London Taught Course Centre: Graph Theory
Exam Solutions
2021

Question 1

Let G_1, G_2, G_3 be three simple graphs on the same vertex set. We denote by a *parallel edge-colouring* of G_1, G_2, G_3 a colouring of the edges $E(G_1) \cup E(G_2) \cup E(G_3)$ that gives a proper edge-colouring of each of $E(G_1), E(G_2), E(G_3)$, i.e., in none of the graphs are there two edges of the same colour that are adjacent. (A parallel edge-colouring of G_1, G_2, G_3 does not necessarily need to be a proper edge-colouring of the graph with edge set $E(G_1) \cup E(G_2) \cup E(G_3)$.)

We let $\chi'(3, \Delta)$ be the minimum number of colours needed in a parallel edge-colouring of any choice of three graphs G_1, G_2, G_3, each of which has maximum degree at most Δ. Improve on the trivial upper and lower bounds $\Delta + 1 \leq \chi'(3, \Delta) \leq 3\Delta + 1$.

Solution : This type of colouring is known as simultaneous colouring and the bounds

$$3\lfloor \tfrac{\Delta}{2} \rfloor \leq \chi'(3, \Delta) \leq 3\lfloor \tfrac{\Delta}{2} \rfloor + \Delta + 2\,.$$

can be obtained along the lines of the arguments by Bosquet and Durain in arXiv:2001.01463.

We first give a construction that proves $\chi'(3, \Delta) \geq 3\lfloor \frac{\Delta}{2} \rfloor$. Consider three sets of vertices A_1, A_2, A_3 of size $\lfloor \frac{\Delta}{2} \rfloor$ and one additional vertex v. Let $G_{i,j}$ for $i \neq j$ be the graph with vertex set $A_i \cup A_j \cup \{v\}$ and edges vu for $u \in A_i \cup A_j$. Each $G_{i,j}$ has maximum degree Δ and any pair of edges of $G = G_{1,2} \cup G_{1,3} \cup G_{2,3}$ appears in one of the graphs. Therefore, a parallel edge-colouring of G needs to be a proper edge-colouring and uses at least $|A_1| + |A_2| + |A_3| \geq 3\lfloor \frac{\Delta}{2} \rfloor$ colours.

Next we will prove that $\chi'(3, \Delta) \leq 3\lfloor \frac{\Delta}{2} \rfloor + \Delta + 2$. Let G_1, G_2, G_3 be three graphs of maximum degree Δ on the same vertex set and consider their union $G = G_1 \cup G_2 \cup G_3$. Then let H_2 be the edges of G that appear in at least two graphs from G_1, G_2, G_3 and let H_1 be the edges of G that appear in only one graph from G_1, G_2, G_3. As each edge of H_2 appears in at least two graphs we get that H_2 has maximum degree at most $3\lfloor \frac{\Delta}{2} \rfloor$. Therefore, using Vizings's Theorem from the lecture, we can colour H_2 with $3\lfloor \frac{\Delta}{2} \rfloor + 1$ colours. Similarly, $\Delta + 1$ colours are sufficient for a parallel colouring of the edges of G_1, G_2, G_3 in H_1. For this it suffices to apply Vizing's Theorem to each part $G_i \cap H_1$ independently and use that is has maximum degree Δ. Together this gives a parallel colouring of G_1, G_2, G_3 with $3\lfloor \frac{\Delta}{2} \rfloor + \Delta + 2$ colours.

Question 2

With $t \geq 2$ we let $Z_t(n)$ be the maximum number of edges in a simple graph on n vertices that does not contain a copy of H_t, where H_t is obtained from the complete bipartite graph $K_{t,t}$ by removing a perfect matching. Prove that there are constants c and C independent of n such that

$$cn^{2-2(t-1)/(t^2-t-1)} \leq Z_t(n) \leq Cn^{2-1/t}\,.$$

Hint: Try a probabilistic construction for the lower bound. For the upper bound, try considering a supergraph of H_t and counting copies of $K_{1,t}$.

