Equilibrium Interest Rate and Liquidity Premium with Transaction Costs*

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Abstract

In this paper we study the effects of transaction costs on asset prices. We assume an overlapping generations economy with two riskless assets. The first asset is liquid while the second asset carries proportional transaction costs. We show that agents buy the liquid asset for short-term investment and the illiquid asset for long-term investment. When transaction costs increase, the price of the liquid asset increases. The price of the illiquid asset decreases if the asset is in small supply, but may increase if the supply is large. These results have implications for the effects of transaction taxes and commission deregulation.

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1 Introduction

Transaction costs such as bid-ask spreads, brokerage commissions, exchange fees, and transaction taxes, are important in many financial markets.[1] Considerable attention has focused on their impact on asset prices and subsequent investment decisions. For instance, how would a transaction tax, such as the one that has been proposed for the US, affect asset prices?[2] How would information technology and financial market deregulation, both of which reduce transaction costs, affect asset prices?[3]

Although transaction costs are mentioned in many asset pricing debates, they are generally absent from asset pricing models. Starting with Constantinides (1986), some papers study the optimal policy of an agent who invests in a riskless, liquid asset, and a risky, illiquid asset.[4] These papers treat asset prices as exogenous. Amihud and Mendelson (1986), Aiyagari and Gertler (1991), Huang (1998), and Vayanos (1998) endogenize asset prices, assuming a riskless, liquid asset, and one or more illiquid assets.[5] However, these papers treat the price of the liquid asset as exogenous, and only determine the price of the illiquid asset *relative* to the liquid asset. Heaton and Lucas (1996) assume a riskless, liquid asset, and a risky, illiquid asset. They endogenize the prices of both assets, but have to resort to numerical methods.

In this paper we develop a general equilibrium model with transaction costs. We assume an overlapping generations economy with two riskless assets and a numéraire consumption good. The first asset is liquid while the second asset carries proportional transaction costs. Agents receive labor income and trade the assets for life-cycle purposes. In contrast to other equilibrium models with transaction costs, we endogenize the prices of both the liquid and the illiquid asset. Moreover, our model is very simple and tractable. Our assumptions of riskless assets and life-cycle trading, which are made for tractability, are admittedly special. At the same time, our assumptions on agents' preferences and labor income streams are very general. The only restriction that we impose, is that without transaction costs there exists an equilibrium where agents' wealth is increasing and then decreasing with age.

We show that with transaction costs, agents first buy the illiquid asset, next buy the liquid asset, then sell the liquid asset, and finally sell the illiquid asset. For a short holding period transaction costs are important and the liquid asset is the better investment, despite being more expensive than the illiquid asset. For a long period transaction costs are less important and the illiquid asset is the better investment. As in Amihud and Mendelson (1986), each asset has its own clientele.

When the costs of trading the illiquid asset increase, the price of the liquid asset increases. The price of the illiquid asset changes because of two effects that work in opposite directions. The first effect is that the price of the illiquid asset decreases *relative* to the price of the liquid asset. The second effect is that the price of the liquid asset increases. If the supply of the illiquid asset is small relative to the supply of the liquid asset, the second effect is weak and the price of the illiquid asset decreases. However, if the supply is large, the second effect is stronger and, surprisingly, the price of the illiquid asset may increase. Even when it decreases, it generally changes less than the price of the liquid asset.

Our results imply that a change in transaction costs for a significant fraction of assets may have a stronger effect on the remaining assets rather than on those subject to the change. Therefore a "partial equilibrium" analysis that assumes that the price of the remaining assets stays constant, will be incorrect. As a practical application, consider a transaction tax on most financial securities. We show that the tax increases the price of the non-taxed securities by as much as it decreases the price of the taxed ones. Therefore an analysis that erroneously assumes that the price of the non-taxed securities stays constant in spite of the tax, will overestimate the effect of the tax by 100%.

The rest of the paper is structured as follows. In section 2, we present the model. In section 3 we state the agents' optimization problem, and the market-clearing conditions. In section 4 we study the benchmark case of zero transaction costs. In section 5 we consider the case of nonzero transaction costs. We first study the agents' optimization problem and construct an equilibrium. We then study the effects of transaction costs on asset prices and prove our main results. Section 6 concludes, and all proofs are in the Appendix.

2 The Model

We consider a continuous time overlapping generations economy. Time, t, goes from $-\infty$ to ∞. There is a continuum of agents. Each agent lives for an interval of length T. Between times t and $t + dt$, dt/T agents are born and dt/T die. The total population is thus 1.

2.1 Financial Structure

Agents can invest in two financial assets. Both assets pay dividends at a constant rate D. The first asset is liquid and does not carry transaction costs, while the second asset is illiquid. The total number of shares (i.e. the total supply) of the two assets is normalized to 1. The supply of the liquid asset is $1 - k$ ($0 < k < 1$), its price is p, and its rate of return is $r = D/p$. Similarly, the price of the illiquid asset is P and its rate of return is $R = D/P$. The illiquid asset carries transaction costs that are proportional to the value traded, i.e. the costs of buying or selling x shares of the illiquid asset are $\epsilon x P$, with $\epsilon \geq 0$. We assume that transaction costs are "real", i.e. transactions consume resources. In section 5.3.4 we examine the case where transaction costs are due to taxes that are distributed back to the agents. Finally, to capture the fact that short-sale costs are much higher than transaction costs,[6] we assume that the assets cannot be sold short.

An agent of age t holds x_t and X_t shares of the liquid and illiquid assets, respectively. His "liquid wealth", i.e. the dollar value of the shares of the liquid asset, is $a_t = px_t$, his "illiquid wealth" is $A_t = PX_t$, and his total wealth is $w_t = a_t + A_t$. His dollar investment in the liquid asset is $i_t = p(dx_t/dt)$, and his dollar investment in the illiquid asset is $I_t = P(dX_t/dt)$.

Note that we are assuming that asset prices are constant, and are thus focusing on stationary equilibria. This assumption is not without loss of generality. It is well-known[7] that overlapping generations models may have bubbles (such as money) and multiple equilibria (such as cycles and sunspots). The presence of long-lived assets rules out bubbles but not multiple equilibria.

2.2 Preferences and Endowments

Agents derive utility from lifetime consumption of a consumption good. An agent of age t consumes at a rate c_t, where $c_t \geq 0$. Utility over consumption is

$$\int_0^T u(c_t, t)dt.$$

We assume that the felicity function, $u(c,t)$, is C^2 on $(0,\infty) \times [0,T]$, and that

$$\frac{\partial u}{\partial c}(c,t) > 0, \qquad \frac{\partial^2 u}{\partial c^2}(c,t) < 0, \qquad \lim_{c \to 0} \frac{\partial u}{\partial c}(c,t) = \infty.$$

Therefore the function $c \to \partial u(c,t)/\partial c \equiv q$ is invertible, the inverse function, $v(q,t)$, is C^1, and

$$\frac{\partial v}{\partial q}(q,t) < 0.$$

Agents are born with zero financial wealth and receive labor income over their lifetimes. An agent of age t receives labor income at a rate y_t, where $y_t \geq 0$ is a C^1 function of t.

3 Optimization and Market-Clearing

In this section we state the agents' optimization problem, and the market-clearing conditions.

3.1 The Optimization Problem

An agent's optimization problem, $(\mathcal{P})$, is

$$\sup_{(i_t,I_t)} \int_0^T u(c_t,t)dt,$$

subject to (i_t, I_t) piecewise continuous,

$$\frac{da_t}{dt} = i_t, \qquad a_0 = 0, \tag{3.1}$$

$$\frac{dA_t}{dt} = I_t, \qquad A_0 = 0, \tag{3.2}$$

$$c_t = ra_t + RA_t + y_t - i_t - I_t - \epsilon|I_t|, \tag{3.3}$$

and the short-sale constraints, $a_t \geq 0$ and $A_t \geq 0$. Equation 3.3 states that consumption, c_t, is dividend income, $ra_t + RA_t = D(x_t + X_t)$, plus labor income, y_t, minus investment in the liquid and illiquid assets, i_t and I_t, minus transaction costs, $\epsilon|I_t| = \epsilon P|dX_t/dt|$. We call feasible a control (i_t, I_t) that satisfies all the constraints.

3.2 The Market-Clearing Conditions

The market-clearing condition for the liquid asset is that the total number of shares that agents hold at a given time is equal to the asset's supply, $1-k$. To compute the total number of shares, we note that there are dt/T agents with age between t and $t+dt$, and each holds x_t shares. Therefore the market-clearing condition is

$$\int_0^T x_t \frac{dt}{T} = (1-k). \tag{3.4}$$

Multiplying both sides of equation 3.4 by p, we get

$$\int_0^T a_t \frac{dt}{T} = (1-k)p = (1-k)\frac{D}{r}. \tag{3.5}$$

Equation 3.5 states that the total liquid wealth at a given time is equal to the market value of the liquid asset. Similarly, we can write the market-clearing condition for the illiquid asset as

$$\int_0^T A_t \frac{dt}{T} = k\frac{D}{R}. \tag{3.6}$$

An equilibrium consists of a quadruplet of rates of return and investments, (r, R, i_t, I_t), such that (i) i_t and I_t solve $(\mathcal{P})$ for r and R, and (ii) r, R, and the liquid and illiquid wealth, a_t and A_t, defined by equations 3.1 and 3.2, satisfy the market-clearing conditions 3.5 and 3.6.

4 Equilibrium Without Transaction Costs

In this section we study the benchmark case of zero transaction costs ($\epsilon = 0$). The liquid and illiquid assets are then identical and $r = R$. For simplicity, we make a "life-cycle" (LC) assumption, and ensure existence of an equilibrium where agents' total wealth, w_t, is (strictly) increasing and then decreasing with age. To motivate the formal statement of our (LC) assumption, we study the agents' optimization problem and the market-clearing conditions. An agent cares only about his total wealth, $w_t = a_t + A_t$, and chooses total investment, $i_t + I_t$, or equivalently consumption, c_t. The optimization problem, $(\mathcal{P})$, simplifies into

$$\sup_{c_t} \int_0^T u(c_t, t) dt$$

subject to

$$\frac{dw_t}{dt} = rw_t + y_t - c_t, \quad w_0 = 0, \tag{4.1}$$

and the short-sale constraint, $w_t \geq 0$. According to our (LC) assumption, optimal consumption is such that wealth is increasing and then decreasing. Therefore, the short-sale constraint is not binding in $(0, T)$, and optimal consumption is simply given by

$$\frac{\partial u}{\partial c}(c_t, t) = \lambda e^{-rt} \Rightarrow c_t = v(\lambda e^{-rt}, t), \tag{4.2}$$

where the Lagrange multiplier, λ, is determined by the intertemporal budget constraint

$$\int_0^T (c_t - y_t) e^{-rt} dt = 0. \tag{4.3}$$

Since the assets are identical, there is only one market-clearing condition. Total wealth at a given time is equal to the market value of the two assets, i.e.

$$\int_0^T w_t \frac{dt}{T} = \frac{D}{r}. \tag{4.4}$$

Our assumption thus is

Life-Cycle (LC) Assumption *There exists r^* such that if we define c_t^*, λ^*, and w_t^* by equations 4.2, 4.3, and 4.1 respectively, then the following are true. First, there exists $\tau^* \in (0, T)$ such that*

$$\frac{dw_t^*}{dt} > 0 \quad for\, all\;\; t \in [0, \tau^*), \quad and \quad \frac{dw_t^*}{dt} < 0 \quad for\, all\;\; t \in (\tau^*, T]. \tag{4.5}$$

Second, the market-clearing condition 4.4 holds.

