An Institutional Theory of Momentum and Reversal

Dimitri Vayanos*
LSE, CEPR and NBER

Paul Woolley†
LSE

February 9, 2013‡

Abstract

We propose a theory of momentum and reversal based on flows between investment funds. Flows are triggered by changes in fund managers' efficiency, which investors either observe directly or infer from past performance. Momentum arises if flows exhibit inertia, and because rational prices under-react to expected future flows. Reversal arises because flows push prices away from fundamental values. Besides momentum and reversal, flows generate comovement, lead-lag effects and amplification, with these being larger for high-idiosyncratic-risk assets. A calibration of our model using evidence on mutual-fund returns and flows generates sizeable Sharpe ratios for momentum and value strategies.

Keywords: Asset pricing, momentum, reversal, fund flows, limits of arbitrage

*d.vayanos@lse.ac.uk

†p.k.woolley@lse.ac.uk

‡We thank Nick Barberis, Jonathan Berk, Bruno Biais, Pierre Collin-Dufresne, Peter DeMarzo, Xavier Gabaix, John Geanakoplos, Jennifer Huang, Ravi Jagannathan, Jakub Jurek, Peter Kondor, Arvind Krishnamurthy, Dong Lou, Toby Moskowitz, Anna Pavlova, Lasse Pedersen, Christopher Polk, Matthew Pritzker, Tarun Ramadorai, Tano Santos, Jeremy Stein, Pietro Veronesi, Luigi Zingales, seminar participants at Amsterdam, Bergen, Berkeley, Brunel, Chicago, Columbia, Lausanne, Leicester, LSE, MIT, Munich, Northwestern, NYU, Oslo, Oxford, Paris, Stanford, Sydney, Toulouse, Zurich, and participants at the American Economic Association 2010, American Finance Association 2010, CREST-HEC 2010, CRETE 2009, EFMA 2011, Gerzensee 2008 and NBER Asset Pricing 2009 conferences for helpful comments.

1 Introduction

Two of the most prominent financial-market anomalies are momentum and reversal. Momentum is the tendency of assets with good (bad) recent performance to continue overperforming (underperforming) in the near future. Reversal concerns predictability based on a longer performance history: assets that performed well (poorly) over a long period tend to subsequently underperform (overperform). Closely related to reversal is the value effect, whereby the ratio of an asset's price relative to book value is negatively related to subsequent performance. Momentum and reversal have been documented extensively and for a wide variety of assets.[1]

Momentum and reversal are viewed as anomalies because they are hard to explain within the standard asset-pricing paradigm with a rational representative agent. The prevalent explanations of these phenomena are behavioral.[2] In this paper we show that momentum and reversal can result from flows between investment funds in markets where fund investors and managers are rational. Besides momentum and reversal, fund flows generate comovement, lead-lag effects, and amplification, with all these being larger for assets with high idiosyncratic risk. A calibration of our model using evidence on mutual-fund returns and flows generates sizeable Sharpe ratios for momentum and value strategies.

Our explanation of momentum and reversal is as follows. Suppose that a negative shock hits the fundamental value of some assets. Investment funds holding these assets realize low returns, triggering outflows by investors who update negatively about the efficiency of the managers running these funds. As a consequence of the outflows, funds sell assets they own, and this depresses further the prices of the assets hit by the original shock. Momentum arises if the outflows are gradual, and if they trigger a gradual price decline and a drop in expected returns. Reversal arises because outflows push prices below fundamental values, and so expected returns eventually rise. Gradual outflows can be the consequence of investor inertia or institutional constraints, and we simply assume them. We explain, however, why gradual outflows can trigger a gradual decline in rational prices and a drop in expected returns. This result, key to momentum, is surprising. Indeed, why do rational investors absorb the outflows, buying assets whose expected returns have decreased?

[1] Jegadeesh and Titman (1993) document momentum for individual US stocks, predicting returns over horizons of 3-12 months by returns over the past 3-12 months. DeBondt and Thaler (1985) document reversal, predicting returns over horizons of up to 5 years by returns over the past 3-5 years. Fama and French (1992) document the value effect. This evidence has been extended to stocks in other countries (Fama and French 1998, Rouwenhorst 1998), industry-level portfolios (Grinblatt and Moskowitz 1999), country indices (Asness, Liew, and Stevens 1997, Bhojraj and Swaminathan 2006), bonds (Asness, Moskowitz and Pedersen 2011), currencies (Bhojraj and Swaminathan 2006) and commodities (Gorton, Hayashi and Rouwenhorst 2012).

[2] See, for example, Barberis, Shleifer and Vishny (1998), Daniel, Hirshleifer and Subrahmanyam (1998), Hong and Stein (1999), and Barberis and Shleifer (2003).