Can you improve the exponent in either of the bounds for some values of t?

Solution : This is a question about the extremal number of bipartite graphs, for which the asymptotic behaviour is not known in general. We start with a probabilistic construction for the lower bound. In the binomial random graph $G(n,p)$ the expected number of edges is at least $pn^2/4$. The expected number of copies of H_t is at most $(2t)!\, n^{2t}p^{t(t-1)}$. The goal is to pick p such that this is at most $pn^2/8$, because then we can delete one edge from each copy of H_t and still keep the other $pn^2/8$ edges. We choose $p = \frac{1}{(8\cdot 2t)!}n^{-(2t-2)/(t(t-1)-1)}$. With this choice of p the number of edges minus the number of copies of H_t is in expectation at least $pn^2/8$. Therefore, there exists a graph G, where by removing one edge from each copy of H_t, we get a H_t-free graph with $pn^2/8 \geq cn^{2-2(t-1)/(t^2-t-1)}$ edges for some appropriately chosen c. This implies $Z_t(n) \geq cn^{2-2(t-1)/(t^2-t-1)}$.

For the upper bound we only assume that we have a graph G on n vertices that does not contain a copy of $K_{t,t}$, because this contains H_t. We can additionally assume that all vertex degrees in G are at least t (because vertices of smaller degree can not be contained in a copy of $K_{t,t}$ and removing them deletes less than t edges). Then we let T be the number of copies of $K_{1,t}$ in G and obtain $T = \sum_{v\in V(G)} \binom{\deg(v)}{t}$. As there is no copy of $K_{t,t}$ in G no t vertices can be the set of leaves of $K_{1,t}$ in t different copies of $K_{1,t}$. Therefore, we also have $T \leq \binom{n}{t}(t-1)$. We can then estimate

$$(t-1)\binom{n}{t} \geq \sum_{v\in V(G)} \binom{\deg(v)}{t} \geq n\binom{\sum_{v\in V(G)}\deg(v)/n}{t} = n\binom{2e(G)/n}{t} \geq n\frac{(2e(G)/n-t)^t}{t!}\,,$$

where we use Jensen's inequality in the second estimate and that the vertex degrees are at least t. From this we obtain

$$e(G) \leq \frac{n}{2}\left(\frac{(t-1)\binom{n}{t}}{nt!}\right)^{1/t} + \frac{2n}{t} \leq Cn^{2-1/t}\,,$$

for some appropriately chosen C. This implies that $Z_t(n) \leq Cn^{2-1/t}$.

There are no better bounds known, expect for some small values of t. First note that H_2 consists of 2 independent edges and, therefore, $Z_2(n) = n-1$ follows from the star $K_{1,n-1}$. Next, we have that H_3 is C_6 and, therefore, $Z_3(n) = \Theta(n^{4/3})$ is known. Finally, H_4 is the cube on 8 vertices, where the best known lower bound $Z_4(n) \geq cn^{3/2}$ comes from C_4 and the best known upper bound is $Z_4(n) \leq Cn^{8/5}$.

Question 3

In the exercises we proved that 2-SAT is in P. Use this to show that the following problem PRE-3-COL is in P. The input for PRE-3-COL is a graph G with some vertices which are pre-coloured with a colour from $\{1,2,3\}$, such that any vertex of G is either pre-coloured, or has a pre-coloured neighbour. The question to be answered in PRE-3-COL then is if the given pre-colouring can be extended to a proper (vertex) 3-colouring of G.

Solution : For this question one should recall the reduction from GRAPH-3-COL to SAT that we discussed in a lecture: This reduction proceeds by introducing for each vertex x three boolean variables x_1, x_2, x_3 where we interpret x_c being true as "vertex x has colour c". Our SAT-formula then is a conjunction of the following clauses: For each vertex x we have the clause $x_1 \vee x_2 \vee x_3$ (meaning that x gets assigned a colour) and the clauses $\neg x_c \vee \neg x_{c'}$ for any pair $c \neq c'$ of colours (meaning that x does not get more than one colour). Moreover, for each edge xy and every colour c we have a clause $\neg x_c \vee \neg y_c$ (meaning that x and y do not get the same colours).