Assumption (LC) is a joint assumption on the felicity function, $u(c,t)$, the labor income, y_t, and the dividend, D. In proposition 4.1 we show that assumption (LC) is satisfied when there is exponential discounting ($u(c,t) = u(c)e^{-\beta t}$), labor income, y_t, is decreasing, and financial wealth is large relative to labor income. The proposition is proven in appendix A.

Proposition 4.1 *Assumption (LC) is satisfied when $u(c,t) = u(c)e^{-\beta t}$, y_t is decreasing, and*

$$D > \frac{\int_0^T y_t e^{-\beta t} dt}{\int_0^T e^{-\beta t} dt} - \int_0^T y_t \frac{dt}{T}. \tag{4.6}$$

To prove proposition 4.1, we first show that wealth is increasing and then decreasing, if savings, defined as the difference between labor income and consumption, are decreasing. Since labor income is decreasing, savings are decreasing if consumption is increasing. Using exponential discounting, and denoting the inverse of $u'(c)$ by $v(q)$, we can write equation 4.2 as

$$c_t = v(\lambda e^{(\beta - r)t}). \tag{4.7}$$

Consumption is thus increasing if $r > \beta$. Equation 4.6 restricts financial wealth to be large, and ensures that in equilibrium $r > \beta$. Note that when labor income is constant, equation 4.6 becomes $D > 0$, and is always satisfied.

When financial wealth is small, r may be smaller than β, and savings may not be decreasing. Assumption (LC) may or may not be satisfied. In proposition 4.2 we show that assumption (LC) is satisfied for any financial wealth, when $u(c)$ exhibits constant elasticity of substitution ($u(c) = c^{1-A}/(1-A)$, $A \geq 0$),[8] and labor income declines exponentially ($y_t = ye^{-\delta t}$, $\delta \geq 0$). We also present a numerical example where assumption (LC) is not satisfied. In this example financial wealth is small, $u(c) = \log(c)$, and labor income declines linearly. Proposition 4.2 is proven in appendix A.

Proposition 4.2 *Assumption (LC) is satisfied when $u(c,t) = c^{1-A}/(1-A)e^{-\beta t}$, $A \geq 0$, and $y_t = ye^{-\delta t}$, $\delta \geq 0$. Assumption (LC) is not satisfied when $T = 50$, $u(c,t) = \log(c)e^{-\beta t}$, $\beta = 0.04$, $y_t = y(1 - t/T)$, and $D/y = 0.005$.*

5 Equilibrium With Transaction Costs

In this section we study the case of nonzero transaction costs ($\epsilon > 0$). We first study the agents' optimization problem and derive optimality conditions, i.e. sufficient conditions for a control (i_t, I_t) to be optimal. We then use these conditions, together with the market-clearing conditions, to construct an equilibrium for small transaction costs. We finally study the effects of transaction costs on asset prices and prove our main results.

5.1 Optimality Conditions

We study the optimization problem, $(\mathcal{P})$, for rates of return r and R such that $R > r$. We first derive the optimality conditions intuitively, and then show that they indeed imply that the control (i_t, I_t) solves $(\mathcal{P})$. We define μ and m by $\mu = R - r$ and $m = \mu/\epsilon$. μ is the *liquidity premium* and m is the liquidity premium per unit of transaction costs.

The optimal investment depends on the holding period. For a short period transaction costs are important, and the liquid asset is the better investment despite its lower rate of return. For a long period transaction costs are less important and the illiquid asset is the better investment. Therefore an agent buys the illiquid asset until an age τ_1, next buys the liquid asset, then sells the liquid asset until an age $\tau_1 + \Delta$, and finally sells the illiquid asset. The optimal control (i_t, I_t) is such that

$$i_t = 0 \text{ for all } t \in [0, \tau_1) \cup [\tau_1 + \Delta, T], \tag{5.1}$$

and

$$I_t > 0 \text{ for all } t \in [0, \tau_1), \quad I_t = 0 \text{ for all } t \in [\tau_1, \tau_1 + \Delta), \quad I_t < 0 \text{ for all } t \in [\tau_1 + \Delta, T]. \tag{5.2}$$

The agent buys his last share of the illiquid asset at age τ_1 and sells it at age $\tau_1 + \Delta$. The minimum holding period of the illiquid asset is thus Δ. To compute Δ, we note that at age τ_1 the agent is indifferent between the liquid and the illiquid asset. More precisely, he is indifferent between two investments. The first investment is to buy one share of the illiquid asset and sell it at $\tau_1 + \Delta$. The cost of this investment is

$(1+\epsilon)P$, and the cash flows are the dividend, D, between τ_1 and $\tau_1+\Delta$, and the price net of transaction costs, $(1-\epsilon)P$, at $\tau_1+\Delta$. The second investment is an investment in the liquid asset that generates the same cash flows as the first investment. The cost of this investment is the present value of the cash flows. Therefore

$$(1+\epsilon)P = \int_{\tau_1}^{\tau_1+\Delta} De^{-r(t-\tau_1)}dt + (1-\epsilon)Pe^{-r\Delta}. \tag{5.3}$$

Dividing by P and using the definition of m, we get

$$m = r\frac{1+e^{-r\Delta}}{1-e^{-r\Delta}}. \tag{5.4}$$

Equation 5.4 shows that Δ is decreasing in m, i.e. is decreasing in the liquidity premium, μ, and increasing in transaction costs, ϵ. We should note that at age $\tau_1' < \tau_1$ the illiquid asset is a strictly better investment than the liquid asset. Indeed, the holding period is $\Delta' > \Delta$ and the RHS of equation 5.3 is strictly greater than the LHS. The opposite is true at age $\tau_1' > \tau_1$.

Optimal consumption is given by

$$\frac{\partial u}{\partial c}(c_t, t) = \lambda e^{-\rho(t)} \Rightarrow c_t = v(\lambda e^{-\rho(t)}, t). \tag{5.5}$$

Equation 5.5 is analogous to equation 4.2 in the no transaction costs case. The Lagrange multiplier, λ, determines the level of c_t, and the discount rate, $\rho(t)$, its slope. The discount rate $\rho(t)$ is given by

$$\rho(t) = \int_0^t r(s)ds. \tag{5.6}$$

The function $r(t)$ is the rate of return relevant for age t, and is given by

$$r(t) = R_B \equiv \frac{R}{1+\epsilon} \text{ for all } t \in [0, \tau_1), \qquad r(t) = r \text{ for all } t \in [\tau_1, \tau_1+\Delta),$$

$$r(t) = R_S \equiv \frac{R}{1-\epsilon} \text{ for all } t \in [\tau_1+\Delta, T]. \tag{5.7}$$

In the no transaction costs case, $\rho(t) = rt$ and $r(t) = r$. With transaction costs, r is the relevant rate of return only for $t \in [\tau_1, \tau_1+\Delta)$, i.e. when the agent invests in the liquid asset. For $t \in [0, \tau_1)$ the relevant rate of return is R_B. Indeed, suppose that the agent decides to postpone consumption of one dollar from age t to $t+dt$. He then buys $1/((1+\epsilon)P)$ shares of the illiquid asset at age t rather than at $t+dt$, and receives a cash flow of $Ddt/((1+\epsilon)P) = R_B dt$. Similarly, for $t \in [\tau_1 + \Delta, T]$ the relevant rate of return is R_S. Indeed, now the agent sells $1/((1-\epsilon)P)$ shares of the illiquid asset at age $t + dt$ rather than at t, and receives a cash flow of $Ddt/((1-\epsilon)P) = R_S dt$.

Our optimality conditions will be 5.1, 5.2, 5.4, 5.5, and the conditions that the agent has zero liquid and illiquid wealth at age T. In proposition 5.1 we show that these conditions imply that the control (i_t, I_t) solves $(\mathcal{P})$. Proposition 5.1 is proven in appendix B.

Proposition 5.1 *Consider a feasible control* (i_t, I_t)*. If conditions 5.1, 5.2, 5.4, 5.5, and* $a_T = A_T = 0$ *are satisfied, then* (i_t, I_t) *solves* $(\mathcal{P})$*.*

5.2 Equilibrium Existence

In proposition 5.2 we construct an equilibrium for small transaction costs. The proposition is proven in appendix C.

Proposition 5.2 *For sufficiently small* $\epsilon > 0$*, there exists an equilibrium,* (r, R, i_t, I_t)*.*

To construct the equilibrium we first determine the parameters r, m,[9] τ_1, Δ, and λ. We use equations 5.4 and $a_T = A_T = 0$ of proposition 5.1, and the market-clearing conditions 3.5 and 3.6. We then construct the control (i_t, I_t) using equations 5.1, 5.2, and 5.5 of proposition 5.1. More precisely, we first determine c_t by equation 5.5. We then determine i_t and I_t by the budget constraints 3.1, 3.2, and 3.3, and equations 5.1 and 5.2. For $t \in [0, \tau_1)$, for instance, $i_t = 0$ and I_t is determined by

$$\frac{dA_t}{dt} = R_B A_t + \frac{y_t - c_t}{1+\epsilon}, \quad A_0 = 0. \tag{5.8}$$

(Equation 5.8 follows from equations 3.2, 3.3, and the fact that $I_t > 0$.)

To show that equations 5.4, $a_T = A_T = 0$, 3.5, and 3.6 have a solution, we show that they have a solution for $\epsilon = 0$, and apply the implicit function theorem. The solution for $\epsilon = 0$ is the following. First, $r = r^*$ and $\lambda = \lambda^*$, for the r^* and λ^* of assumption (LC). Next, $m = m^*$, where m^* is given by equation 5.4 for $r = r^*$ and $\Delta = \Delta^*$. Finally, $\tau_1 = \tau_1^*$ and $\Delta = \Delta^*$, where τ_1^* and Δ^* are the solution to

$$w^*_{\tau_1^*} = w^*_{\tau_1^* + \Delta^*} \tag{5.9}$$

and

$$\int_{\tau_1^*}^{\tau_1^*+\Delta^*} (w_t^* - w_{\tau_1^*}^*)\frac{dt}{T} = (1-k)\frac{D}{r^*}. \tag{5.10}$$

To motivate equations 5.9 and 5.10, suppose that agents follow the policy of investing in the "illiquid" asset first, even when $\epsilon = 0$. The liquid wealth of an agent is non-zero only for $t \in (\tau_1^*, \tau_1^* + \Delta)$, where it is $w_t^* - w_{\tau_1^*}^*$. Equation 5.9 implies that at age $\tau_1^* + \Delta^*$ liquid wealth is zero, and that equation $a_T = 0$ is satisfied. Equation 5.10 implies that the market-clearing condition for the liquid asset, 3.5, is satisfied. Figure 1 illustrates the determination of τ_1^* and Δ^*. The area under w_t^* is equal to DT/r^* (from equation 4.4) and the shaded area is equal to $(1-k)DT/r^*$.