Rational investors in our model buy assets whose expected returns have decreased because of what we term the "bird-in-the-hand" effect. Assets that experience a price drop and are expected to continue underperforming in the short run are those held by investment funds expected to experience outflows. The anticipation of outflows causes these assets to be underpriced and to guarantee investors an attractive return (bird in the hand) over a long horizon. Investors could earn an even more attractive return on average (two birds in the bush), by buying these assets after the outflows occur. This, however, exposes them to the risk that the outflows might not occur, in which case the assets would cease to be underpriced.[3]

The bird-in-the-hand effect, and its implication that rational prices under-react to expected future flows, extend beyond institutional flows; all that is needed is that flows are gradual, uncertain, and can push prices away from fundamental values. Provided that flows have these properties, momentum and reversal can arise even in a model with one risky asset and no delegated fund management.[4] Our theory has thus two distinct parts: first show the bird-in-the-hand effect and rational momentum, and second do this in a model with multiple risky assets and delegation. Multiple assets allow us to derive additional phenomena such as comovement and lead-lag effects. Delegation provides us with a driver of flows: investor response to changes in fund manager efficiency. It also allows us to tie the pricing phenomena that we derive to a growing empirical literature on institutional flows, summarized later in the Introduction. At the same time, delegation gives rise to a "career-concern" effect, which works against the bird-in-the-hand effect and eliminates momentum for some parameter values: because fund managers care about fund flows in addition to investment performance, they hedge against the risk that outflows might occur and that mispricings increase.

Section 2 presents our model. We consider an infinite-horizon economy with multiple risky assets, which we refer to as stocks, and one riskless asset. A competitive investor can access the stocks only through two investment funds. We assume that one of the funds tracks mechanically a market index, so portfolio optimization concerns only the other fund, which we refer to as the active fund. To ensure that the active fund can add value over the index fund, we assume that the market index differs from the true market portfolio characterizing equilibrium asset returns. Our assumptions on the index and the active fund are meant to capture in a stylized and parsimonious

[3]The bird-in-the-hand effect can be illustrated in the following simple example. An asset is expected to pay off 100 in Period 2. If outflows do not occur in Period 1 the price will be 100, but if they occur the price will drop to 80. Each scenario is equally likely. Buying the asset in Period 0 at 92 earns an investor a two-period expected capital gain of 8. Buying in Period 1 earns an expected capital gain of 20 if outflows occur and 0 if they do not. A risk-averse investor might prefer earning 8 rather than 20 or 0 with equal probabilities, even though the expected capital gain between Periods 0 and 1 is negative. The bird-in-the-hand effect can be reinterpreted in the formal language of Merton's ICAPM: the investor buys an underpriced asset even though the price is expected to drop even further in the short run, to hedge against a reduction in the mispricing.

[4]See Vayanos and Woolley (2013).

manner a setting in which investors have access to multiple funds with different portfolios. Flows between the active and the index fund could be interpreted more generally as between active funds pursuing different strategies.

The active fund is run by a competitive manager, who can also invest his personal wealth through the fund. The latter assumption is for parsimony: in addition to generating a simple objective that the manager maximizes when choosing the fund's portfolio, it ensures that the manager acts as trading counterparty to the investor's flows. The manager can be interpreted more generally as an aggregate of all agents absorbing the flows, and this eliminates the need to introduce additional agents into the model. Both investor and manager are risk-averse. To ensure that the investor has a motive to move across funds, and so generate flows, we assume that she suffers a time-varying cost from holding the active fund. The interpretation of the cost that best fits our model is as a managerial perk, although other interpretations such as managerial ability are possible within more complicated versions of the model.

Section 3 solves for equilibrium in the case of symmetric information, where the cost of holding the active fund is observable by both the investor and the manager. Section 4 considers the more complicated case of asymmetric information, where the investor does not observe the cost and must infer it from fund returns. Asymmetric information is more realistic, since investors typically do not observe managerial perks or ability, and yields a richer set of results.

Momentum and reversal arise even under symmetric information. For example, following an increase in the cost, the investor flows out of the active and into the index fund, effectively selling stocks that the active fund overweights relative to the index. Because flows are gradual, the bird-in-the-hand effect implies that the price of these stocks declines gradually, yielding momentum. Flows also generate comovement and lead-lag effects, i.e., cross-asset predictability. Since, for example, outflows from the active fund lower the prices of stocks that it overweights and raise those that it underweights, they increase comovement within each group while reducing comovement across groups. Moreover, since a price drop of an overweighted stock is correlated with outflows, it forecasts low expected returns of other overweighted stocks in the short run and high returns in the long run.

