Price Multipliers of Anticipated and Unanticipated Shocks to Demand and Supply

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1 Introduction

This note presents a model of how anticipated and unanticipated shocks to demand and supply move prices in rational markets. That unanticipated shocks can move prices is not surprising, assuming that the market has limited risk-absorption capacity. That anticipated shocks can move prices is more surprising. The explanation for the price effect of anticipated shocks presented in this note is based on the theory of rational momentum in Vayanos and Woolley (2013, VW).

This note is an extended version of my discussion of Hartzmark and Salomon (2021) at the NBER Asset Pricing meeting in Fall 2021. HS show that the aggregate stock market rises on days when aggregate dividend payments are large, even though the size of these payments is known more than a month in advance. HS's explanation, which is backed by a number of additional tests, is that many institutional investors reinvest dividend payments, generating buying pressure. HS also show that stocks of firms with large stock grants to their employees drop on the days after earnings announcements, even though the size of these grants is known well in advance. HS's explanation is that employees are allowed to sell their shares after earnings announcements, generating selling pressure.

In principle, the dividend reinvestment effect in HS could attract *round-trip arbitrage*. Realizing that stock prices rise predictably on days of large dividend payments, arbitrageurs could buy the aggregate stock market on the day before the payments and sell on the day after. Transaction costs dent significantly the profitability of this strategy, however. HS find that the average abnormal return across the fifty largest payment days is 6bps. Trading commissions for S& P500 futures are 0.25bps and price impact costs for a \$100 million order are 1.25bps.[1] This results in round-trip transaction costs of 3bps. While these costs are low in absolute terms, they eat up half of the 6bps profit, leaving an abnormal return of only 150bps (=3× 50) per year for the 50 largest payment days.

[1] https://www.cmegroup.com/trading/equity-index/report-a-cost-comparison-of-futures-and-etfs.html

Although transaction costs dent significantly the profitability of round-trip arbitrage, they have no effect on *timing arbitrage*. Suppose that a market order to buy hits the market on the day before a large dividend payment. Why should a liquidity provider sell to meet that order knowing that prices will rise predictably on the next day? Transaction costs are immaterial here: the liquidity provider will incur them one day or the next. Hence, HS's finding that the aggregate stock market rises on days when aggregate dividend payments are large seems puzzling in a rational world where investors know about the dividend payments. This note suggests a resolution to this puzzle. It lays out a simple model in which there are anticipated demand shocks, and derives their price multiplier. It also compares the price multiplier of anticipated shocks to that of unanticipated shocks, and determines the drivers of each multiplier.

2 Model

There is an infinite number of discrete trading periods $t = 0, 1, ...$ The riskless rate between periods is constant and equal to $r > 0$. A risky asset pays a dividend d_t in period t, and is in supply of $\theta > 0$ shares. The dividend follows the random walk

$$d_{t+1} = d_t + \epsilon_{d,t+1}, \tag{1}$$

where the shocks $\epsilon_{d,t}$ are normally distributed with mean zero and variance σ_d^2, and are independent across periods.

There is a single competitive investor with CARA utility over intertemporal consumption

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \exp(-\alpha c_t - \beta t).$$

In period $t_0 > 0$, there is a supply shock of u shares. This shock is generated by unmodelled noise traders. The shock is normally distributed with mean $\bar{u}$ and variance σ_u^2, and is independent of the dividend shocks $\epsilon_{d,t}$. If $\bar{u} \neq 0$, the shock is anticipated, in the sense that the investor expects sales if $\bar{u} > 0$ and purchases if $\bar{u} < 0$. The exact size of the shock is uncertain as long as $\sigma_u^2 > 0$. The shock reverts to zero at a rate κ_u. The mean-reversion rate can represent the entry of offsetting orders in the market, as in Grossman and Miller (1988) and Duffie (2010).