INSERT FIGURE 1 SOMEWHERE HERE

5.3 Asset Prices

In this section we study the effects of small transaction costs on asset prices. We first study the effect on the price difference between the two assets or, equivalently, on the liquidity premium. We then study the effect on the price of each asset. We finally examine the case where transaction costs are not "real" but due to taxes, and illustrate our results with a numerical example.

5.3.1 Liquidity Premium

Equation 5.4 implies the following simple relation between the liquidity premium, μ, (i.e. the difference between the rates of return on the two assets), the minimum holding period of the illiquid asset, Δ, and the transaction costs, ϵ:

$$\mu = r\frac{1+e^{-r\Delta}}{1-e^{-r\Delta}}\epsilon. \tag{5.11}$$

In section 5.1 we interpreted equation 5.11 as an *optimality* condition determining the optimal Δ as a function of μ and ϵ. We can also interpret equation 5.11 as a *pricing* equation, determining the equilibrium μ as a function of ϵ and the equilibrium Δ. Using the prices, p and P, rather than the rates of return, we can rewrite equation 5.11 as

$$p - P = P\frac{1+e^{-r\Delta}}{1-e^{-r\Delta}}\epsilon = \sum_{\ell=0}^{\infty} P\epsilon(1+e^{-r\Delta})e^{-r\ell\Delta}. \tag{5.12}$$

Equation 5.12 has a very simple interpretation. An agent with a holding period Δ is indifferent between the liquid and the illiquid asset, i.e. is the *marginal* investor. For the marginal investor to buy one share of the illiquid asset, the price has to fall by the present value of the transaction costs that he incurs, $P\epsilon(1+e^{-r\Delta})$. Moreover, since he will sell this share to a new marginal investor, the price has also to fall by the present value of the transaction costs that the new investor incurs, $P\epsilon(1+e^{-r\Delta})e^{-r\Delta}$, and so on. Equation 5.12 states that the price has to fall by the present value of the transaction costs that a sequence of marginal investors incur. This result is consistent with Amihud and Mendelson (1986).

To determine the liquidity premium we thus need to determine the holding period of the marginal investor, Δ. For small transaction costs, Δ is close to Δ^*, defined in section 5.2. Therefore

$$\mu = m^*\epsilon + o(\epsilon) = r^*\frac{1+e^{-r^*\Delta^*}}{1-e^{-r^*\Delta^*}}\epsilon + o(\epsilon). \tag{5.13}$$

Equations 5.9 and 5.10 imply that Δ^* decreases in the supply of the illiquid asset, k. Intuitively, if the illiquid asset is in greater supply, agents have to hold it for shorter periods. Therefore, equation 5.13 implies that the liquidity premium increases in k.

5.3.2 Liquid Asset

In proposition 5.3 we show that transaction costs increase the price of the liquid asset, i.e. decrease its rate of return. The proposition is proven in appendix D.

Proposition 5.3 *For sufficiently small $\epsilon > 0$, the rate of return on the liquid asset, r, decreases in ϵ.*

A simple explanation for proposition 5.3 is that with transaction costs agents substitute away from the illiquid asset, and towards the liquid asset whose price increases. However this explanation is incomplete. Indeed, the price of the liquid asset may remain unchanged and the price of illiquid asset may sufficiently decrease, so that agents do not substitute away from either asset. To show why the price of the liquid asset has to increase, we will show that if it remains unchanged, total asset demand (i.e. demand for the two assets) will exceed total asset supply.

Suppose that transaction costs increase from 0 to ϵ and that the rate of return on the liquid asset remains equal to r^*. The rate of return on the illiquid asset has to increase by approximately $m^*\epsilon$, as we showed in section 5.3.1. Agents can still obtain the rate of return r^* by investing in the liquid asset. In addition, the illiquid asset becomes a strictly better investment than the liquid asset for holding periods longer than Δ^*, as we showed in section 5.1. Since agents face better investment opportunities, they are more wealthy and consume more at each age t. This is the *wealth* effect. Agents also substitute consumption over their lifetime. In order to benefit from the higher rate of return on the illiquid asset, they buy more shares, consuming less at the beginning of their lifetime and more towards the end. This is the *substitution* effect. Both the wealth and substitution effects imply that agents consume more towards the end of their lifetime. Since agents finance consumption at the end of their lifetime by selling shares, they have to buy more shares. Therefore, total asset demand exceeds total asset supply. Note that, because of transaction costs, the proceeds of selling shares of the illiquid asset are lower. This *direct transaction costs* effect also increases asset demand. In appendix D we decompose the effect of transaction costs into the wealth, substitution, and direct transaction costs effects.

5.3.3 Illiquid Asset

We now study the effect of transaction costs on the illiquid asset. The effects of sections 5.3.1 and 5.3.2 work in opposite directions, i.e. transaction costs decrease the price of the illiquid asset *relative to the price of the liquid asset* (section 5.3.1) but increase the price of the liquid asset (section 5.3.2). The overall effect is unambiguous if the supply of the illiquid asset is small relative to the supply of the liquid asset, i.e. if k is small. Since transaction costs concern a small fraction of the assets, they have a weak effect on the remaining assets, i.e. on the liquid asset. Therefore they decrease the price of the illiquid asset. However, if k is large, the effect is ambiguous.

In order to study the effect of transaction costs on the illiquid asset for large k, we proceed more directly and perform the exercise of section 5.3.2. We assume that transaction costs increase from 0 to ϵ and that the the rate of return on the *illiquid* asset remains equal to r^*. We then study whether asset demand increases or decreases. For simplicity we assume that k is very close to 1, i.e. there is no liquid asset. Because of transaction costs, the illiquid asset becomes a less attractive investment. Therefore agents buy fewer shares, and substitute consumption towards the beginning of their lifetimes. However, to economize on transaction costs they sell these shares more slowly, substituting consumption towards the end of their lifetimes (and away from the middle). Indeed, the relevant rate of return for the interval during which agents sell the illiquid asset is $R_S = r^*/(1-\epsilon) > r^*$. (Since k is very close to 1 and ϵ is small, this interval is approximately $[\tau^*, T]$, for the τ^* of section 4.) Since agents buy more shares but hold them for longer periods, the substitution effect has an ambiguous implication for asset demand. The wealth and direct transaction costs effects also have an ambiguous implication in the sense that the first increases and the second decreases asset demand. The wealth effect is that agents consume less at each age t because of transaction costs. It decreases asset demand since agents have to finance lower future consumption. The direct transaction costs effect is that the proceeds from selling shares are lower. It increases asset demand since agents need to buy more shares to finance a given future consumption.

Our exercise does not determine whether transaction costs increase or decrease the price of the illiquid asset. However, it points to a case where transaction costs may increase the price. Suppose that the interval during which agents sell the illiquid asset is very long. Then the effect that agents sell shares more slowly may dominate the effect that they buy fewer shares, i.e. the substitution effect may increase asset demand. In proposition 5.4 we show that this indeed happens when $u(c,t) = (c^{1-A}/(1-A))e^{-\beta t}$, $A \geq 0$, and $y_t = ye^{-\delta t}$, $\delta \geq 0$. The proposition is proven in appendix E.

Proposition 5.4 *Suppose that* $u(c,t) = (c^{1-A}/(1-A))e^{-\beta t}$ *and* $y_t = ye^{-\delta t}$ *with* $D/y < 1$, $\delta = \hat{\delta}/T$, *and* $\hat{\delta}$ *a large constant. Then if* T *is sufficiently large,* k *sufficiently close to 1, and* $\epsilon > 0$ *sufficiently small, the rate of return on the illiquid asset,* R, *decreases in* ϵ.

In proposition 5.4 we assume that agents' lifetime, T, is long, so that the interval during which agents sell the illiquid asset can be very long. We also assume that $\delta = \hat{\delta}/T$. This assumption implies that agents' labor income at a given fraction,

f, of their lifetime, i.e. at age fT, is independent of T. It also implies that τ^*/T is approximately independent of T, i.e. agents buy and sell the illiquid asset during constant fractions of their lifetime. In the proposition we finally assume that $D/y < 1$, i.e. financial wealth is small, and that $\hat{\delta}$ is large, i.e. labor income declines fast. These assumptions imply that agents buy the illiquid asset only during a small fraction of their lifetime. In appendix E we show that the substitution effect dominates the other effects, and increases asset demand.

5.3.4 Transaction Taxes

So far we assumed that transaction costs are "real", i.e. transactions consume resources. However, transaction costs are sometimes due to taxes that are distributed back to the agents. Equation 5.12, that gives the price difference between the two assets as a function of the horizon of the marginal investor, is derived from the optimization problem and does not depend on the origin of transaction costs. The origin of transaction costs matters for the effect on the price of each asset. If transaction costs are due to taxes, they can affect asset demand also through the distribution of taxes back to the agents. If the distribution benefits more older (younger) agents, it will decrease (increase) asset demand and decrease (increase) the price.

5.3.5 A Numerical Example

We illustrate our results with a numerical example. We assume that $T = 50$, $u(c,t) = \log(c)e^{-\beta t}$, $\beta = 0.04$, $y_t = ye^{-\delta t}$, $\delta = 0.04$, and $\epsilon = 3\%$. We also assume that $D/(\int_0^T ye^{-\delta t}dt/T) = 1/3$, i.e. dividends are 1/3 of total labor income. In figure 2 we plot the rates of return on the two assets as a function of the supply of the illiquid asset, k. The solid line represents the benchmark case where there are no transaction costs and the two assets are identical. The lines with the short and long dashes represent the rates of return on the liquid and the illiquid asset, respectively, with transaction costs. Figure 2 shows that the liquidity premium increases in k, consistent with section 5.3.1. It also shows that the rate of return on the liquid asset decreases with transaction costs, consistent with section 5.3.2. The effect of transaction costs on the rate of return on the liquid asset is stronger, the higher k is. This is intuitive since an increase in k is a different way to increase transaction costs. In figure 2 the rate of return on the illiquid asset increases with transaction costs. This is consistent with proposition 5.4, since $\hat{\delta} = \delta/T = 0.0008$, i.e. labor income does not decline very fast.