The key new feature of asymmetric information is that fund flows not only cause stock returns, as under symmetric information, but are also caused by them. For example, a negative cashflow shock to a stock that the active fund overweights lowers the active fund's performance relative to the index fund. The investor then infers that the cost has increased and flows out of the active and into the index fund. This lowers the stock's price, amplifying the effect of the original

shock. Amplification generates new channels of momentum, reversal and comovement. For example, momentum and reversal arise conditional not only on past returns, as under symmetric information, but also on past cashflow shocks. Moreover, a new channel of comovement is that a cashflow shock to one stock induces fund flows which affect the prices of other stocks.

Momentum, reversal, lead-lag effects and comovement are larger for stocks with high idiosyncratic risk. This result holds under both symmetric and asymmetric information, with the intuition being different in the two cases. For example, in the case of asymmetric information, a cashflow shock to a stock with high idiosyncratic risk generates a large discrepancy between the performance of the active and of the index fund. This causes large fund flows and price effects.

Our model's implications for return predictability map naturally into implications for active portfolio management. We sketch some of them in Section 5, where we calibrate our model using evidence on mutual-fund returns and flows. Assuming a Sharpe ratio of 30% for the market index, we find Sharpe ratios of 40% for a cross-sectional momentum strategy and 26% for a cross-sectional value strategy. Thus, a significant fraction of these strategies' actual Sharpe ratios can perhaps be explained based only on institutional flows and rational behavior.

Empirical support for our theory comes from Lou (2012), who examines more generally the implications of predictable fund flows. Lou forecasts flows in or out of mutual funds based on the funds' past performance, and imputes these fund-level flows into flows in or out of individual stocks held by the funds. Consistent with our theory, the stock-level flows explain a significant part of stock-level momentum, especially for large stocks and in recent data where mutual funds are more prevalent. Other predictions of our theory are also supported in the data. For example, Coval and Stafford (2007) find that mutual funds experiencing large outflows engage in distressed selling of their stock portfolios, and this generates significant price pressure and return predictability. Anton and Polk (2012) and Greenwood and Thesmar (2011) find that comovement between stocks is larger when these are held by many mutual funds in common, controlling for style characteristics. Jotikasthira, Lundblad and Ramadorai (2012) find price pressure and comovement in an international context.

Behavioral models of momentum and reversal include Barberis, Shleifer and Vishny (1998), Daniel, Hirshleifer and Subrahmanyam (1998), Hong and Stein (1999), and Barberis and Shleifer (BS 2003). BS is the closest to our work. They assume that stocks belong to styles and are traded between switchers and fundamental investors. Following a stock's bad performance, switchers assume that the corresponding style will perform poorly in the future, and switch to other styles.

Switching is assumed to be gradual, and leads to momentum because fundamental investors are assumed not to anticipate the switchers' predictable flows. Switching also generates comovement of stocks within a style, lead-lag effects, and amplification. We show that these effects can be consistent with rational behavior. We also study the effects of idiosyncratic risk and career concerns, neither of which is examined in BS. Rational models of momentum include Berk, Green and Naik (1999), Johnson (2002) and Shin (2006), which assume symmetric information, and Albuquerque and Miao (2012) and Cespa and Vives (2012), which assume asymmetric information. In these papers a risky asset's expected return decreases following a low return typically because volatility or asset supply decreases.

The equilibrium implications of delegated portfolio management are the subject of a growing literature. In Shleifer and Vishny (1997), fund flows are an exogenous function of the funds' past performance, and amplify the effects of cashflow shocks. In He and Krishnamurthy (2012ab) and Brunnermeier and Sannikov (2012) the equity stake of fund managers must exceed a lower bound because of optimal contracting under moral hazard, and amplification effects can again arise.[5] In Dasgupta, Prat and Verardo (2011), reputation concerns cause managers to herd, and this generates momentum under the additional assumption that the market makers trading with the managers are either monopolistic or myopic. In Basak and Pavlova (2012), flows by investors benchmarked against an index cause stocks in the index to comove.[6] We contribute a number of new results to this literature, e.g., momentum with competitive and rational agents, and larger effects for high-idiosyncratic-risk assets. We also bring the analysis of delegation and fund flows within a flexible normal-linear framework that yields closed-form solutions.

2 Model

2.1 Assets

Time t is continuous and goes from zero to infinity. There are N risky assets and a riskless asset. We refer to the risky assets as stocks, but they could also be interpreted as industry-level portfolios or asset classes. The riskless asset has an exogenous, continuously compounded return r. The

[5] Amplification effects can also arise when agents face margin constraints or have wealth-dependent risk aversion. See the survey by Gromb and Vayanos (2010) and the references therein.

