four-vertex cycle. For each $i \geq 1$ we define a graph G_i as follows. Suppose the vertices of G_{i-1} are $\{u_1, \dots, u_k\}$; we take a copy of G_{i-1}, and add to it vertices $v_1, \dots, v_k, w$. We add edges $u_i v_j$ for pairs (i, j) such that $u_i u_j \in E(G_{i-1})$, and edges $v_i w$ for each $1 \leq i \leq k$.

What is the chromatic number of G_i?

2 Given a graph G with vertex set $\{1, \dots, n\}$, the adjacency matrix A of G is the $n \times n$ matrix such that $A_{ij} = 1$ if and only if $ij \in E(G)$.

(a) Prove that all eigenvalues of $A(G)$ are real.

For $k \geq 0$, a *walk* in G of length k is a sequence $(u_1, \dots, u_{k+1})$ of not necessarily distinct vertices of G such that $u_i u_{i+1} \in E(G)$ for each $1 \leq i \leq k$.

(b) Show that the number of walks of length k in G whose first vertex is i and last vertex is j is $(A^k)_{ij}$.

The distance between two vertices i and j of G is the minimum length of a walk in G whose first vertex is i and last is j. If there is no such walk, we say the distance is ∞. The diameter of G is the maximum, over pairs (i, j) of vertices of G, of the distance from i to j.

(c) Find an algorithm which, for input G, finds the diameter of G. You should determine the worst-case running time of your algorithm in the form $O(n^x)$ where x is a constant. Points will be awarded as follows: one point for finding such an algorithm, plus $\lceil \frac{1}{x-2} \rceil$ points.

3 (a) Given any $\eta > 0$, show that there is $C > 0$ such that the following holds for any $p \geq Cn^{-1}$. With probability tending to 1 as $n \to \infty$, $G(n, p)$ does not contain any disjoint vertex sets X, Y with $|X|, |Y| \geq \eta n$ such that $e(X, Y) \geq 2p|X||Y|$.

Given a graph G and $0 < p \leq 1$, let A and B be disjoint vertex sets in G. We define $d_p(A, B) := \frac{e(A,B)}{p|A||B|}$. We say (A, B) is (ε, p)-regular if the following holds. For any $A' \subset A$ and $B' \subset B$, such that $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$, we have $d_p(A', B') = d_p(A, B) \pm \varepsilon$.

(b) Given any $\varepsilon > 0$, show that there are $C > 0$ and $K \in \mathbb{N}$ such that for any $p \geq Cn^{-1}$, with probability tending to 1 as $n \to \infty$, the random graph $\Gamma = G(n, p)$ has the following property. If G is any n-vertex subgraph of Γ, then there is a partition of $V(G)$ into parts $V_0, V_1, \dots, V_t$ where $\varepsilon^{-1} \leq t \leq K$, such that $|V_0| \leq \varepsilon n$ and $|V_1| = \dots = |V_t|$, and in addition for all but at most εt^2 pairs (V_i, V_j) the pair (V_i, V_j) is (ε, p)-regular.

(c) Is it true, for $p = n^{-0.9}$ and $\varepsilon = 10^{-1000}$, that with probability tending to 1 as $n \to \infty$, the random graph $\Gamma = G(n, p)$ has the following property. Suppose V_1, V_2, V_3 are three disjoint vertex sets in Γ each of size $\lfloor \frac{n}{3} \rfloor$. Suppose G is any subgraph of Γ such that (V_i, V_j) is (ε, p)-regular and $d_p(V_i, V_j) \geq \frac{1}{2}$. Then G contains at least $\frac{1}{1000} p^3 n^3$ copies of K_3.

(d) Is the statement of (c) true if $p = n^{-0.1}$?