3 Equilibrium

3.1 No Supply Shock

Consider first the case where there is no supply shock. This corresponds to the special case of the model where $\bar{u} = \sigma_u = 0$. We look for an equilibrium price of the risky asset of the form

$$S_t = \frac{d_t}{r} - Z, \tag{2}$$

where Z is a constant. The price in period t is equal to the present value of expected future dividends discounted at the riskless rate r, minus a discount Z. We look for a value function for the investor of the form

$$V(W_t) = -\exp(-AW_t - B), \tag{3}$$

where (A, B) are constants.

Proposition 1. *The equilibrium price and value function are given by (2) and (3), respectively, with*

$$Z = \frac{1}{r} A\theta \frac{(1+r)^2}{r^2} \sigma_d^2, \tag{4}$$

$$A = \frac{r\alpha}{1+r}, \tag{5}$$

$$B = \frac{1}{r} \left[\beta + r\log(r) - (1+r)\log(1+r) + \frac{1}{2} A^2 \theta^2 \frac{(1+r)^2}{r^2} \sigma_d^2 \right]. \tag{6}$$

In equilibrium, the price discount Z is constant over time. The expected return of risky asset in excess of riskless rate is

$$\mathbb{E}_t(R_{t+1}) = \mathbb{E}_t \left[d_{t+1} + S_{t+1} - (1+r)S_t \right] = rZ = A\theta \frac{(1+r)^2}{r^2} \sigma_d^2,$$

and is also constant over time.

3.2 Supply Shock – Dynamics After the Shock

Consider next the case where there is a supply shock. The shock evolves according to $u_{t_0} = u$ and

$$u_{t+1} = u_t(1 - \kappa_u), \tag{7}$$

for $t \geq t_0$. We look for an equilibrium price of the risky asset at $t \geq t_0$ of the form

$$S_t = \frac{d_t}{r} - Z - Z_u u_t, \tag{8}$$

where Z is given by (4) and Z_u is a constant. We look for a value function for the investor at $t \geq t_0$ of the form

$$V(W_t) = -\exp\left(-AW_t - B - B_u u_t - B_{uu} u_t^2\right), \tag{9}$$

where A and B are given by (5) and (6), respectively, and (B_u, B_{uu}) are constants.

Proposition 2. *The equilibrium price and value function are given by (8) and (9), respectively, with*

$$Z_u = \frac{1}{r+\kappa_u} A \frac{(1+r)^2}{r^2} \sigma_d^2, \tag{10}$$

$$B_u = \frac{1}{r+\kappa_u} A^2 \theta \frac{(1+r)^2}{r^2} \sigma_d^2, \tag{11}$$

$$B_{uu} = \frac{1}{2(r+2\kappa_u-\kappa_u^2)} A^2 \frac{(1+r)^2}{r^2} \sigma_d^2. \tag{12}$$

Following a positive supply shock u, the price (8) of the risky asset decreases, and then increases gradually to its no-shock value. The asset's expected return

$$\mathbb{E}_t(R_{t+1}) = A(\theta + u_t) \frac{(1+r)^2}{r^2} \sigma_d^2$$

increases following the shock, and then decreases gradually to its no-shock value.

The constant Z_u is equal to the price multiplier of an unanticipated shock. Indeed, we can define the price effect of the shock as $S_{t_0} - (1+r)S_{t_0-1}$, i.e., the difference in price adjusted by riskless discounting. For an unanticipated shock, the price S_{t_0-1} is given by (2) as in the no-shock case. Therefore, (8) and (2) imply that the shock's price effect is

$$S_{t_0} - (1+r)S_{t_0-1} = Z_u u_t.$$

Equation (10) shows that the price multiplier of an unanticipated shock increases in the risk-aversion coefficient A in the value function, the variance σ_d^2 of dividend shocks, and the persistence $\frac{1}{\kappa_u}$ of the supply shock.

3.3 Supply Shock – Dynamics Before the Shock

We next determine the equilibrium before the shock hits. We look for an equilibrium price of the risky asset at $t < t_0$ of the form

$$S_t = \frac{d_t}{r} - Z - z_t, \tag{13}$$

where Z is given by (4) and z_t is a deterministic function of t. We look for a value function for the investor at $t < t_0$ of the form

$$V(W_t) = -\exp(-AW_t - B - b_t), \tag{14}$$

where A and B are given by (5) and (6), respectively, and b_t is a deterministic function of t.