[6] Other models exploring equilibrium implications of delegated portfolio management include Brennan (1993), Vayanos (2004), Dasgupta and Prat (2008), Petajisto (2009), Cuoco and Kaniel (2011), Malliaris and Yan (2011), Guerreri and Kondor (2012), and Kaniel and Kondor (2012). See also Berk and Green (2004), in which fund flows are driven by fund performance because investors learn about managers' ability, and feed back into performance because of decreasing returns to managing a large fund.

stocks pay dividends over time, and their prices are determined endogenously in equilibrium. We denote by D_{nt} the cumulative dividend per share of stock $n = 1, .., N$, by S_{nt} the stock's price, and by π_n the stock's supply in terms of number of shares. The exact specification of the dividend process has little bearing on our results, so we consider a simple *i.i.d.* specification. We assume that the vector $D_t \equiv (D_{1t}, .., D_{Nt})'$ follows the process

$$dD_t = \bar{F}dt + \sigma dB_t^D, \tag{2.1}$$

where $\bar{F}$ is a constant vector, σ is a constant matrix of diffusion coefficients, B_t^D is a d-dimensional Brownian motion, and v' is the transpose of the vector v. Thus, the instantaneous dividends dD_t paid between t and $t + dt$ are independent over time and identically distributed, with expectation $\bar{F}dt$ and covariance matrix $\Sigma dt \equiv \sigma\sigma' dt$.

2.2 Investment Funds

A competitive investor can invest in the riskless asset and in the stocks. The investor, however, faces the friction that she can access the stocks only through two investment funds. This could reflect the cost of learning about individual stocks and trading them. We assume that one investment fund is passively managed and tracks mechanically a market index. This is for simplicity, so that portfolio optimization concerns only the other fund, which we term the active fund. We assume that the market index includes a fixed number η_n of shares of stock n. Thus, if the vectors $\pi \equiv (\pi_1, .., \pi_N)$ and $\eta \equiv (\eta_1, .., \eta_N)$ are collinear, the market index is capitalization-weighted and coincides with the market portfolio.

To ensure that the active fund can add value over the index fund, we assume that the market index differs from the true market portfolio characterizing equilibrium asset returns. This can be because the market index does not include some stocks, and so the vectors π and η are not collinear. Alternatively, the market index can coincide with the market portfolio, but unmodelled buy-and-hold investors, such as firms' managers or founding families, can hold a portfolio different from the market portfolio. That is, buy-and-hold investors hold $\hat{\pi}_n$ shares of stock n, and the vectors π and $\hat{\pi} \equiv (\hat{\pi}_1, .., \hat{\pi}_N)$ are not collinear. To nest the two cases, we define a vector $\theta \equiv (\theta_1, .., \theta_N)$ to coincide with π in the first case and $\pi - \hat{\pi}$ in the second. The vector θ represents the residual supply left over from buy-and-hold investors, and is the true market portfolio characterizing equilibrium asset returns. We assume that θ is not collinear with the market index η, and set

$$\Delta \equiv \theta\Sigma\theta'\eta\Sigma\eta' - (\eta\Sigma\theta')^2 > 0.$$

Our assumptions on the index and the active fund are meant to capture in a stylized and parsimonious manner a setting in which investors have access to multiple funds with different portfolios. The main intuitions coming out of our model are likely to carry over to alternative settings, e.g., no index fund but two active funds that specialize in different segments of the market.

2.3 Agents

The investor determines how to allocate her wealth between the riskless asset, the index fund, and the active fund. She maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-E\int_0^\infty \exp(-\alpha c_t - \beta t)dt, \tag{2.2}$$

where α is the coefficient of absolute risk aversion, c_t is consumption, and β is the discount rate. The investor's control variables are consumption c_t and the number of shares x_t and y_t of the index and active fund, respectively.

The active fund is run by a competitive manager, who can also invest his personal wealth in the fund. The manager determines the active portfolio and the allocation of his wealth between the riskless asset and the fund. He maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-E\int_0^\infty \exp(-\bar{\alpha}\bar{c}_t - \bar{\beta}t)dt, \tag{2.3}$$

where $\bar{\alpha}$ is the coefficient of absolute risk aversion, $\bar{c}_t$ is consumption, and $\bar{\beta}$ is the discount rate. The manager's control variables are consumption $\bar{c}_t$, the number of shares $\bar{y}_t$ of the active fund, and the active portfolio $z_t \equiv (z_{1t}, .., z_{Nt})$, where z_{nt} denotes the number of shares of stock n included in one share of the active fund.