Now we observe that among our clauses the only ones that contain more than 2 literals are the clauses that list the colours that are possible for a vertex: $x_1 \vee x_2 \vee x_3$. Now, in our scenario each vertex x is either already pre-coloured with some colour c in which case we can replace the clause $x_1 \vee x_2 \vee x_3$ with the clause x_c; or it has a neighbour that is pre-coloured with some colour c, in which case x cannot be coloured with c and hence we can replace the clause $x_1 \vee x_2 \vee x_3$ by the clause $x_{c'} \vee x_{c''}$ such that $\{c, c', c''\} = \{1, 2, 3\}$. After performing all these replacements (which we can easily do in polynomial time) we end up with a 2-SAT formula that is satisfyable if and only if the given pre-colouring of the given graph can be extended to a proper 3-colouring. Checking if this 2-SAT formula is satisfyable, which we can do in polynomial time, thus solves the given problem.

Question 4

Let us call the graph on vertex set $\{1, 2, 3, 4\}$ with edges $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$, $\{3, 4\}$ the *lollipop* graph. Prove a counting lemma for the lollipop graph, that is, show: If G is a is a graph with $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ (where the V_i are pairwise disjoint), $|V_i| = n$ for each $i \in \{1, 2, 3, 4\}$, and in which (V_1, V_2), (V_2, V_3), (V_1, V_3), (V_3, V_4) are ε-regular pairs, of densities d_{12}, d_{23}, d_{13}, d_{34}, respectively, and G has no other edges, then G contains

$$\big(d_{12}d_{23}d_{13}d_{34} + 2d_{12}^2 d_{23}d_{13} + 2d_{12}d_{23}^2 d_{13} + 2d_{12}d_{23}d_{13}^2 \pm 1000\varepsilon\big)n^4$$

copies of the lollipop graph.

Solution: We first observe that we can assume $\varepsilon \leq 1/100$, because otherwise the statement is trivially true. (This is helpful with some calculations below.)

Proving this is somewhat technical, and there are different ways of doing it. One way is to use the counting lemma for triangles as a black box, which we do here. One important observation here is that (up to isomorphism) there are 7 different ways of mapping the lollipop graph vertices to the clusters V_1, V_2, V_3, V_4 so that edges are mapped to regular pairs: The vertex 3 could be mapped to V_1, to V_2, or to V_3; the two other triangle-vertices 1 and 2 then have to be mapped to the other two of these three clusters (and, up to isomorphism, it does not matter which way around we map them); the vertex 4 then can be mapped as follows: If 3 is mapped to V_1, then 4 can be mapped to V_2 or V_3; If 3 is mapped to V_2, then 4 can be mapped to V_1 or V_3; If 3 is mapped to V_3, then 4 can be mapped to V_1, V_2 or V_4.

Among these 7 mappings there is one mapping where we get $(d_{12}d_{23}d_{13}d_{34} \pm 20\varepsilon)n^4$ lollipop-copies obeying this mapping, two mappings where we get $(d_{12}^2 d_{23}d_{13} \pm 20\varepsilon)n^4$, two mappings where we get $(d_{12}d_{23}^2 d_{13} \pm 20\varepsilon)n^4$, and two mappings where we get $(d_{12}d_{23}d_{13}^2 \pm 20\varepsilon)n^4$. We only show this here for the first of these cases: the mapping ϕ that maps vertex i to V_i; for the others the calculations are similar. This implies that in total we get

$$(d_{12}d_{23}d_{13}d_{34} + 2d_{12}^2 d_{23}d_{13} + 2d_{12}d_{23}^2 d_{13} + 2d_{12}d_{23}d_{13}^2 \pm 140\varepsilon)n^4$$

lollipop-copies. So here the error term $\pm 140\varepsilon$ seems to be enough; the question is written with $\pm 1000\varepsilon$ instead, to give some wriggle-room for the calculations, and because in applications it is not important what the precise constant here is.