Proposition 3. *The equilibrium price and value function are given by (13) and (14), respectively, with*

$$z_t = \begin{cases} \frac{z_{t_0-1}}{(1+r)^{t_0-1-t}} & \text{for } t < t_0 - 1 \\ \frac{Z_u \bar{u}}{1+r} \frac{1}{1+x} & \text{for } t = t_0 - 1 \end{cases}, \tag{15}$$

$$b_t = \begin{cases} \frac{b_{t_0-1}}{(1+r)^{t_0-1-t}} & \text{for } t < t_0 - 1 \\ \frac{(B_u + B_{uu}\bar{u})\bar{u}\sigma_u^2}{(1+r)(1+2B_{uu}\sigma_u^2)} + \frac{1}{2}\log\left(1 + 2B_{uu}\sigma_u^2\right) & \text{for } t = t_0 - 1 \end{cases}, \tag{16}$$

$$x = \frac{r + \kappa_u}{r + 2\kappa_u - \kappa_u^2} A Z_u \sigma_u^2. \tag{17}$$

In anticipation of a positive supply shock u, i.e., when the shock's expectation $\bar{u}$ is positive, the price (8) of the risky asset decreases gradually until $t_0 - 1$ because the discount z_t increases. The price then decreases more sharply at t_0. The asset's expected return is equal to its no-shock value

$$\mathbb{E}_t(R_{t+1}) = A\theta \frac{(1+r)^2}{r^2} \sigma_d^2$$

for $t < t_0 - 1$, and to

$$\mathbb{E}_{t_0-1}(R_{t_0}) = A\theta \frac{(1+r)^2}{r^2} \sigma_d^2 - Z_u \bar{u} \frac{x}{1+x}$$

for $t = t_0 - 1$.

Equations (8) and (13) imply that the expected price effect of an anticipated shock is

$$\mathbb{E}_{t_0-1}S_{t_0} - (1+r)S_{t_0-1} = Z_u \bar{u} \frac{x}{1+x}.$$

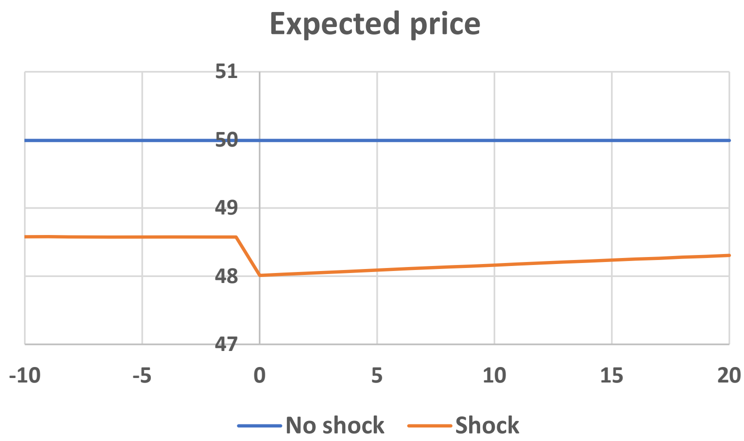
The effect is non-zero, with price multiplier $Z_u \frac{x}{1+x}$. The price multiplier of an anticipated shock is equal to the price multiplier Z_u of an unanticipated shock, times $\frac{x}{1+x}$. The ratio $\frac{x}{1+x}$ of price multipliers increases in x. In turn, x increases in the risk-aversion coefficient A in the value function, the variance σ_u^2 of the supply shock, and the price multiplier Z_u of an unanticipated shock. Two comparative statics that ensue are (i) the price multiplier of an anticipated shock increases in the shock's variance σ_u^2, and (ii) the price multiplier of an anticipated shock is a convex function of market liquidity, as measured by the price multiplier Z_u of an unanticipated shock.