INSERT FIGURE 2 SOMEWHERE HERE

Figure 2 shows that if k is large (larger than 0.9), an increase in transaction costs has a stronger effect on the liquid asset than on the illiquid asset. In other words, an increase in transaction costs for a significant fraction of assets, may have a stronger effect on the remaining assets rather than on those subject to the change. Therefore a "partial equilibrium" analysis that assumes that the rates of return on the remaining assets stay constant, will be incorrect. In fact, the increase in transaction costs can even concern a smaller fraction of assets, provided that these assets were among the most liquid. Consider, for instance, a transaction tax on financial securities. Before the tax the illiquid asset consists of housing and other real estate, so k is around 0.5. After the tax the illiquid asset also includes most financial securities, and let's assume that k is around 0.95. Before the tax the rate of return on financial securities is given by the value of the line with the short dashes for $k = 0.5$. After the tax the rate of return on the taxed (non-taxed) securities is given by the value of the line with the long (short) dashes for $k = 0.95$. Figure 2 shows that the tax decreases the rate of return on the non-taxed securities by as much as it increases the rate of return on the taxed ones. Therefore an analysis that erroneously assumes that the rate of return on the non-taxed securities stays constant in spite of the tax, will overestimate the effect of the tax by 100%.

6 Concluding Remarks

In this paper we develop a general equilibrium model with transaction costs. We assume an overlapping generations economy with two riskless assets. The first asset is liquid while the second asset carries proportional transaction costs. Agents receive labor income and trade the assets for life-cycle purposes. Their preferences and labor income streams can be very general. We show that agents first buy the illiquid asset, next buy the liquid asset, then sell the liquid asset, and finally sell the illiquid asset. When transaction costs increase, the price of the liquid asset increases. The price of the illiquid asset decreases if the asset is in small supply, but may increase if the supply is large. Our results imply that a change in transaction costs for a significant fraction of assets may have a stronger effect on the remaining assets rather than on those subject to the change. This point is important when evaluating the effects of a transaction tax or of information technology and financial market deregulation.

Appendix

A Assumption (LC)

We first prove proposition 4.1.

Proof: We first show that for $r > \beta$, w_t is increasing and then decreasing. Consider $\tau \in (0, T)$ such that $dw_t/dt|_{t=\tau} = 0$. Differentiating equation 4.1 w.r.t. t at τ, and using $dw_t/dt|_{t=\tau} = 0$, we get

$$\left.\frac{d^2 w_t}{dt^2}\right|_{t=\tau} = \left.\frac{d(y_t - c_t)}{dt}\right|_{t=\tau}.$$

Equation 4.7 and the fact that $r > \beta$, imply that $dc_t/dt > 0$. Since, in addition, $dy_t/dt \leq 0$, we have $d^2 w_t/dt^2|_{t=\tau} < 0$. Therefore, $dw_t/dt > 0$ for t smaller than and close to τ, and $dw_t/dt < 0$ for t greater than and close to τ. This means that there exists at most one τ such that $dw_t/dt|_{t=\tau} = 0$. Moreover, $dw_t/dt > 0$ for $t < \tau$ and $dw_t/dt < 0$ for $t > \tau$. Since $w_0 = 0$ and, by the intertemporal budget constraint, $w_T = 0$, there exists one such τ

We now show that there exists $r^* > \beta$ such that the market-clearing condition 4.4 holds. Integrating both sides of equation 4.1 and using $w_0 = w_T = 0$, we conclude that condition 4.4 is equivalent to

$$\int_0^T (c_t - y_t)\frac{dt}{T} = D. \tag{A.1}$$

Equation A.1 is the market-clearing condition for the consumption good. It states that total consumption at a given time is equal to total labor income plus total dividend. To show that there exists $r^* > \beta$ such that equation A.1 is satisfied, we show that the LHS is smaller than the RHS for $r = \beta$, and goes to ∞ as r goes to ∞. We then conclude by continuity.

For $r = \beta$, consumption is constant. The intertemporal budget constraint implies that this constant is

$$c = \frac{\int_0^T y_t e^{-\beta t} dt}{\int_0^T e^{-\beta t} dt}. \tag{A.2}$$

Equations 4.6 and A.2 imply that the LHS of equation A.1 is smaller than the RHS. To show that

$$\lim_{r \to \infty} \int_0^T c_t \frac{dt}{T} = \infty,$$

we first note that from Jensen's inequality

$$\int_0^T u(c_t)e^{-\beta t}dt \leq u(\frac{\int_0^T c_t e^{-\beta t}dt}{\int_0^T e^{-\beta t}dt})\int_0^T e^{-\beta t}dt \leq u(\frac{\int_0^T c_t dt}{\int_0^T e^{-\beta t}dt})\int_0^T e^{-\beta t}dt.$$

Therefore, it suffices to show that

$$\lim_{r\to\infty}\int_0^T u(c_t)e^{-\beta t}dt = \lim_{c\to\infty} u(c)\int_0^T e^{-\beta t}dt \tag{A.3}$$

i.e. that as the interest rate goes to infinity, the consumer achieves the maximum possible utility. To derive equation A.3 we consider the following suboptimal policies $\hat{c}_t(h)$

$$\hat{c}_t(h) = \frac{y_t}{2} \text{ for all } t \in [0, h],$$

$$\hat{c}_t(h) = rw_h = r\int_0^h \frac{y_s}{2}e^{r(h-s)}ds \text{ for all } t \in (h, T].$$

Since $\hat{c}_t(h)$ is feasible and c_t is optimal

$$\int_0^T u(c_t)e^{-\beta t}dt \geq \int_0^T u(\hat{c}_t(h))e^{-\beta t}dt = \int_0^h u(\frac{y_t}{2})e^{-\beta t}dt + u(rw_h)\int_h^T e^{-\beta t}dt.$$

Fixing $h > 0$ and letting r go to ∞ we get

$$\lim_{r\to\infty}\int_0^T u(c_t)e^{-\beta t}dt \geq \int_0^h u(\frac{y_t}{2})e^{-\beta t}dt + \lim_{c\to\infty} u(c)\int_h^T e^{-\beta t}dt.$$

Letting h go to 0, we obtain equation A.3. Q.E.D.

We now prove proposition 4.2.

Proof: We first show that assumption (LC) is satisfied when labor income declines exponentially. Assuming that the rate of return r is such that $\delta > \omega \equiv (\beta - r)/A$, we show that total wealth is increasing and then decreasing. We then show that there exists r^* such that the market-clearing condition 4.4 holds.

Equation 4.2 implies that

$$c_t = \lambda^{-\frac{1}{A}}e^{-\omega t}. \tag{A.4}$$

Combining equations 4.3 and A.4, we get

$$c_t = y\frac{\omega + r}{\delta + r}\frac{1 - e^{-(\delta+r)T}}{1 - e^{-(\omega+r)T}}e^{-\omega t}. \tag{A.5}$$

Equation 4.1 implies that

$$w_t = \frac{y}{\delta + r}e^{rt}\left(\frac{1 - e^{-(\delta+r)T}}{1 - e^{-(\omega+r)T}}(e^{-(\omega+r)t} - e^{-(\omega+r)T}) - (e^{-(\delta+r)t} - e^{-(\delta+r)T})\right). \tag{A.6}$$

Differentiating, we get

$$\frac{dw_t}{dt} = \frac{y}{(\delta+r)}e^{rt}\left(\frac{1-e^{-(\delta+r)T}}{1-e^{-(\omega+r)T}}(-\omega e^{-(\omega+r)t} - re^{-(\omega+r)T}) + (\delta e^{-(\delta+r)t} + re^{-(\delta+r)T})\right). \tag{A.7}$$

Therefore dw_t/dt has the same sign as the function

$$f(t) = \frac{1-e^{-(\delta+r)T}}{1-e^{-(\omega+r)T}}(-\omega e^{-(\omega+r)t} - re^{-(\omega+r)T}) + (\delta e^{-(\delta+r)t} + re^{-(\delta+r)T})$$

that we now study. Since the function $x \to xe^{-x}/(1-e^{-x})$ is strictly decreasing, and $\delta > \omega$, we have $f(T) < 0$. $f'(t)$ has the same sign as

$$\frac{1-e^{-(\delta+r)T}}{1-e^{-(\omega+r)T}}\omega(\omega+r) - \delta(\delta+r)e^{-(\delta-\omega)t}. \tag{A.8}$$

Since $\delta > \omega$, expression A.8 is increasing. Therefore $f'(t)$ is positive, or negative and then positive, or negative. If $f'(t)$ is positive, $f(t)$ is increasing, and, since $f(T) < 0$, $f(t)$ is negative. Therefore w_t is strictly decreasing, which is a contradiction since $w_0 = 0$ and $w_T = 0$. If $f'(t)$ is negative and then positive, $f(t)$ is decreasing and then increasing. Since $f(t)$ cannot be always negative, it is positive and then negative. Therefore, w_t is increasing and then decreasing. The case where $f'(t)$ is negative, is identical.

Instead of using the market-clearing condition 4.4, we will use equation A.1. Using equation A.5, we can write equation A.1 as

$$\frac{\omega+r}{\delta+r}\frac{1-e^{-(\delta+r)T}}{1-e^{-(\omega+r)T}}\frac{1-e^{-\omega T}}{\omega T} - \frac{1-e^{-\delta T}}{\delta T} = \frac{D}{y}. \tag{A.9}$$

The LHS of equation A.9 is continuous in r, and is equal to 0 for $r = 0$, and for r such that $\delta = \omega$. Moreover it goes to ∞ as r goes to ∞. (To prove this we distinguish the cases $A < 1$, $A = 1$, and $A > 1$. If $A < 1$, for instance, the LHS is approximately

$$\frac{r(1-\frac{1}{A})}{r}\frac{1}{-e^{(\frac{r-\beta}{A}-r)T}}\frac{e^{\frac{r-\beta}{A}T}}{\frac{r}{A}} - \frac{1-e^{-\delta T}}{\delta} = \frac{1-A}{r}e^{rT} - \frac{1-e^{-\delta T}}{\delta}.$$

The cases $A = 1$ and $A > 1$ are similar.) Therefore there exists r^* such that equation A.9 holds.

We finally show that assumption (LC) is not satisfied when labor income declines linearly. Proceeding as in the exponential case, we get

$$c_t = y\frac{\omega+r}{1-e^{-(\omega+r)T}}\frac{rT-(1-e^{-rT})}{r^2T}e^{-\omega t}. \tag{A.10}$$

Using equation A.10, we can write equation A.1 as

$$\frac{\omega + r}{1 - e^{-(\omega+r)T}} \frac{rT - (1 - e^{-rT})}{r^2 T} \frac{1 - e^{-\omega T}}{\omega T} - \frac{1}{2} = \frac{D}{y}. \tag{A.11}$$

Solving equation A.11 numerically, we find that $r^* = 0.006519$. Plugging back into equation A.10 we find that $c_0^* = 1.04047y > y = y_0$. Therefore, $dw_t^*/dt|_{t=0} < 0$, and assumption (LC) is not satisfied. Q.E.D.

B Proof of Proposition 5.1

We first note that since condition 5.4 holds for some $\Delta > 0$, $R_B \equiv R/(1+\epsilon) > r$ and, as a result, $R_S \equiv R/(1-\epsilon) > r$.