The assumption that the manager can invest his personal wealth in the active fund is for parsimony: it generates a simple objective that the manager maximizes when choosing the fund's portfolio, and ensures that the manager acts as trading counterparty to the investor's flows.[7] Under

[7]Restricting the manager not to invest his personal wealth in the index fund is also in the spirit of generating a simple objective. Indeed, in the absence of this restriction, the active portfolio would be indeterminate: the manager could mix a given active portfolio with the index, and make that the new active portfolio, while achieving the same personal portfolio through an offsetting short position in the index. Note that restricting the manager not to invest in the index only weakly constrains his personal portfolio since he can always modify the portfolio of the active fund and his stake in that fund.

the alternative assumption that the manager must invest his wealth in the riskless asset, we would need to introduce two new elements into the model: a performance fee to provide the manager with incentives for portfolio choice, and an additional set of "smart-money" agents with the expertise to invest in individual stocks, identify the investor's flows and act as counterparty to them. This would complicate the model without changing the main intuitions. The manager in our model can be interpreted as an aggregate of all smart-money agents.

2.4 Cost of Active Fund

Under the assumptions introduced so far, and in the absence of other frictions, the equilibrium takes a simple form. As we show in Section 3, the investor holds stocks only through the active fund since its portfolio dominates the index portfolio. As a consequence, the active fund holds the true market portfolio θ, and there are no flows between the two funds.

To generate fund flows, we assume that the investor suffers a time-varying cost from investing in the active fund. Empirical evidence on the existence of such a cost is provided in a number of papers. For example, Grinblatt and Titman (1989), Wermers (2000), and Kacperczyk, Sialm and Zheng (KSZ 2008) study the return gap, defined as the difference between a mutual fund's return over a given quarter and the return of a hypothetical portfolio invested in the stocks that the fund holds at the beginning of the quarter. The return gap varies significantly across funds and over time. It is also persistent, with a half-life of about 2.5 years according to KSZ. The high persistence indicates that the return gap is linked to underlying fund characteristics, and indeed such a correlation exists in the data.

We model the return gap in a simple manner: we assume that the investor's return from the active fund is equal to the gross return, made of the dividends and capital gains of the stocks held by the fund, net of a time-varying cost. Empirical studies typically attribute the return gap to agency costs, operational costs, and managerial stock-picking ability. The interpretation that best fits our model is agency costs: the cost is a perk that the manager can extract from the investor.[8] Operational costs and stock-picking ability fit better the more complicated version of our model where the manager must invest his wealth in the riskless asset and other smart-money agents absorb the investor's flows. Indeed, when the manager also invests in the active fund, it would be natural

[8]Examples of perks in a delegated portfolio management context are late trading and soft-dollar commissions. Managers engaging in late trading use their privileged access to the fund to buy or sell fund shares at stale prices. Late trading was common in many funds and led to the 2003 mutual-fund scandal. Soft-dollar commissions is the practice of inflating funds' brokerage commissions to pay for services that mainly benefit managers, e.g., promote the fund to new investors.

to assume that his investment is affected by changes in operational costs or stock-picking ability. We are assuming, however, that the manager does not incur the cost on his investment.

We assume that the index fund entails no cost, so its gross and net returns coincide. This is for simplicity, but also fits the interpretations of the return gap. Indeed, managing an index fund involves no stock-picking ability, and operational and agency costs are smaller than for active funds.

We model the cost as a flow (i.e., the cost between t and $t+dt$ is of order dt), and assume that the flow cost is proportional to the number of shares y_t that the investor holds in the active fund. We denote the coefficient of proportionality by C_t and assume that it follows the process

$$dC_t = \kappa(\bar{C} - C_t)dt + sdB_t^C, \tag{2.4}$$

where κ is a mean-reversion parameter, $\bar{C}$ is a long-run mean, s is a positive scalar, and B_t^C is a Brownian motion independent of B_t^D. The mean-reversion of C_t is not essential for momentum and reversal, which occur even when $\kappa = 0$.

To remain consistent with the managerial-perk interpretation of the cost, we allow the manager to derive a benefit from the investor's participation in the active fund. We model the benefit in the same manner as the cost, i.e., a flow which is proportional to the number of shares y_t that the investor holds in the active fund. If the cost is a perk that the manager can extract efficiently, then the coefficient of proportionality for the benefit is C_t. We allow more generally the coefficient of proportionality to be λC_t, where $\lambda \geq 0$ is a constant that can be interpreted as the efficiency of perk extraction. Varying λ generates a rich specification of the manager's objective. When $\lambda = 0$, the manager cares about fund performance only through his personal investment in the fund, and his objective is similar to the fund investor's. When instead $\lambda > 0$, the manager has career concerns, i.e., cares about the risk that the investor might reduce her participation in the fund. As we show in Section 3.3, momentum arises when λ is not too large, i.e., perk extraction is inefficient. Momentum would also arise under non-perk interpretations of the cost (operational costs or stock-picking ability) since λ would then be zero.