Figure 1 plots the price effect of an anticipated shock. In the no-shock case, the average price, represented by the blue line, is constant over time. In the case where there is a shock, the average price declines slightly until the day before the shock hits (Day -1), and more sharply on the day when the shock hits (Day 0). It then increases gradually to its no-shock value. The price effect of the anticipated shock is 1.22 and the price effect of a non-anticipated shock of equal magnitude is 1.98.

Figure 1: Price Effect of Anticipated Shock

Figure 2 plots the price effect of an anticipated shock when the shock's standard deviation is half as large. The effect of the shock is better anticipated in the price before the shock. The price effect of the anticipated shock drops to 0.57. The price effect of a non-anticipated shock does not change.

Figure 2: Price Effect of Anticipated Shock



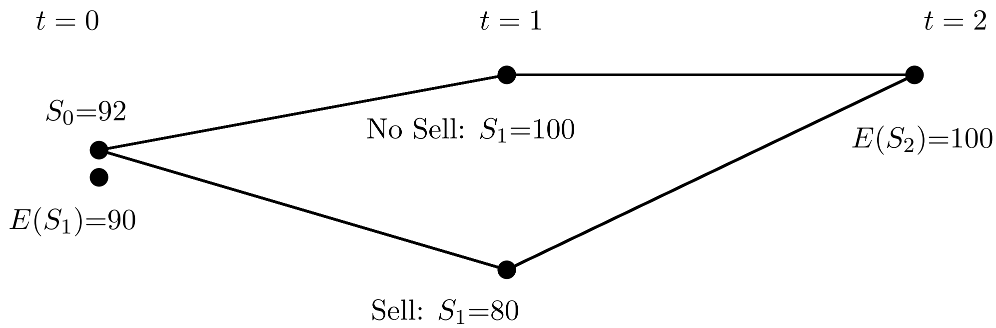
3.4 Bird-in-the-Hand Effect

Why is an anticipated supply shock not fully reflected into the price just before the shock hits? Why is its price multiplier non-zero? Why do rational traders trade against the shock before the shock hits, rather than waiting to trade at a better price on average after it hits? The explanation provided by the model presented in this note goes back to the theory of rational momentum in Vayanos and Woolley (2013, VW).

Consider the following three-period example, shown in Figure 3. An asset is expected to pay off 100 in Period 2. If a large sell order does not materialize in Period 1, then the price will be 100. If instead the sell order materializes, then the price will drop to 80. Each scenario is equally likely. Buying the asset in Period 0 at 92 earns an investor a two-period expected capital gain of 8. Buying in Period 1 earns an expected capital gain of 20 if outflows occur and 0 if they do not. A risk-averse investor might prefer earning 8 rather than 20 or 0 with equal probabilities. Hence, the investor might prefer to buy the asset before the sell order hits rather than waiting to buy after it hits, even though the price will decline on average when it hits. VW term this the *bird-in-the-hand* effect: the investor goes for the bird in the hand by buying at $t = 0$, rather than for two birds in the bush by buying at $t = 1$.

The bird-in-the-hand effect can be interpreted formally in the language of Merton's ICAPM. The investor buys an underpriced asset even though the price is expected to drop even further in the short term, to hedge against a reduction in the mispricing. The state in period 1 where the sell order hits is a low marginal utility one for the investor, as the investor gains from large mispricing.

Figure 3: Bird-in-the-Hand Effect



Coversely, the state in period 1 where the sell order does not hit is a high marginal utility one because there is no mispricing to be exploited.

The mapping with the ICAPM in context of the model presented in this note can be seen through the value function (9) at time $t \geq t_0$ after the shock hits. The value function depends on the supply shock u_t, which is a state variable in addition to wealth W_t. Large values of u_t in either direction raise the value function because of the quadratic term $B_{uu}u_t^2$. Intuitively, the value function is high because the investor can achieve large gains by trading a highly mispriced asset.

