London Taught Course Centre

2021 Examination

Graph Theory

Instructions to candidates

This open-book exam has 3 questions. Some parts are harder than others. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed and you may use results you find there (remember though that you should show own work for each question), but you should reference your sources.

Question 1

(a) The $(m \times n)$-*grid* is the graph with vertex set $\{(x, y)\colon 1 \le x \le m, 1 \le y \le n\}$ and edge set

$$\Big\{(x_1, y_1)(x_2, y_2)\colon \text{ either } |x_1 - x_2| = 1 \text{ or } |y_1 - y_2| = 1 \text{ (but not both)}\Big\}.$$

Determine the list chromatic number of the $(m \times n)$-grid.

(b) A *double torus* is a surface that is homeomorphic to the sphere with 2 handles, that is, a double torus is an orientable surface with genus 2. A *tcdt-graph* (two-cell double-toroidal graph) is a (simple) graph which has a 2-cell embedding on a double torus.

State and prove an upper bound on the chromatic number of tcdt-graphs.

Question 2

Let $F_{n,n}$ be a graph with vertex set $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ whose edge set is randomly constructed as follows. For each $1 \le i < j \le n$ independently, we consider the four potential edges

$$x_i x_j\,, \quad x_i y_j\,, \quad y_i x_j\,, \quad y_i y_j\,,$$

pick one uniformly at random that we *omit*, and then add the three remaining ones as edges to $F_{n,n}$.

(a) Prove that for every $\varepsilon > 0$ with probability tending to 1 as n tends to infinity, the graph $F_{n,n}$ does not contain a complete graph K_s with $s \ge \varepsilon n$ as a subgraph.

(b) Prove that for every $\varepsilon > 0$ with probability tending to 1 as n tends to infinity, the graph $F_{n,n}$ does not contain a K_t-minor with $t \ge (\frac{2}{3} + \varepsilon)n$ as a subgraph.

Question 3

For U and V disjoint with n vertices each, let (U, V) be an ε-regular pair with density at least d. Solve the following questions "by hand", that is, without using any existing lemmas that directly provide the answers (you may use results along the way that you find, however). A *binary tree* is a tree in which at most 1 vertex has degree 2 and all other vertices have degree 3 or 1.

(a) What is the longest path you can find in (U, V)?

(b) What is the longest cycle you can find in (U, V)?

(c) What is the biggest binary tree you can find in (U, V)?

(d) What is the biggest complete bipartite graph you can find in (U, V)?

END OF PAPER