The cost and benefit are assumed proportional to y_t for analytical convenience. At the same time, these variables are sensitive to how shares of the active fund are defined (e.g., they change with a stock split). We define one share of the fund by the requirement that its market value equals the equilibrium market value of the entire fund. Under this definition, the number of fund shares

held by the investor and the manager in equilibrium sum to one, i.e.,

$$y_t + \bar{y}_t = 1. \tag{2.5}$$

We define one share of the index fund to coincide with the market index η.

We assume that the investor can adjust her active-fund holdings y_t to new information only gradually. Gradual adjustment can result from contractual restrictions or institutional decision lags. For simplicity we model these frictions as a flow cost $\psi(dy_t/dt)^2/2$ that the investor must incur when changing y_t.[9]

The manager observes all the variables in the model. The investor observes the returns and share prices of the index and active funds, but not the same variables for the individual stocks. We study both the case of symmetric information, where the investor observes the cost C_t, and that of asymmetric information, where C_t is observable only by the manager.

3 Symmetric Information

In this section we study the case of symmetric information, where the investor observes the cost C_t. We look for an equilibrium in which stock prices $S_t \equiv (S_{1t}, .., S_{Nt})'$ take the form

$$S_t = \frac{\bar{F}}{r} - (a_0 + a_1 C_t + a_2 y_t), \tag{3.1}$$

where (a_0, a_1, a_2) are constant vectors. The first term is the present value of expected dividends, discounted at the riskless rate r, and the second term is a risk discount linear in (C_t, y_t). The risk discount moves in response to fund flows, as we show later in this section. The rate $v_t \equiv dy_t/dt$ at which the investor changes her active-fund holdings in our conjectured equilibrium is

$$v_t = b_0 - b_1 C_t - b_2 y_t, \tag{3.2}$$

where (b_0, b_1, b_2) are constants. We expect (b_1, b_2) to be positive, i.e., the investor disinvests faster from the active fund when C_t and y_t are large. We refer to an equilibrium satisfying (3.1) and (3.2) as linear.

[9] An example of contractual restrictions is lock-up periods, often imposed by hedge funds, which require investors not to withdraw capital for a pre-specified time period. Institutional decision lags can arise for investors such as pension funds, foundations or endowments, where decisions are made by boards of trustees that meet infrequently. We employ the quadratic specification $\psi(dy_t/dt)^2/2$ to model these frictions because it preserves the linearity of the model.

3.1 Manager's Optimization

The manager chooses the active fund's portfolio z_t, the number $\bar{y}_t$ of fund shares that he owns, and consumption $\bar{c}_t$. The manager's budget constraint is

$$dW_t = rW_t dt + \bar{y}_t z_t (dD_t + dS_t - rS_t dt) + \lambda C_t y_t dt - \bar{c}_t dt. \tag{3.3}$$

The first term is the return from the riskless asset, the second term is the return from the active fund in excess of the riskless asset, the third term is the manager's benefit from the investor's participation in the fund, and the fourth term is consumption. To compute the return from the active fund, we note that since one share of the fund corresponds to z_t shares of the stocks, the manager's effective stock holdings are $\bar{y}_t z_t$ shares. These holdings are multiplied by the vector $dR_t \equiv dD_t + dS_t - rS_t dt$ of the stocks' excess returns per share (referred to as returns, for simplicity).

The manager's optimization problem is to choose controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (2.3) subject to the budget constraint (3.3) and the investor's holding policy (3.2). The active fund's portfolio z_t satisfies, in addition, the normalization

$$z_t S_t = (\theta - x_t \eta) S_t. \tag{3.4}$$

This is because one share of the active fund is defined so that its market value equals the equilibrium market value of the entire fund. Moreover, the latter is $(\theta - x_t\eta)S_t$ because in equilibrium the active fund holds the true market portfolio θ minus the investor's holdings $x_t\eta$ of the index fund. We conjecture that the manager's value function is

$$\bar{V}(W_t, \bar{X}_t) \equiv -\exp\left[-\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right)\right], \tag{3.5}$$

where $\bar{X}_t \equiv (C_t, y_t)'$, $(\bar{q}_0, \bar{q}_1, \bar{q}_2)$ are constants, and $\bar{Q}$ is a constant symmetric 2×2 matrix. The Bellman equation is

$$\max_{\bar{c}_t, \bar{y}_t, z_t} \left[-\exp(-\bar{\alpha}\bar{c}_t) + \mathcal{D}\bar{V} - \bar{\beta}\bar{V}\right] = 0, \tag{3.6}$$

where $\mathcal{D}\bar{V}$ is the drift of the process $\bar{V}$ under the controls $(\bar{c}_t, \bar{y}_t, z_t)$.

