LTCC Course: Graph Theory 2022/23
§0: Introduction

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Notes 0: Introduction

These are lecture notes for a short course in Graph Theory at the *London Taught Course Centre* (LTCC). The course is intended for first-year research students in Mathematics, especially those specialising in Pure Mathematics. The course consists of five two-hour sessions.

It is expected that the audience will include some students intending to specialise in Graph Theory, Combinatorics, or a closely related area, along with others who wish to broaden their mathematical knowledge and learn new techniques. We aim the course to be suitable for all these students.

The course is not a comprehensive treatment of Graph Theory: such a thing would be impossible within the time constraints. Moreover, it is likely that most students taking the course will have encountered the basics of the subject at some earlier point in their career. So the course will not start from scratch with the definition of a graph, or the definition of a path or a cycle in a graph, or a proof that trees on n vertices have $n-1$ edges, etc. The next section shall point the reader to various sources where such basic material can be found. The plan in this course is to concentrate on themes and techniques, rather than on results. Anyone looking for a definitive source on the main results of Graph Theory should consult a (recent) textbook; that's what they are for!

Over the years, this course has been given by several lecturers. Peter Allen, Julia Böttcher, Graham Brightwell, Olaf Parczyk, and Jozef Skokan have all contributed to the notes in the form they are now. Nevertheless, every year there is some fine-tuning and rewriting of the material, and sometimes some reorganisation of the topics in the course. This means that the production of notes is being carried out in accordance with the "just-in-time" paradigm: There will be weekly notes, but don't expect them to be available much earlier than the lectures.

The notes will include references to books and online material, and some exercises. Solutions to those exercises will be published on the course webpage at the appropriate time.

Pre-requisites

There are many introductory texts on Graph Theory. Any one of these will provide an adequate account of the basics of the subject, although the reader is warned that notation and terminology is not completely standard. Different books do different things, and while the notation in the lectures will be internally consistent, there is no promise it is consistent with any given book.

Here is a list of notions a student on this course is expected to be familiar with already.

- The basic definition of a *graph* G as a pair (V, E), where V is the set of *vertices* and E the set of *edges* of G. We sometimes use $V(G)$ and $E(G)$ to make it clear what graph we are referring to.

- The meaning of terms such as *adjacent* and *neighbour*. We will write $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ for the neighbourhood of a vertex, and drop the subscript whenever G is clear from the context.
- *Trees*: equivalent definitions and basic properties.
- Definitions of a *path* and a *cycle* in a graph (no repeated vertices).
- *(Vertex) degree*, and notation $d(v)$. Also, $\delta(G)$, and $\Delta(G)$ denote the *minimum* and *maximum* degrees of the graph G.
- *Connectedness* and connected *components*.
- *Bipartite* graphs, and the idea of a *matching* in a bipartite graph.
- *Subgraphs*, and the distinction between a subgraph and an *induced* subgraph.
- *Complete* and *empty* graphs; a complete subgraph of a graph is called a *clique*, and a set of vertices inducing an empty graph is an *independent set* or a *stable set*.

Other notions will be defined as they come up.

For those missing some or all of this background: don't worry! Most of it is easy and in general the terminology is very intuitive. Simply look it up in one of the recommended books.

Recommended books and online sources

Here are some books that are particularly recommended.

- (**Diestel**) Reinhard Diestel, *Graph Theory* (1st–5th edition). Springer (1997–2016).

  Although this book is still in print, the author has made sure that it is available in several versions online as well. See `diestel-graph-theory.com`. The free downloadable version has low quality, but is perfectly readable. Most editions are suitable for this course. References in the notes will refer to the 5th edition (which is the same as the one you can download chapter by chapter).

- (**Bondy & Murty**) J.A. Bondy and U.S.R. Murty, *Graph Theory*. Springer (2008).

  See `www.springer.com/book/9781846289699`.
  A thorough and well-written textbook covering most parts of modern graph theory. In many institutes you will be able to read this book online.

- (**Bollobás**) Béla Bollobás, *Modern Graph Theory* (1st–3rd edition). Springer (1998–2002).

  See `www.springer.com/gb/book/9780387984889`.
  This is another classic textbook aimed at students at this level, and is suitable for the course.

- (**Alon & Spencer**) Noga Alon and Joel H. Spencer, *The Probabilistic Method* (1st–4th edition). Wiley (1991–2016).

  A great, clearly written, and well-motivated book with applications of probability in combinatorics. Many exercises. You can download the 3rd edition via `math.bme.hu/~gabor/oktatas/SztoM/AlonSpencer.ProbMethod3ed.pdf`.

Other books may be recommended for use in individual weeks; these will be mentioned at the appropriate points.