London Taught Course Centre

2021 Examination

Graph Theory

Instructions to candidates

This open-book exam has 4 questions. Some parts are harder than others. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed. You are also allowed to use any theorems you find, provided they are properly referenced.

Question 1

Let G_1, G_2, G_3 be three simple graphs on the same vertex set. We denote by a *parallel edge-colouring* of G_1, G_2, G_3 a colouring of the edges $E(G_1) \cup E(G_2) \cup E(G_3)$ that gives a proper edge-colouring of each of $E(G_1), E(G_2), E(G_3)$, i.e., in none of the graphs are there two edges of the same colour that are adjacent. (A parallel edge-colouring of G_1, G_2, G_3 does not necessarily need to be a proper edge-colouring of the graph with edge set $E(G_1) \cup E(G_2) \cup E(G_3)$.)

We let $\chi'(3, \Delta)$ be the minimum number of colours needed in a parallel edge-colouring of any choice of three graphs G_1, G_2, G_3, each of which has maximum degree at most Δ. Improve on the trivial upper and lower bounds $\Delta + 1 \leq \chi'(3, \Delta) \leq 3\Delta + 1$.

Question 2

With $t \geq 2$ we let $Z_t(n)$ be the maximum number of edges in a simple graph on n vertices that does not contain a copy of H_t, where H_t is obtained from the complete bipartite graph $K_{t,t}$ by removing a perfect matching. Prove that there are constants c and C independent of n such that

$$cn^{2-2(t-1)/(t^2-t-1)} \leq Z_t(n) \leq Cn^{2-1/t}\,.$$

Hint: Try a probabilistic construction for the lower bound. For the upper bound, try considering a supergraph of H_t and counting copies of $K_{1,t}$.

Can you improve the exponent in either of the bounds for some values of t?

Question 3

In the exercises we proved that 2-SAT is in P. Use this to show that the following problem PRE-3-COL is in P. The input for PRE-3-COL is a graph G with some vertices which are pre-coloured with a colour from $\{1, 2, 3\}$, such that any vertex of G is either pre-coloured, or has a pre-coloured neighbour. The question to be answered in PRE-3-COL then is if the given pre-colouring can be extended to a proper (vertex) 3-colouring of G.

Question 4

Let us call the graph on vertex set $\{1, 2, 3, 4\}$ with edges $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$, $\{3, 4\}$ the *lollipop* graph. Prove a counting lemma for the lollipop graph, that is, show: If G is a is a graph with $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ (where the V_i are pairwise disjoint), $|V_i| = n$ for each $i \in \{1, 2, 3, 4\}$, and in which (V_1, V_2), (V_2, V_3), (V_1, V_3), (V_3, V_4) are ε-regular pairs, of densities d_{12}, d_{23}, d_{13}, d_{34}, respectively, and G has no other edges, then G contains

$$\big(d_{12}d_{23}d_{13}d_{34} + 2d_{12}^2 d_{23}d_{13} + 2d_{12}d_{23}^2 d_{13} + 2d_{12}d_{23}d_{13}^2 \pm 1000\varepsilon\big)n^4$$

copies of the lollipop graph.

END OF PAPER