Consider a feasible control $(i_t + \hat{i}_t, I_t + \hat{I}_t)$ which induces liquid wealth $a_t + \hat{a}_t$, illiquid wealth $A_t + \hat{A}_t$, and consumption $c_t + \hat{c}_t$. We will show that it gives lower utility than (i_t, I_t). Equation 3.3 implies that

$$\hat{c}_t = r\hat{a}_t + R\hat{A}_t - \hat{i}_t - \hat{I}_t - \epsilon(|I_t + \hat{I}_t| - |I_t|).$$

Since $u(c,t)$ is concave in c, we get

$$u(c_t + \hat{c}_t, t) \leq u(c_t, t) + \frac{\partial u}{\partial c}(c_t, t)\hat{c}_t$$

$$= u(c_t, t) + \frac{\partial u}{\partial c}(c_t, t)\left(r\hat{a}_t + R\hat{A}_t - \hat{i}_t - \hat{I}_t - \epsilon(|I_t + \hat{I}_t| - |I_t|)\right). \tag{B.1}$$

Integrating B.1 from 0 to T, we get

$$\int_0^T u(c_t + \hat{c}_t, t)dt \leq \int_0^T u(c_t, t)dt + K_i + K_I$$

with

$$K_i = \int_0^T \frac{\partial u}{\partial c}(c_t, t)(r\hat{a}_t - \hat{i}_t)dt$$

and

$$K_I = \int_0^T \frac{\partial u}{\partial c}(c_t, t)(R\hat{A}_t - \hat{I}_t - \epsilon(|I_t + \hat{I}_t| - |I_t|))dt.$$

We will show that K_i and K_I are negative.

Integrating the second term of K_i by parts, and noting that $\hat{a}_t = \int_0^t \hat{i}_s ds$, we get

$$K_i = \int_0^T \frac{\partial u}{\partial c}(c_t, t)r\hat{a}_t dt - \left[\frac{\partial u}{\partial c}(c_t, t)\hat{a}_t\right]_0^T + \int_0^T \frac{d(\frac{\partial u}{\partial c}(c_t, t))}{dt}\hat{a}_t dt.$$

Equation 5.5 implies that

$$K_i = \int_0^T \frac{\partial u}{\partial c}(c_t, t)(r - r(t))\hat{a}_t dt - \frac{\partial u}{\partial c}(c_T, T)\hat{a}_T.$$

The function $r - r(t)$ is non-zero only in $[0, \tau_1] \cup (\tau_1 + \Delta, T]$, where it is strictly negative since $R_B > r$ and $R_S > r$. Moreover, $a_t = 0$ for all $t \in [0, \tau_1] \cup [\tau_1 + \Delta, T]$, since $i_t = 0$ and $a_0 = a_T = 0$. The short-sale constraint implies then that $\hat{a}_t \geq 0$ for all $t \in [0, \tau_1] \cup [\tau_1 + \Delta, T]$. Therefore K_i is negative.

Integrating the first term of K_I by parts, and noting that $d\hat{A}_t/dt = \hat{I}_t$, we get

$$K_I = -\left[\left(\int_t^T \frac{\partial u}{\partial c}(c_s, s)ds + \frac{\partial u}{\partial c}(c_T, T)\frac{1-\epsilon}{R}\right) R\hat{A}_t\right]_0^T +$$

$$\int_0^T \left(\int_t^T \frac{\partial u}{\partial c}(c_s, s)ds + \frac{\partial u}{\partial c}(c_T, T)\frac{1-\epsilon}{R}\right) R\hat{I}_t dt - \int_0^T \frac{\partial u}{\partial c}(c_t, t)(\hat{I}_t + \epsilon(|I_t + \hat{I}_t| - |I_t|))dt.$$

Since $A_T = 0$, the short-sale constraint implies that $\hat{A}_T \geq 0$. Since in addition $\hat{A}_0 = 0$, the term in brackets is positive, i.e. the first term is negative. To study the remaining terms, we define the functions $K_B(t)$ and $K_S(t)$ by

$$K_B(t) = \left(\int_t^T \frac{\partial u}{\partial c}(c_s, s)ds\right) R + \frac{\partial u}{\partial c}(c_T, T)(1-\epsilon) - \frac{\partial u}{\partial c}(c_t, t)(1+\epsilon)$$

and

$$K_S(t) = \left(\int_t^T \frac{\partial u}{\partial c}(c_s, s)ds\right) R + \frac{\partial u}{\partial c}(c_T, T)(1-\epsilon) - \frac{\partial u}{\partial c}(c_t, t)(1-\epsilon).$$

Inequalities

$$|I_t + \hat{I}_t| - |I_t| \geq \hat{I}_t \text{ for } I_t > 0$$

and

$$|I_t + \hat{I}_t| - |I_t| \geq -\hat{I}_t \text{ for } I_t < 0,$$

imply that the remaining terms are smaller than

$$\int_0^{\tau_1} K_B(t)\hat{I}_t dt + \int_{\tau_1+\Delta}^T K_S(t)\hat{I}_t dt + \int_{\tau_1}^{\tau_1+\Delta} (K_B(t)\hat{I}_t 1_{\{\hat{I}_t \geq 0\}} + K_S(t)\hat{I}_t 1_{\{\hat{I}_t < 0\}})dt, \quad \text{(B.2)}$$

where 1_X is the indicator function of the set X. We first prove that the terms in $K_S(t)$ in expression B.2 are negative. Equation 5.5 implies that

$$\frac{dK_S(t)}{dt} = \frac{\partial u}{\partial c}(c_t, t)((1-\epsilon)r(t) - R).$$

Since $K_S(T) = 0$ and $R_S > r$, $K_S(t) = 0$ for all $t \in [\tau_1 + \Delta, T]$ and $K_S(t) \geq 0$ for all $t \in [\tau_1, \tau_1 + \Delta]$. Therefore the terms in $K_S(t)$ are negative. We finally show that the terms in $K_B(t)$ are negative. The definitions of $K_B(t)$ and $K_S(t)$ imply that

$$K_B(\tau_1) - K_S(\tau_1+\Delta) = \int_{\tau_1}^{\tau_1+\Delta} \frac{\partial u}{\partial c}(c_s, s)ds - \frac{\partial u}{\partial c}(c_{\tau_1}, \tau_1)(1+\epsilon) + \frac{\partial u}{\partial c}(c_{\tau_1+\Delta}, \tau_1+\Delta)(1-\epsilon).$$

Noting that $K_S(\tau_1 + \Delta) = 0$, and using equations 5.3 and 5.5, we conclude that $K_B(\tau_1) = 0$. Moreover, equation 5.5 implies that

$$\frac{dK_B(t)}{dt} = \frac{\partial u}{\partial c}(c_t, t)((1+\epsilon)r(t) - R).$$

Since $K_B(\tau_1) = 0$ and $R_B > r$, $K_B(t) = 0$ for all $t \in [0, \tau_1]$ and $K_B(t) \leq 0$ for all $t \in [\tau_1, \tau_1 + \Delta]$. Therefore the terms in $K_B(t)$ are also negative. Q.E.D.

C Proof of Proposition 5.2

We first prove lemma C.1 and then the proposition.

Lemma C.1 *Consider three piecewise continuous functions, χ, ψ, and ω, such that χ and ψ are strictly positive, ψ is decreasing, and ω is increasing. Then*

$$\frac{\int_0^T \chi(t)\omega(t)dt}{\int_0^T \chi(t)dt} \geq \frac{\int_0^T \chi(t)\psi(t)\omega(t)dt}{\int_0^T \chi(t)\psi(t)dt}.$$

Proof: We consider the probability distributions P_χ and $P_{\chi\psi}$ in $[0,T]$, with cumulative distribution functions

$$F_\chi(t) = \frac{\int_0^t \chi(s)ds}{\int_0^T \chi(t)dt} \quad \text{and} \quad F_{\chi\psi}(t) = \frac{\int_0^t \chi(s)\psi(s)ds}{\int_0^T \chi(t)\psi(t)dt},$$

respectively. Proving the lemma is equivalent to proving that P_χ first-order stochastically dominates $P_{\chi\psi}$, i.e. $F_\chi(t) \leq F_{\chi\psi}(t)$ for all $t \in [0,T]$. This inequality is equivalent to

$$\frac{\int_0^t \chi(s)ds}{\int_t^T \chi(s)ds} \leq \frac{\int_0^t \chi(s)\psi(s)ds}{\int_t^T \chi(s)\psi(s)ds},$$

which holds since $\psi(s) \geq \psi(t)$ for $s \in [0,t]$ and $\psi(s) \leq \psi(t)$ for $s \in [t,T]$. Q.E.D.

We now come to the proof of the proposition.

Proof: We first determine the parameters r, m, τ_1, Δ, and λ, using the implicit function theorem. Our equations are 5.4, and four equations equivalent to $a_T = A_T = 0$, 3.5, and 3.6. We then construct the control (i_t, I_t). Finally, we show that the control is feasible and that equations $a_T = A_T = 0$, 3.5, and 3.6 are satisfied.

Determination of r, m, τ_1, Δ, and λ

We will apply the implicit function theorem to the function F defined by

$$F(r,\lambda,m,\tau_1,\Delta,\epsilon) = \begin{pmatrix} \int_0^T (c_t - y_t)e^{-\rho(t)}dt \\ \int_0^T (c_t - y_t)\frac{dt}{T} + \frac{2\epsilon}{T}A_{\tau_1} - D \\ m - r\frac{1+e^{-r\Delta}}{1-e^{-r\Delta}} \\ \int_{\tau_1}^{\tau_1+\Delta}(c_t - y_t - RA_{\tau_1})e^{-r(t-\tau_1)}dt \\ \int_{\tau_1}^{\tau_1+\Delta}(c_t - y_t)\frac{dt}{T} - \frac{R\Delta}{T}A_{\tau_1} - (1-k)D \end{pmatrix}. \tag{C.1}$$

at the point $\mathcal{A} = (r^*, \lambda^*, m^*, \tau_1^*, \Delta^*, 0)$. In equation C.1, R is given by $R = r + m\epsilon$, c_t by equation 5.5, and A_{τ_1} by

$$A_{\tau_1} = \int_0^{\tau_1} \frac{y_t - c_t}{1+\epsilon} e^{R_B(\tau_1 - t)} dt. \tag{C.2}$$

Equation $F_3 = 0$ is 5.4. Equation $F_1 = 0$ is analogous to the intertemporal budget constraint in the no transaction costs case. The difference is that the discount rate is $\rho(t)$ instead of rt. Equation $F_1 = 0$ will imply the "sum" of $a_T = 0$ and $A_T = 0$. Equation $F_2 = 0$ is the market-clearing condition for the consumption good, first derived in appendix A as equation A.1. It is modified to include the total transaction costs that are incurred at a given time. Equation $F_2 = 0$ will imply the "sum" of 3.5 and 3.6. Equation $F_4 = 0$ will imply that $a_T = 0$. It is derived from the counterpart of equation 5.8 for the interval $[\tau_1, \tau_1 + \Delta)$. Finally, equation $F_5 = 0$ will imply equation 3.5.