Now let us consider the mapping ϕ. As indicated before, we want to apply the counting lemma for triangles: This could be used to count the triangles in (V_1, V_2, V_3) – but then it would be more difficult to argue which of these triangles can be extended to how many copies of the lollipop graph; to avoid this, it helps to first perform some cleaning. To this end, let $V_3^* \subseteq V_3$ be the set of those vertices in V_3 that contain $(d_{34} \pm \varepsilon)n$ neighbours in V_4. By the definition of regularity, we have $|V_3 \setminus V_3^*| \leq 2\varepsilon n$. Now consider the pair (V_1, V_3^*): We want to show that it is ε^*-regular for some constant ε^* (so that we can later still apply the counting lemma using this subset). Indeed, since (V_1, V_3) is ε-regular we know for subsets $V_1' \subseteq V_1$ and $V_3' \subseteq V_3^*$ of size at least $\varepsilon n \geq 2\varepsilon |V_3^*|$ that $d(V_1', V_3') = d_{13} \pm \varepsilon$, and hence $|d(V_1', V_3') - d(V_1, V_3^*)| \leq 2\varepsilon$. It follows that (V_1, V_3^*) is ε^*-regular with $\varepsilon^* = 2\varepsilon$ and has density $d_{13}^* = d_{13} \pm \varepsilon$. Similarly, (V_2, V_3^*) is ε^*-regular with $\varepsilon^* = 2\varepsilon$ and has density $d_{13}^* = d_{13} \pm \varepsilon$.

Now we can apply the triangle counting lemma to (V_1, V_2, V_3^*), concluding that that we find $(d_{12} d_{13}^* d_{23}^* \pm 10\varepsilon) n^2 (n \pm 2\varepsilon n)$ triangles there. By the definition of V_3^*, each of these triangles extends to $(d_{34} \pm \varepsilon)n$ lollipop copies mapping vertex 4 to V_4. So we get $(d_{34} \pm \varepsilon) n (d_{12} d_{13}^* d_{23}^* \pm 10\varepsilon) n^2 (n \pm 2\varepsilon n)$ lollipop copies obeying ϕ in (V_1, V_2, V_3^*, V_4). In (V_1, V_2, V_3, V_4) we certainly cannot have less such copies, but we could have up to $2\varepsilon n^4$ more. We conclude that the number of ϕ-obeying lollipop copies in (V_1, V_2, V_3, V_4) is

$$\begin{aligned}
&(d_{34} \pm \varepsilon) n (d_{12} d_{13}^* d_{23}^* \pm 10\varepsilon) n^2 (n \pm 2\varepsilon n) \pm 2\varepsilon n^4 \\
=&(d_{34} \pm \varepsilon)\big(d_{12}(d_{13} \pm \varepsilon)(d_{23} \pm \varepsilon) \pm 10\varepsilon\big) n^3 (n \pm 2\varepsilon n) \pm 2\varepsilon n^4 \\
=&(d_{34} \pm \varepsilon)\big(d_{12} d_{13} d_{23} \pm 3\varepsilon \pm 10\varepsilon\big) n^3 (n \pm 2\varepsilon n) \pm 2\varepsilon n^4 \\
=&\big(d_{34} d_{12} d_{13} d_{23} \pm 15\varepsilon\big) n^3 (n \pm 2\varepsilon n) \pm 2\varepsilon n^4 \\
=&\big(d_{34} d_{12} d_{13} d_{23} \pm 18\varepsilon\big) n^4 \pm 2\varepsilon n^4 \\
=&\big(d_{34} d_{12} d_{13} d_{23} \pm 20\varepsilon\big) n^4 \,.
\end{aligned}$$

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