VW build a theory of rational momentum and reversal based on the price effect of anticipated demand/supply shocks. In VW, flows between mutual funds chase performance, and are gradual. Following underperformance by an asset, funds holding this asset experience gradual outflows. These outflows cause the prices of the assets held by the funds to drop gradually and predictably because of the bird-in-the-hand effect. The drop concerns also the asset hit by the original shock, causing asset-level momentum. Note that whether flows are between institutions rather than between retail investors is not key to the argument: what is key is that flows chase performance and are gradual. Empirical studies linking momentum to flows through this mechanism are Lou (2012) and Ben-David, Li, Rossi, and Song (2021).

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Appendix

Proof of Proposition 1: Denoting by x_t the number of shares of the risky asset held by the investor in period t, we can write the investor's budget constraint as

$$W_{t+1} = (1+r)(W_t - c_t) + x_t \left[d_{t+1} + S_{t+1} - (1+r)S_t\right]. \tag{A.1}$$

Substituting the price S_t from (2) into (A.1), and using (1), we find

$$W_{t+1} = (1+r)(W_t - c_t) + x_t \left(\frac{1+r}{r}\epsilon_{d,t+1} + rZ\right). \tag{A.2}$$

The investor's Bellman equation is

$$V(W_t) = \max_{c_t, x_t} \left[-\exp(-\alpha c_t) + \exp(-\beta)\mathbb{E}_t V(W_{t+1})\right]. \tag{A.3}$$

Substituting the value function $V(W_t)$ from (3) and the wealth W_{t+1} from (A.2) into (A.3), we find

$$\begin{aligned} -\exp(-AW_t - B) = \max_{c_t, x_t} \Bigg[& -\exp(-\alpha c_t) \\ & -\mathbb{E}_t \exp\left(-A\left[(1+r)(W_t - c_t) + x_t\left(\frac{1+r}{r}\epsilon_{d,t+1} + rZ\right)\right] - B - \beta\right)\Bigg]. \end{aligned} \tag{A.4}$$

Taking expectations in the right-hand side of (A.4) over the shock $\epsilon_{d,t+1}$, which is normally distributed with mean zero and variance σ_d^2, we find

$$- \exp(-AW_t - B) = \max_{c_t, x_t} \Bigg[- \exp(-\alpha c_t)$$

$$- \exp\left(-A \left[(1+r)(W_t - c_t) + x_t r Z - \frac{1}{2} x_t^2 A \frac{(1+r)^2}{r^2} \sigma_d^2 \right] - B - \beta \right) \Bigg]. \tag{A.5}$$

The first-order condition with respect to x_t yields

$$rZ = x_t A \frac{(1+r)^2}{r^2} \sigma_d^2. \tag{A.6}$$

The first-order condition with respect to c_t yields

$$\alpha \exp(-\alpha c_t) = A(1+r) \exp\left(-A \left[(1+r)(W_t - c_t) + x_t r Z - \frac{1}{2} x_t^2 A \frac{(1+r)^2}{r^2} \sigma_d^2 \right] - B - \beta \right) \tag{A.7}$$

$$\Rightarrow c_t = \frac{\log\left(\frac{\alpha}{A(1+r)}\right) + A\left[(1+r) W_t + x_t r Z - \frac{1}{2} x_t^2 A \frac{(1+r)^2}{r^2} \sigma_d^2 \right] + B + \beta}{\alpha + A(1+r)}. \tag{A.8}$$

Market clearing implies

$$x_t = \theta. \tag{A.9}$$

Substituting x_t from (A.9) into (A.6), we find (4). Substituting $\exp(-\alpha c_t)$ from (A.7) into (A.5), and using (A.6) and (A.9), we find

$$\exp(-AW_t - B) = \frac{\alpha + A(1+r)}{\alpha} \exp\left(-A \left[(1+r)(W_t - c_t) + \frac{1}{2} A \theta^2 \frac{(1+r)^2}{r^2} \sigma_d^2 \right] - B - \beta \right). \tag{A.10}$$