We first show that $F = 0$ at $\mathcal{A}$. Assumption (LC) implies that $F_1 = 0$ and that the market-clearing condition 4.4 holds. Condition 4.4 implies in turn that $F_2 = 0$, as we showed in appendix A. The definition of m^* implies that $F_3 = 0$. To show that $F_4 = F_5 = 0$, we consider equation 4.1 for w_t^* and write it as

$$\frac{d(w_t^* - w_{\tau_1^*}^*)}{dt} = r(w_t^* - w_{\tau_1^*}^*) + r w_{\tau_1^*}^* + y_t - c_t. \tag{C.3}$$

Integrating equation C.3 from τ_1^* to $\tau_1^* + \Delta^*$, and using equation 5.9 and the fact that $w_{\tau_1^*}^* = A_{\tau_1^*}$, we get $F_4 = 0$. Finally, integrating both sides of equation C.3 from τ_1^* to $\tau_1^* + \Delta^*$, and using equations 5.9 and 5.10, we get $F_5 = 0$.

We now show that the Jacobian matrix of F w.r.t. r, λ, m, τ_1, and Δ, at $\mathcal{A}$ is invertible. For $\epsilon = 0$, m enters only in F_3, and τ_1 and Δ enter only in F_3, F_4, and F_5. Moreover $\partial F_3 / \partial m = 1 \neq 0$. Therefore it suffices to show that the Jacobian matrix of F_1 and F_2 w.r.t. r and λ is invertible, and that the Jacobian matrix of F_4 and F_5 w.r.t. τ_1 and Δ is invertible. We compute the determinants of these matrices. To simplify notation we denote $\partial v(q,t)/\partial q$ by $v_q(q,t)$, and omit the superscript $*$ from r, λ, τ_1, Δ, c_t, and w_t.

Jacobian w.r.t. r and λ

Using equation 5.5, we get

$$\begin{vmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \lambda} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \lambda} \end{vmatrix} = \begin{vmatrix} -\int_0^T v_q(\lambda e^{-rt},t)\lambda e^{-2rt}tdt - \int_0^T (c_t - y_t)e^{-rt}tdt & \int_0^T v_q(\lambda e^{-rt},t)e^{-2rt}dt \\ -\int_0^T v_q(\lambda e^{-rt},t)\lambda e^{-rt}t\frac{dt}{T} & \int_0^T v_q(\lambda e^{-rt},t)e^{-rt}\frac{dt}{T} \end{vmatrix}. \tag{C.4}$$

The determinant C.4 has the same sign as

$$\lambda\left(\frac{\int_0^T v_q(\lambda e^{-rt},t)e^{-rt}tdt}{\int_0^T v_q(\lambda e^{-rt},t)e^{-rt}dt} - \frac{\int_0^T v_q(\lambda e^{-rt},t)e^{-2rt}tdt}{\int_0^T v_q(\lambda e^{-rt},t)e^{-2rt}dt}\right) - \frac{\int_0^T (c_t - y_t)e^{-rt}tdt}{\int_0^T v_q(\lambda e^{-rt},t)e^{-2rt}dt}. \tag{C.5}$$

Setting $\chi(t) = -v_q(\lambda e^{-rt},t)e^{-rt} > 0$, $\psi(t) = e^{-rt}$, and $\omega(t) = t$, we conclude from lemma C.1 that the first term of expression C.5 is positive. To show that expression C.5 is strictly positive, it suffices to show that $\int_0^T (c_t - y_t)e^{-rt}tdt$ is strictly positive, since $v_q < 0$. Using integration by parts, we can write this term as

$$\left[t\int_0^t (c_s - y_s)e^{-rs}ds\right]_0^T - \int_0^T \left(\int_0^t (c_s - y_s)e^{-rs}ds\right)dt =$$

$$[-tw_te^{-rt}]_0^T + \int_0^T w_te^{-rt}dt = \int_0^T w_te^{-rt}dt > 0.$$

Jacobian w.r.t. τ_1 and Δ

Differentiating F_4 w.r.t. τ_1, we get

$$\frac{\partial F_4}{\partial \tau_1} = (c_{\tau_1+\Delta} - y_{\tau_1+\Delta} - rA_{\tau_1})e^{-r\Delta} - (c_{\tau_1} - y_{\tau_1} - rA_{\tau_1}) - r\frac{dA_{\tau_1}}{d\tau_1}\frac{1-e^{-r\Delta}}{r} + rF_4. \tag{C.6}$$

Differentiating equation C.2 w.r.t. τ_1, we get

$$\frac{dA_{\tau_1}}{d\tau_1} = rA_{\tau_1} + y_{\tau_1} - c_{\tau_1}. \tag{C.7}$$

Combining equations C.6 and C.7, and using the fact that $F_4 = 0$ at $\mathcal{A}$, we get

$$\frac{\partial F_4}{\partial \tau_1} = (c_{\tau_1+\Delta} - y_{\tau_1+\Delta} - (c_{\tau_1} - y_{\tau_1}))e^{-r\Delta}. \tag{C.8}$$

Using equation C.8, we get

$$\begin{vmatrix} \frac{\partial F_4}{\partial \tau_1} & \frac{\partial F_4}{\partial \Delta} \\ \frac{\partial F_5}{\partial \tau_1} & \frac{\partial F_5}{\partial \Delta} \end{vmatrix} =$$

$$\begin{vmatrix} (c_{\tau_1+\Delta} - y_{\tau_1+\Delta} - (c_{\tau_1} - y_{\tau_1}))e^{-r\Delta} & (c_{\tau_1+\Delta} - y_{\tau_1+\Delta} - rA_{\tau_1})e^{-r\Delta} \\ \frac{1}{T}(c_{\tau_1+\Delta} - y_{\tau_1+\Delta} - (c_{\tau_1} - y_{\tau_1})) - \frac{r\Delta}{T}(rA_{\tau_1} + y_{\tau_1} - c_{\tau_1}) & \frac{1}{T}(c_{\tau_1+\Delta} - y_{\tau_1+\Delta} - rA_{\tau_1}) \end{vmatrix}. \tag{C.9}$$

To compute this last determinant we multiply the first row by $e^{r\Delta}/T$ and subtract it from the second. We get

$$\frac{r\Delta}{T}(rA_{\tau_1} + y_{\tau_1} - c_{\tau_1})(c_{\tau_1+\Delta} - y_{\tau_1+\Delta} - rA_{\tau_1})e^{-r\Delta} = -\frac{r\Delta}{T}e^{-r\Delta} \left.\frac{dw_t}{dt}\right|_{t=\tau_1} \left.\frac{dw_t}{dt}\right|_{t=\tau_1+\Delta}.$$

Since $w_{\tau_1} = w_{\tau_1+\Delta}$ and $\Delta > 0$, assumption (LC) implies that the derivative of w_t is strictly positive at τ_1 and strictly negative at $\tau_1 + \Delta$. Therefore expression C.9 is strictly positive.

Construction of the Control

We define c_t by equation 5.5. We also define i_t, I_t, and the continuous functions a_t, A_t, as follows. For $t \in [0, \tau_1)$, $i_t = da_t/dt = 0$,

$$I_t = \frac{dA_t}{dt} = R_B A_t + \frac{y_t - c_t}{1+\epsilon}, \tag{C.10}$$

and $a_0 = A_0 = 0$. For $t \in [\tau_1, \tau_1 + \Delta)$,

$$i_t = \frac{da_t}{dt} = ra_t + RA_{\tau_1} + y_t - c_t \tag{C.11}$$

and $I_t = dA_t/dt = 0$. Finally, for $t \in [\tau_1 + \Delta, T]$, $i_t = da_t/dt = 0$ and

$$I_t = \frac{dA_t}{dt} = R_S A_t + \frac{y_t - c_t}{1-\epsilon}. \tag{C.12}$$

We show in turn that $a_T = A_T = 0$, that (i_t, I_t) is feasible, and that the market-clearing conditions 3.5 and 3.6 are satisfied.

Wealth at T

Equation $a_T = 0$ follows from the definition of a_t and equation $F_4 = 0$. (Equation C.10 implies that A_{τ_1} is the same as in equation C.2.) The definition of A_t implies that

$$A_T = A_{\tau_1} e^{R_S(T-(\tau_1+\Delta))} + \int_{\tau_1+\Delta}^{T} \frac{y_t - c_t}{1-\epsilon} e^{R_S(T-t)} dt. \tag{C.13}$$

To show that $A_T = 0$, we compute in two ways the following expression

$$\int_0^{\tau_1} \frac{y_t - c_t}{1+\epsilon} e^{-\rho(t)} dt + \int_{\tau_1}^{\tau_1+\Delta} ((R-r)A_{\tau_1} + y_t - c_t) e^{-\rho(t)} dt + \int_{\tau_1+\Delta}^{T} \frac{y_t - c_t}{1-\epsilon} e^{-\rho(t)} dt. \tag{C.14}$$

Using equation C.2 to rewrite the first term and equation $F_4 = 0$ to rewrite the second, we get

$$A_{\tau_1} e^{-\rho(\tau_1)} - \int_{\tau_1}^{\tau_1+\Delta} rA_{\tau_1} e^{-\rho(t)} dt + \int_{\tau_1+\Delta}^{T} \frac{y_t - c_t}{1-\epsilon} e^{-\rho(t)} dt. \tag{C.15}$$

Integrating the second term and using equation C.13, we get $A_T e^{-\rho(T)}$. Alternatively, we note that equation $F_3 = 0$ implies that

$$\int_{\tau_1}^{\tau_1+\Delta} (R - r) A_{\tau_1} e^{-\rho(t)} dt = \epsilon A_{\tau_1} (e^{-\rho(\tau_1)} + e^{-\rho(\tau_1+\Delta)}). \tag{C.16}$$

Using equations C.2 and C.16, we can write expression C.14 as

$$\int_0^T (y_t - c_t) e^{-\rho(t)} dt + \epsilon A_{\tau_1} e^{-\rho(\tau_1+\Delta)} + \int_{\tau_1+\Delta}^T \frac{\epsilon(y_t - c_t)}{1-\epsilon} e^{-\rho(t)} dt. \tag{C.17}$$

Using equations $F_1 = 0$ and C.13 we finally get $\epsilon A_T e^{-\rho(T)}$. Therefore $A_T = 0$.

Feasibility

The definitions of i_t, I_t, a_t, and A_t imply equations 3.1 and 3.2. They also imply equation 3.3, provided that $I_t > 0$ for $t \in [0, \tau_1)$ and $I_t < 0$ for $t \in [\tau_1 + \Delta, T]$. This property of I_t is true for $\epsilon = 0$, since $I_t = dw_t^*/dt$ and $\tau_1^* < \tau^* < \tau_1^* + \Delta^*$. To prove that it is true for ϵ close to 0, we proceed by continuity and note that $I_t + i_t$ converges uniformly to dw_t^*/dt, and that $|dw_t^*/dt| > \eta > 0$ for $t \in [0, \tau_1^*] \cup [\tau_1^* + \Delta^*, T]$. Coming to the short-sale constraints, $A_t \geq 0$ since $I_t > 0$ for $t \in [0, \tau_1)$ and $I_t < 0$ for $t \in [\tau_1 + \Delta, T]$. To show that $a_t \geq 0$, we again proceed by continuity.