Proposition 3.1 *The value function (3.5) satisfies the Bellman equation (3.6) if* $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{Q})$ *satisfy a system of six scalar equations.*

In the proof of Proposition 3.1 we show that the optimization over $(\bar{c}_t, \bar{y}_t, z_t)$ can be reduced to optimization over the manager's consumption $\bar{c}_t$ and effective stock holdings $\hat{z}_t \equiv \bar{y}_t z_t$. Given $\hat{z}_t$, the decomposition between $\bar{y}_t$ and z_t is determined by the normalization (3.4). The first-order condition with respect to $\hat{z}_t$ is

$$E_t(dR_t) = r\bar{\alpha}Cov_t(dR_t, \hat{z}_t dR_t) + (\bar{q}_1 + \bar{q}_{11}C_t + \bar{q}_{12}y_t)Cov_t(dR_t, dC_t). \tag{3.7}$$

Eq. (3.7) links expected stock returns to the risk faced by the manager. The expected return that the manager requires from a stock depends on the stock's covariance with the manager's portfolio $\hat{z}_t$ (first term in the right-hand side), and on the covariance with changes to the cost C_t (second term). The latter effect reflects a hedging demand by the manager. We derive the implications of (3.7) for the cross section of expected returns later in this section.

3.2 Investor's Optimization

The investor chooses a number of shares x_t in the index fund and y_t in the active fund, and consumption c_t. The investor's budget constraint is

$$dW_t = rW_t dt + x_t \eta dR_t + y_t \left(z_t dR_t - C_t dt\right) - \frac{1}{2}\psi v_t^2 dt - c_t dt. \tag{3.8}$$

The first three terms are the returns from the riskless asset, the index fund, and the active fund (net of the cost C_t), respectively. The fourth term is the cost of adjustment and the fifth term is consumption. The investor's optimization problem is to choose controls (c_t, x_t, y_t) to maximize the expected utility (2.2) subject to the budget constraint (3.8). The investor takes the active fund's portfolio z_t as given and equal to its equilibrium value $\theta - x_t\eta$. We study this optimization problem in two steps. In a first step, we optimize over (c_t, x_t), assuming that v_t is given by (3.2). We solve this problem using dynamic programming, and conjecture the value function

$$V(W_t, X_t) \equiv -\exp\left[-\left(r\alpha W_t + q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t\right)\right], \tag{3.9}$$

where $X_t \equiv (C_t, y_t)'$, (q_0, q_1, q_2) are constants, and Q is a constant symmetric 2×2 matrix. The Bellman equation is

$$\max_{c_t, x_t} \left[-\exp(-\alpha c_t) + \mathcal{D}V - \beta V\right] = 0, \tag{3.10}$$

where $\mathcal{D}V$ is the drift of the process V under the controls (c_t, x_t). In a second step, we derive conditions under which the control v_t given by (3.2) is optimal.

Proposition 3.2 *The value function (3.9) satisfies the Bellman equation (3.10) if* (q_0, q_1, q_2, Q) *satisfy a system of six scalar equations. The control* v_t *given by (3.2) is optimal if* (b_0, b_1, b_2) *satisfy a system of three scalar equations.*

3.3 Equilibrium

In equilibrium, the active fund's portfolio z_t is equal to $\theta - x_t\eta$, and the shares held by the manager and the investor sum to one. Combining these equations with the first-order conditions and value-function equations (Propositions 3.1 and 3.2), yields a system of equations characterizing a linear equilibrium. Proposition 3.3 shows that a unique linear equilibrium exists when the diffusion coefficient s of C_t is small. This is done by computing explicitly the linear equilibrium for $s = 0$ and applying the implicit function theorem. Our numerical solutions for general values of s in Section 5 seem to also generate a unique linear equilibrium. Moreover, the properties that we derive analytically for small s in the rest of this section seem to hold for general values of s.[10]

Proposition 3.3 *For small* s*, there exists a unique linear equilibrium. The constants* (b_1, b_2) *are positive, and the vectors* (a_1, a_2) *are given by*

$$a_i = \gamma_i \Sigma p_f', \tag{3.11}$$

where γ_1 *is a positive and* γ_2 *a negative constant, and*

$$p_f \equiv \theta - \frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta \tag{3.12}$$

is the "flow portfolio." Eq. (3.11) holds in any linear equilibrium for general values of s*.*

Proposition 3.3 can be specialized to the benchmark case of costless delegation, where the investor's cost C_t of investing in the active fund is constant over time and equal to zero. This case can be derived by setting C_t, as well as its long-run mean $\bar{C}$ and diffusion coefficient s, to zero.