Substituting c_t from (A.8) into (A.10), and using (A.6) and (A.9), we find

$$\exp(-AW_t - B) = \frac{\alpha + A(1+r)}{\alpha}$$

$$\times \exp\left(- \frac{\alpha A \left[(1+r) W_t + \frac{1}{2} A \theta^2 \frac{(1+r)^2}{r^2} \sigma_d^2 \right] + \alpha(B+\beta) - A(1+r) \log\left(\frac{\alpha}{A(1+r)}\right)}{\alpha + A(1+r)} \right). \tag{A.11}$$

Identifying terms in W_t on both sides of (A.11), we find

$$A = \frac{\alpha A(1+r)}{\alpha + A(1+r)},$$

which implies (5). Identifying the remaining terms, we find

$$B = \log\left(\frac{\alpha}{\alpha + A(1+r)}\right) + \frac{\frac{1}{2}\alpha A^2\theta^2\frac{(1+r)^2}{r^2}\sigma_d^2 + \alpha(B+\beta) - A(1+r)\log\left(\frac{\alpha}{A(1+r)}\right)}{\alpha + A(1+r)},$$

which, combined with (5), implies (6). ∎

Proof of Proposition 2: The counterparts of (A.2) and (A.5) are

$$W_{t+1} = (1+r)(W_t - c_t) + x_t\left(\frac{1+r}{r}\epsilon_{d,t+1} + rZ + (r+\kappa_u)Z_u u_t\right) \tag{A.12}$$

and

$$\begin{aligned} -\exp\bigl(-AW_t - B - B_u u_t - B_{uu}u_t^2\bigr) = \max_{c_t,x_t}\Biggl[-\exp(-\alpha c_t) - \exp\Biggl(-A\Biggl[(1+r)(W_t - c_t) \\ +x_t[rZ + (r+\kappa_u)Z_u u_t] - \frac{1}{2}x_t^2 A\frac{(1+r)^2}{r^2}\sigma_d^2\Biggr] - B - B_u u_t(1-\kappa_u) - B_{uu}u_t^2(1-\kappa_u)^2 - \beta\Biggr)\Biggr], \end{aligned} \tag{A.13}$$

respectively.

The counterparts of (A.6) and (A.9) are

$$rZ + (r+\kappa_u)Z_u u_t = x_t A\frac{(1+r)^2}{r^2}\sigma_d^2 \tag{A.14}$$

and

$$x_t = \theta + u_t, \tag{A.15}$$

respectively. Substituting x_t from (A.15) into (A.14), and identifying terms in u_t, we find (10). Identifying the remaining terms, we find (4).

The counterpart of (A.11) is

$$\exp\left(-AW_t - B - B_u u_t - B_{uu}u_t^2\right) = \frac{\alpha + A(1+r)}{\alpha}\exp\left(-\frac{\alpha A\left[(1+r)W_t + \frac{1}{2}A(\theta+u_t)^2\frac{(1+r)^2}{r^2}\sigma_d^2\right]}{\alpha + A(1+r)}\right)$$

$$\times \exp\left(-\frac{\alpha(B + B_u u_t(1-\kappa_u) + B_{uu}u_t^2(1-\kappa_u)^2 + \beta) - A(1+r)\log\left(\frac{\alpha}{A(1+r)}\right)}{\alpha + A(1+r)}\right). \tag{A.16}$$

Identifying terms in W_t on both sides of (A.11), we find (5). Identifying terms in u_t, we find

$$B_u = \frac{\alpha A^2\theta\frac{(1+r)^2}{r^2}\sigma_d^2 + \alpha B_u(1-\kappa_u)}{\alpha + A(1+r)},$$

which combined with (5) implies (11). Identifying terms in u_t^2, we find

$$B_{uu} = \frac{\frac{1}{2}\alpha A^2\frac{(1+r)^2}{r^2}\sigma_d^2 + \alpha B_{uu}(1-\kappa_u)^2}{\alpha + A(1+r)},$$

which combined with (5) implies (12). Identifying the remaining terms and using (5), we find (6). ∎

To prove Proposition 3, we recall the following lemma (e.g., Lemma A.1 in Vayanos and Wang (2012)).