Market-Clearing

To show equation 3.5, we integrate both sides of equation C.11 from τ_1^* to $\tau_1^* + \Delta^*$, and use equations $a_{\tau_1} = a_{\tau_1+\Delta} = 0$ and $F_5 = 0$. To show equation 3.6, we multiply both sides of equation C.10 by $(1+\epsilon)/T$ and integrate from 0 to τ_1. We also multiply both sides of equation C.12 by $(1-\epsilon)/T$ and integrate from $\tau_1 + \Delta$ to T. Adding up and using $A_{\tau_1} = A_{\tau_1+\Delta}$, we get

$$\frac{2\epsilon A_{\tau_1}}{T} = \int_{[0,\tau_1]\cup[\tau_1+\Delta,T]} (RA_t + y_t - c_t) \frac{dt}{T}. \tag{C.18}$$

Subtracting equation $F_5 = 0$ from $F_2 = 0$, we get

$$\int_{[0,\tau_1]\cup[\tau_1+\Delta,T]} (c_t - y_t) \frac{dt}{T} + \frac{2\epsilon A_{\tau_1}}{T} - kD + \frac{R\Delta}{T} A_{\tau_1} = 0. \tag{C.19}$$

Equation 3.6 follows from equations C.18 and C.19, and the fact that $A_t = A_{\tau_1}$ for $t \in [\tau_1, \tau_1 + \Delta]$. Q.E.D.

D Proof of Proposition 5.3

We will show that $\partial r/\partial\epsilon|_{\epsilon=0} < 0$ and conclude by continuity. To compute $\partial r/\partial\epsilon$, we differentiate w.r.t. ϵ equations $F_1 = 0$ and $F_2 = 0$. Noting that for $\epsilon = 0$, F_1 and F_2 are independent of m, τ_1, and Δ, we get

$$\frac{\partial F_1}{\partial r}\frac{\partial r}{\partial \epsilon} + \frac{\partial F_1}{\partial \lambda}\frac{\partial \lambda}{\partial \epsilon} + \frac{\partial F_1}{\partial \epsilon} = 0$$

and the same for F_2. Combining these equations, we get

$$\frac{\partial r}{\partial \epsilon} = -\frac{\begin{vmatrix} \frac{\partial F_1}{\partial \epsilon} & \frac{\partial F_1}{\partial \lambda} \\ \frac{\partial F_2}{\partial \epsilon} & \frac{\partial F_2}{\partial \lambda} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \lambda} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \lambda} \end{vmatrix}}. \tag{D.1}$$

In appendix C we showed that the denominator in equation D.1 is strictly positive. We will now show that the numerator is also strictly positive. As in appendix C, we denote $\partial v(q,t)/\partial q$ by $v_q(q,t)$, and omit the superscript $*$ from r, λ, τ_1, Δ, c_t, and w_t. We also define the function $g(t)$ by

$$g(t) = (m-r)t \text{ for all } t \in [0, \tau_1), \quad g(t) = (m-r)\tau_1 \text{ for all } t \in [\tau_1, \tau_1 + \Delta),$$

$$g(t) = (m-r)\tau_1 + (m+r)(t - (\tau_1 + \Delta)) \text{ for all } t \in [\tau_1 + \Delta, T].$$

Since $m = r(1 + e^{-r\Delta})(1 - e^{-r\Delta}) > r$, $g(t)$ is increasing. To compute the partial derivatives of F_1 and F_2 w.r.t. ϵ, we use the definition of the discount rate $\rho(t)$, i.e. equations 5.6 and 5.7, and the fact that $R = r + m\epsilon$. The numerator in equation D.1 is

$$\begin{vmatrix} -\int_0^T v_q(\lambda e^{-rt}, t)\lambda e^{-2rt} g(t)dt - \int_0^T (c_t - y_t)e^{-rt} g(t)dt & \int_0^T v_q(\lambda e^{-rt}, t)e^{-2rt}dt \\ -\int_0^T v_q(\lambda e^{-rt}, t)\lambda e^{-rt} g(t)\frac{dt}{T} + \frac{2w_{\tau_1}}{T} & \int_0^T v_q(\lambda e^{-rt}, t)e^{-rt}\frac{dt}{T} \end{vmatrix}. \tag{D.2}$$

Proceeding as in appendix C, the determinant D.2 has the same sign as

$$\lambda\left(\frac{\int_0^T v_q(\lambda e^{-rt}, t)e^{-rt} g(t)dt}{\int_0^T v_q(\lambda e^{-rt}, t)e^{-rt}dt} - \frac{\int_0^T v_q(\lambda e^{-rt}, t)e^{-2rt} g(t)dt}{\int_0^T v_q(\lambda e^{-rt}, t)e^{-2rt}dt}\right)$$

$$-\frac{\int_0^T (c_t - y_t)e^{-rt} g(t)dt}{\int_0^T v_q(\lambda e^{-rt}, t)e^{-2rt}dt} - \frac{2w_{\tau_1}}{\int_0^T v_q(\lambda e^{-rt}, t)e^{-rt}dt}. \tag{D.3}$$

Setting $\chi(t) = -v_q(\lambda e^{-rt}, t)e^{-rt}$, $\psi(t) = e^{-rt}$, and $\omega(t) = g(t)$, we conclude from lemma C.1 that the first term of expression D.3 is positive. To show that expression D.3 is strictly positive, it suffices to show that $\int_0^T (c_t - y_t)e^{-rt}g(t)dt$ is strictly positive, since $v_q < 0$ and $w_{\tau_1} > 0$. Using the definition of $g(t)$ and integrating by parts, we can write this term as

$$\int_0^T (c_t - y_t)e^{-rt}(m-r)\min(t,\tau_1)dt + \int_{\tau_1+\Delta}^T (c_t - y_t)e^{-rt}(m+r)(t-(\tau_1+\Delta))dt$$

$$= [-(m-r)\min(t,\tau_1)w_t e^{-rt}]_0^T + (m-r)\int_0^{\tau_1} w_t e^{-rt}dt$$

$$+[-(m+r)(t-(\tau_1+\Delta))w_t e^{-rt}]_{\tau_1+\Delta}^T + (m+r)\int_{\tau_1+\Delta}^T w_t e^{-rt}dt$$

$$= (m-r)\int_0^{\tau_1} w_t e^{-rt}dt + (m+r)\int_{\tau_1+\Delta}^T w_t e^{-rt}dt. \qquad \text{(D.4)}$$

Since $m > r$, expression D.4 is strictly positive. Q.E.D.

Note that the proof of the proposition uses the market-clearing condition for the consumption good,

$$F_2 = \int_0^T (c_t - y_t)\frac{dt}{T} + \frac{2\epsilon}{T}A_{\tau_1} - D = 0,$$

while the intuition given after the statement of the proposition uses the market-clearing condition for the two assets. There is however a close correspondence between the two conditions. The three effects that increase asset demand, increase demand for the consumption good as well. The wealth effect is that agents consume more at each age t. Therefore, it increases total consumption, $\int_0^T c_t dt$. The substitution effect is that agents consume less at the beginning of their lifetimes and more towards the end. The increase in consumption dominates the decrease because of the interest payments on the savings. Therefore, the substitution effect increases total consumption. The direct transaction costs effect is that transaction costs reduce the proceeds of selling the illiquid asset. Since transaction costs use up part of the consumption good, they increase its demand. The wealth effect corresponds to the second term of expression D.3. As expression D.4 shows, the wealth effect is larger the more wealth, w_t, agents accumulate over their lifetimes. This is intuitive: the more agents invest, the more they will benefit from an improvement in investment opportunities. The substitution effect corresponds to the first term of expression D.3. As expression D.3 shows, the substitution effect is large relative to the other effects if v_q is large. This is intuitive: if v_q is large, consumption, $v(\lambda e^{-\rho(t)}, t)$, responds heavily to changes in the discount rate, $\rho(t)$. Finally, the direct transaction costs effect corresponds to the third term of expression D.3. This term is proportional to $2w_{\tau_1}$ since total transaction costs incurred at a given time are $\int_0^T \epsilon |I_t| dt/T = 2\epsilon w_{\tau_1}/T$.

E Proof of Proposition 5.4

We will show that if T is large, the limit of $\partial R/\partial\epsilon|_{\epsilon=0}$ as k goes to 1 is strictly negative, and conclude by continuity. We compute this limit in proposition E.1. As in appendices C and D, we omit the superscript $*$ from r, λ, τ_1, Δ, c_t, and w_t. We also denote the τ^* of section 4 by τ.

Proposition E.1 *Define the function $h(b)$ by*

$$h(b) = \frac{1}{b^2}\left[2b(e^{-b\tau} - e^{-bT}) + r(1 - 2e^{-b\tau} + e^{-bT} + e^{-bT}b(T - 2\tau))\right].$$

Then the limit of $\partial R/\partial\epsilon|_{\epsilon=0}$ as k goes to 1, has the same sign as the expression

$$\frac{1}{A}\left(\frac{h(\omega)\omega}{1-e^{-\omega T}} - \frac{h(\omega+r)(\omega+r)}{1-e^{-(\omega+r)T}}\right) + \left(\frac{h(\omega+r)(\omega+r)}{1-e^{-(\omega+r)T}} - \frac{h(\delta+r)(\delta+r)}{1-e^{-(\delta+r)T}}\right)$$
$$-\frac{2e^{r\tau}\omega}{(\omega+r)(1-e^{-\omega T})}\left((e^{-(\omega+r)\tau} - e^{-(\omega+r)T}) - \frac{1-e^{-(\omega+r)T}}{1-e^{-(\delta+r)T}}(e^{-(\delta+r)\tau} - e^{-(\delta+r)T})\right). \tag{E.1}$$

Proof: Differentiating the identity $R = r + m\epsilon$ for $\epsilon = 0$, we get $\partial R/\partial\epsilon = \partial r/\partial\epsilon + m$. Equation D.1 implies then that

$$\frac{\partial R}{\partial\epsilon} = \frac{\begin{vmatrix} m\frac{\partial F_1}{\partial r} - \frac{\partial F_1}{\partial\epsilon} & \frac{\partial F_1}{\partial\lambda} \\ m\frac{\partial F_2}{\partial r} - \frac{\partial F_2}{\partial\epsilon} & \frac{\partial F_2}{\partial\lambda}\end{vmatrix}}{\begin{vmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial\lambda} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial\lambda}\end{vmatrix}}. \tag{E.2}$$

In appendix C we showed that the denominator in equation E.2 is strictly positive. Proceeding as in appendices C and D, the numerator has the same sign as

$$\lambda\left(\frac{\int_0^T v_q(\lambda e^{-rt},t)e^{-rt}(mt-g(t))dt}{\int_0^T v_q(\lambda e^{-rt},t)e^{-rt}dt} - \frac{\int_0^T v_q(\lambda e^{-rt},t)e^{-2rt}(mt-g(t))dt}{\int_0^T v_q(\lambda e^{-rt},t)e^{-2rt}dt}\right)$$
$$-\frac{\int_0^T(c_t-y_t)e^{-rt}(mt-g(t))dt}{\int_0^T v_q(\lambda e^{-rt},t)e^{-2rt}dt} + \frac{2w_{\tau_1}}{\int_0^T v_q(\lambda e^{-rt},t)e^{-rt}dt}. \tag{E.3}$$

We will compute the limit of expression E.3 as Δ goes to 0. (Equations 5.9 and 5.10 imply that if k goes to 1, Δ goes to 0.)