Lemma A.1. *Let x be an $n \times 1$ normal vector with mean zero and covariance matrix Σ, A a scalar, B an $n \times 1$ vector, C an $n \times n$ symmetric matrix, I the $n \times n$ identity matrix, and $|M|$ the determinant of a matrix M. Then,*

$$\mathbb{E}_x \exp\left\{-\alpha\left[A + B'x + \frac{1}{2}x'Cx\right]\right\} = \exp\left\{-\alpha\left[A - \frac{1}{2}\alpha B'\Sigma(I+\alpha C\Sigma)^{-1}B\right]\right\}\frac{1}{\sqrt{|I+\alpha C\Sigma|}}. \tag{A.17}$$

Proof: When $C = 0$, (A.17) gives the moment-generating function of the normal distribution. We can always assume $C = 0$ by also assuming that x is a normal vector with mean 0 and covariance matrix $\Sigma(I + \alpha C\Sigma)^{-1}$. ∎

Proof of Proposition 3: Suppose first $t < t_0 - 1$. The counterparts of (A.2) and (A.5) are

$$W_{t+1} = (1+r)(W_t - c_t) + x_t \left(\frac{1+r}{r} \epsilon_{d,t+1} + rZ + (1+r)z_t - z_{t+1} \right) \tag{A.18}$$

and

$$\begin{aligned} - \exp(-AW_t - B - b_t) = \max_{c_t, x_t} \Bigg[&- \exp(-\alpha c_t) - \exp \Bigg(-A \Bigg[(1+r)(W_t - c_t) \\ &+ x_t [rZ + (1+r)z_t - z_{t+1}] - \frac{1}{2} x_t^2 A \frac{(1+r)^2}{r^2} \sigma_d^2 \Bigg] - B - b_{t+1} - \beta \Bigg) \Bigg], \end{aligned} \tag{A.19}$$

respectively.

The counterpart of (A.6) is

$$rZ + (1+r)z_t - z_{t+1} = x_t A \frac{(1+r)^2}{r^2} \sigma_d^2, \tag{A.20}$$

and (A.9) carries through unchanged. Substituting x_t from (A.9) into (A.20), and using (4), we find (15) for $t < t_0 - 1$.

The counterpart of (A.11) is

$$\begin{aligned} \exp(-AW_t - B - b_t) = &\frac{\alpha + A(1+r)}{\alpha} \\ &\times \exp \left(- \frac{\alpha A \left[(1+r)W_t + \frac{1}{2} A \theta^2 \frac{(1+r)^2}{r^2} \sigma_d^2 \right] + \alpha(B + b_{t+1} + \beta) - A(1+r) \log \left(\frac{\alpha}{A(1+r)} \right)}{\alpha + A(1+r)} \right). \end{aligned} \tag{A.21}$$

Identifying terms in W_t on both sides of (A.21), we find (5). Identifying the remaining terms and using (5) and (6), we find (16) for $t < t_0 - 1$.

Suppose next $t = t_0 - 1$. Using (8) for $t = t_0$ and (13) for $t = t_0 - 1$, we can write the counterpart of (A.2) as

$$W_{t_0} = (1+r)(W_{t_0-1} - c_{t_0-1}) + x_{t_0-1} \left(\frac{1+r}{r} \epsilon_{d,t_0} + rZ + (1+r)z_{t_0-1} - Z_u u \right). \tag{A.22}$$

Using (9) for $t = t_0$ and (14) for $t = t_0 - 1$, we can write the counterpart of (A.5) as

$$
\begin{aligned}
-\exp(-AW_{t_0-1} - B - b_{t_0-1}) = \max_{c_{t_0-1}, x_{t_0-1}} \Bigg[-\exp(-\alpha c_{t_0-1}) - \mathbb{E}_u \exp\Bigg(-A \Bigg[(1+r)(W_{t_0-1} - c_{t_0-1}) \\
+ x_{t_0-1}[rZ + (1+r)z_{t_0-1} - Z_u u] - \frac{1}{2} x_{t_0-1}^2 A \frac{(1+r)^2}{r^2} \sigma_d^2 \Bigg] - B - B_u u - B_{uu} u^2 - \beta \Bigg) \Bigg].
\end{aligned}
\tag{A.23}
$$

respectively. The key difference between (A.23) and its counterparts for all other cases is that expectation must be taken over u. To compute that expectation, we use Lemma A.1 and set

$$
\begin{aligned}
& x \equiv u - \bar{u}, \\
& \Sigma \equiv \sigma_u^2, \\
& \alpha \equiv 1, \\
& A \equiv A \Bigg[(1+r)(W_{t_0-1} - c_{t_0-1}) + x_{t_0-1}[rZ + (1+r)z_{t_0-1} - Z_u \bar{u}] - \frac{1}{2} x_{t_0-1}^2 A \frac{(1+r)^2}{r^2} \sigma_d^2 \Bigg] \\
& \qquad + B + B_u \bar{u} + B_{uu} \bar{u}^2 + \beta, \\
& B \equiv B_u + 2B_{uu}\bar{u} - A x_{t_0-1} Z_u, \\
& C \equiv 2B_{uu}.
\end{aligned}
$$

Using Lemma A.1, we can write (A.23) as

$$
\begin{aligned}
-\exp(-AW_{t_0-1} - B - b_{t_0-1}) = \max_{c_{t_0-1}, x_{t_0-1}} \Bigg[-\exp(-\alpha c_{t_0-1}) - \exp\Bigg(-A \Bigg[(1+r)(W_{t_0-1} - c_{t_0-1}) \\
+ x_{t_0-1}[rZ + (1+r)z_{t_0-1} - Z_u \bar{u}] - \frac{1}{2} x_{t_0-1}^2 A \frac{(1+r)^2}{r^2} \sigma_d^2 \Bigg] - B - B_u \bar{u} - B_{uu} \bar{u}^2 - \beta \\
+ \frac{(B_u + 2B_{uu}\bar{u} - A x_{t_0-1} Z_u)^2 \sigma_u^2}{2(1 + 2B_{uu}\sigma_u^2)} \Bigg) \frac{1}{\sqrt{1 + 2B_{uu}\sigma_u^2}} \Bigg].
\end{aligned}
\tag{A.24}
$$

The counterpart of (A.6) is

$$
rZ + (1+r)z_{t_0-1} - Z_u \bar{u} = x_{t_0-1} A \frac{(1+r)^2}{r^2} \sigma_d^2 - \frac{Z_u (B_u + 2B_{uu}\bar{u} - A x_{t_0-1} Z_u) \sigma_u^2}{1 + 2B_{uu}\sigma_u^2}, \tag{A.25}
$$

and (A.9) carries through unchanged. Substituting x_t from (A.9) into (A.25), and using (4), (11) and (12), we find (15) for $t = t_0 - 1$.

The counterpart of (A.11) is

$$\exp(-AW_{t_0-1} - B - b_{t_0-1}) = \frac{\alpha + A(1+r)}{\alpha} \exp\left(-\frac{\alpha A\left[(1+r)W_t + \frac{1}{2}A\theta^2\frac{(1+r)^2}{r^2}\sigma_d^2\right]}{\alpha + A(1+r)}\right)$$

$$\times \exp\left(-\frac{\alpha(B + B_u\bar{u} + B_{uu}\bar{u}^2 + \beta) - A(1+r)\log\left(\frac{\alpha}{A(1+r)}\right)}{\alpha + A(1+r)}\right)$$

$$\times \exp\left(\frac{\alpha\frac{(B_u+2B_{uu}\bar{u}-A\theta Z_u)(B_u+2B_{uu}\bar{u}+A\theta Z_u)\sigma_u^2}{2(1+2B_{uu}\sigma_u^2)}}{\alpha + A(1+r)}\right)\frac{1}{\sqrt{1+2B_{uu}\sigma_u^2}}. \tag{A.26}$$

Identifying terms in W_t on both sides of (A.26), we find (5). Identifying the remaining terms and using (5), (6), (10), (11) and (12) we find (16) for $t = t_0 - 1$. ∎