For $u(c,t) = (c^{1-A}/(1-A))e^{-\beta t}$ and $y_t = ye^{-\delta t}$, consumption, c_t, and wealth, w_t, are given by equations A.4 and A.6, respectively. Equations A.4 and A.5 imply that the Lagrange multiplier, λ, is given by

$$\lambda^{-\frac{1}{A}} = y\frac{\omega + r}{\delta + r}\frac{1 - e^{-(\delta+r)T}}{1 - e^{-(\omega+r)T}}. \tag{E.4}$$

Moreover, since $v(q,t) = q^{-1/A}e^{-\beta t/A}$, we have

$$v_q(\lambda e^{-rt}, t) = -\frac{1}{A}\lambda^{-\frac{1}{A}-1}e^{-\omega t}e^{rt}. \tag{E.5}$$

To compute expression E.3, we thus need to compute

$$\lim_{\Delta\to 0}\int_0^T e^{-bt}(mt - g(t))dt,$$

where b is a constant. The definition of $g(t)$ implies that the integral is equal to

$$\int_{\tau_1}^T e^{-bt}m\min(t-\tau_1,\Delta)dt + \int_0^T e^{-bt}r\min(t,\tau_1)dt - \int_{\tau_1+\Delta}^T e^{-bt}r(t-(\tau_1+\Delta))dt.$$

Integrating by parts the first term, we get

$$[-\frac{1}{b}e^{-bt}m\min(t-\tau_1,\Delta)]_{\tau_1}^T + \frac{1}{b}\int_{\tau_1}^{\tau_1+\Delta} e^{-bt}mdt =$$

$$-\frac{1}{b}e^{-bT}m\Delta + \frac{1}{b^2}m(e^{-b\tau_1} - e^{-b(\tau_1+\Delta)}).$$

We similarly get

$$-\frac{1}{b}e^{-bT}r\tau_1 + \frac{1}{b^2}r(1-e^{-b\tau_1})$$

for the second term, and

$$\frac{1}{b}e^{-bT}r(T-(\tau_1+\Delta)) - \frac{1}{b^2}r(e^{-b(\tau_1+\Delta)} - e^{-bT})$$

for the third term. As Δ goes to 0, τ_1 goes to τ. Moreover, equation 5.4 implies that $m\Delta$ goes to 2. Therefore

$$\lim_{\Delta\to 0}\int_0^T e^{-bt}(mt - g(t))dt = h(b). \tag{E.6}$$

Using equations A.4, A.6, E.4, E.5, and E.6, we can easily compute the limit of expression E.3 as Δ goes to 0. Dividing the result by λA, we get expression E.1. Q.E.D.

We now prove proposition 5.4.

Proof: We first derive asymptotic expressions for r and τ, as T goes to ∞. To determine r we use equation A.9. Setting $r = \beta - A\hat{\omega}/T$, i.e. $\omega = \hat{\omega}/T$, we can write this equation as

$$\frac{\frac{\hat{\omega}}{T} + \beta - A\frac{\hat{\omega}}{T}}{\frac{\hat{\delta}}{T} + \beta - A\frac{\hat{\omega}}{T}} \frac{1 - e^{-(\hat{\delta}+\beta T - A\hat{\omega})}}{1 - e^{-(\hat{\omega}+\beta T - A\hat{\omega})}} \frac{1 - e^{-\hat{\omega}}}{\hat{\omega}} - \frac{1 - e^{-\hat{\delta}}}{\hat{\delta}} = \frac{D}{y}. \tag{E.7}$$

For $T = \infty$, equation E.7 becomes

$$\frac{1 - e^{-\hat{\omega}}}{\hat{\omega}} - \frac{1 - e^{-\hat{\delta}}}{\hat{\delta}} = \frac{D}{y}. \tag{E.8}$$

It is easy to check that the function $x \to (1 - e^{-x})/x$ is strictly decreasing, and is ∞, 1, and 0 when x is $-\infty$, 0, and ∞, respectively. Therefore equation E.8 has a unique solution $\hat{\omega}(\hat{\delta})$. By the implicit function theorem, equation E.7 has a solution $\hat{\omega} = \hat{\omega}(\hat{\delta}) + o(1)$. Since $D/y \in (0, 1)$, $\hat{\omega}(\hat{\delta})$ goes to a strictly positive limit, $\hat{\omega}(\infty)$, as $\hat{\delta}$ goes to ∞. Therefore the function $\hat{\tau}(\hat{\delta})$ defined by

$$\hat{\tau}(\hat{\delta}) = \frac{1}{\hat{\delta} - \hat{\omega}(\hat{\delta})} \log \frac{\hat{\delta}}{\hat{\omega}(\hat{\delta})}$$

goes to 0 as $\hat{\delta}$ goes to ∞ and, as a result, is smaller than 1 for $\hat{\delta}$ large enough. To determine τ we write that dw_t/dt, given by expression A.7, is equal to 0 for $t = \tau$. Setting $\hat{\tau} = \tau/T$, we can write the resulting equation as

$$\frac{1 - e^{-(\hat{\delta}+rT)}}{1 - e^{-(\hat{\omega}+rT)}} \left(-\hat{\omega}e^{-\hat{\omega}\hat{\tau}} - rTe^{-\hat{\omega}}e^{-rT(1-\hat{\tau})}\right) + \left(\hat{\delta}e^{-\hat{\delta}\hat{\tau}} + rTe^{-\hat{\delta}}e^{-rT(1-\hat{\tau})}\right) = 0. \tag{E.9}$$

In equation E.9, r is given by $r = \beta - A\hat{\omega}/T$. For $T = \infty$ and $\hat{\tau} < 1$, equation E.9 becomes

$$-\hat{\omega}(\hat{\delta})e^{-\hat{\omega}(\hat{\delta})\hat{\tau}} + \hat{\delta}e^{-\hat{\delta}\hat{\tau}} = 0,$$

and has the solution $\hat{\tau}(\hat{\delta}) < 1$. By the implicit function theorem, equation E.9 has a solution $\hat{\tau} = \hat{\tau}(\hat{\delta}) + o(1)$.

We now come to expression E.1. Since $\delta = \hat{\delta}/T$, $\omega = \hat{\omega}(\hat{\delta})/T + o(1/T)$, $r = \beta + o(1)$, and $\tau/T = \hat{\tau}(\hat{\delta}) + o(1)$, the third term, that corresponds to transaction costs, goes to 0 as T goes to ∞. Moreover, since

$$\frac{h(\omega + r)(\omega + r)}{1 - e^{-(\omega+r)T}} \quad \text{and} \quad \frac{h(\delta + r)(\delta + r)}{1 - e^{-(\delta+r)T}}$$

go to 1, the second term, that corresponds to the wealth effect, goes also to 0. The first term, that corresponds to the substitution effect, is approximately

$$\frac{2(e^{-\hat{\omega}(\hat{\delta})\hat{\tau}(\hat{\delta})} - e^{-\hat{\omega}(\hat{\delta})}) + \beta T(1 - 2e^{-\hat{\omega}(\hat{\delta})\hat{\tau}(\hat{\delta})} + e^{-\hat{\omega}(\hat{\delta})} + e^{-\hat{\omega}(\hat{\delta})}\hat{\omega}(\hat{\delta})(1 - 2\hat{\tau}(\hat{\delta})))}{A(1 - e^{-\hat{\omega}})} - \frac{1}{A},$$

and has the same sign as

$$1 - 2e^{-\hat{\omega}(\hat{\delta})\hat{\tau}(\hat{\delta})} + e^{-\hat{\omega}(\hat{\delta})} + e^{-\hat{\omega}(\hat{\delta})}\hat{\omega}(\hat{\delta})(1 - 2\hat{\tau}(\hat{\delta})). \tag{E.10}$$

Expression E.10 goes to

$$-1 + e^{-\hat{\omega}(\infty)} + e^{-\hat{\omega}(\infty)}\hat{\omega}(\infty) < 0$$

as $\hat{\delta}$ goes to ∞, and is thus negative for $\hat{\delta}$ large enough. Q.E.D.

Notes

[1]For a complete description of transaction costs see Amihud and Mendelson (1991). For evidence on the magnitude of transaction costs see Aiyagari and Gertler (1991).

[2]Amihud and Mendelson (1986) regress cross-sectional asset returns on bid-ask spreads and betas. Using these results, they argue (Amihud and Mendelson (1990)) that a .5% tax would decrease prices by 13.8%. Barclay, Kandel, and Marx (1998) study changes in bid-ask spreads associated with the use of odd-eighths quotes in the Nasdaq, and the migration of stocks from Nasdaq to NYSE or AMEX. Using their results, they argue that a tax would have much smaller effects. Umlauf (1993), Campbell and Froot (1994), and Stulz (1994) present and discuss empirical evidence on the effects of transaction taxes in Sweden and the U.K.

[3]Jones and Seguin (1998) examine how the abolition of fixed commissions on May 1975, affected NYSE prices.

[4]See, for instance, Duffie and Sun (1990), Davis and Norman (1990), Grossman and Laroque (1990), Dumas and Luciano (1991), Fleming et al. (1992), Shreve and Soner (1992), and Schroder (1998).

[5]Amihud and Mendelson assume risk-neutral agents. Aiyagari and Gertler, and Huang assume one illiquid asset, which is riskless. Vayanos allows for many risky illiquid assets. See also Brennan (1975), Goldsmith (1976), Levy (1978), and Mayshar (1979,1981), for static models with fixed transaction costs.

[6]For evidence on short-sale costs see, for instance, Tuckman and Vila (1992).

[7]See, for instance, Kehoe (1989).

[8]For $A = 1$, $u(c) = \log(c)$.

[9]We can deduce R from m by $R = r + m\epsilon$.

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Figure Legends

There is no legend for figure 1.

The legend for figure 2 is:

Figure 2 plots the rates of return on the two assets as a function of the supply of the illiquid asset, k. The solid line represents the benchmark case where there are no transaction costs and the two assets are identical. The lines with the short and long dashes represent the rates of return on the liquid and illiquid assets, respectively, with transaction costs.