Corollary 3.1 (Costless Delegation) *When* $C_t = \bar{C} = s = 0$*, the investor adjusts her holdings of the active fund to the steady-state value* $\lim_{t\to\infty} y_t = \bar{\alpha}/(\alpha + \bar{\alpha})$ *and those of the index fund to*

[10] This applies to $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 < 0$, and to Corollaries 3.2, 3.5 and 3.6 (with a different threshold λ^R), and (3.7).

$\lim_{t\to\infty} x_t = 0$. *Stocks' expected returns in the steady state are given by the one-factor model*

$$\lim_{t\to\infty} E_t(dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\Sigma\theta' dt = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}} Cov_t(dR_t, \theta dR_t), \tag{3.13}$$

with the factor being the true market portfolio θ.

Corollary 3.1 concerns a steady state reached for $t \to \infty$ because the adjustment cost prevents the investor from adjusting instantaneously to her optimal holdings. In the steady state, the investor holds only the active fund because that fund offers a superior portfolio than the index fund at no cost. The relative shares of the investor and the manager in the active fund are determined by their risk-aversion coefficients, according to optimal risk-sharing. Stocks' expected returns are determined by the covariance with the true market portfolio. The intuition for the latter result is that since the index fund receives zero investment, the true market portfolio coincides with the active portfolio z_t, which is also the portfolio held by the manager. Since the manager determines the cross section of expected returns through the first-order condition (3.7), and there is no hedging demand because C_t is constant, the true market portfolio is the only pricing factor.

When the cost C_t is stochastic, the investor's fund holdings and the stocks' expected returns are stochastic in the steady state. We next determine how shocks to C_t affect fund flows, stock prices and expected returns.

Following an increase in C_t, the investor flows out of the active and into the index fund. Because of the adjustment cost, this flow does not occur instantaneously at time t, but only gradually for $t' > t$. Corollary 3.2 computes the implied change in the number of shares of each stock that the investor holds through the aggregate of the two funds. We consider the expectation as of time t of stock holdings at time $t' > t$, to isolate the effect of the time-t shock from those of subsequent shocks. Because of the linearity of our model, this amounts to setting the realized values of the subsequent shocks to zero.

Corollary 3.2 (Fund Flows) *The change in the investor's expected stock holdings at time $t' > t$, caused by a change in C_t at time t, is proportional to the flow portfolio p_f:*

$$\frac{\partial E_t(x_{t'}\eta + y_{t'}z_{t'})}{\partial C_t} = -\frac{b_1\left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right]}{b_2-\kappa} p_f. \tag{3.14}$$

For small s, the coefficient of p_f is negative.

The change in the investor's stock holdings is proportional to the flow portfolio p_f, defined in (3.12). This portfolio consists of the true market portfolio θ, plus a position in the market index η that renders the covariance with the index equal to zero. Long positions in p_f correspond to large components of the vector θ relative to η, and hence to stocks that the active fund overweights relative to the index fund. Conversely, short positions correspond to stocks that the active fund underweights. In flowing out of the active and into the index fund, the investor is selling a slice of the flow portfolio, thus selling stocks that the active fund overweights and buying stocks that it underweights.

The effect of C_t on stock prices derives from that on fund flows, and is computed in Corollary 3.3. Following an increase in C_t, the investor gradually sells a slice of the flow portfolio. Since the manager takes the other side of this transaction, he becomes increasingly averse to holding the flow portfolio and stocks covarying positively with it. Therefore, the expected returns of these stocks increase and their price decreases. Moreover, the price decreases not only when the manager acquires the flow portfolio, after time t, but also in anticipation of this happening, at time t. Conversely, the time-t price of stocks covarying negatively with the flow portfolio increases.

Corollary 3.3 (Prices) *The change in stock prices at time t, caused by a contemporaneous change in C_t, is proportional to stocks' covariance with the flow portfolio p_f:*

$$\frac{\partial S_t}{\partial C_t} = -\gamma_1 \Sigma p_f' = -\frac{\gamma_1}{1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}} Cov_t(dR_t, p_f dR_t) = -\frac{\gamma_1}{1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}} Cov_t(d\epsilon_t, p_f d\epsilon_t). \tag{3.15}$$

where $d\epsilon_t \equiv (d\epsilon_{1t}, .., d\epsilon_{Nt})'$ denotes the residual from a regression of stock returns dR_t on the market-index return ηdR_t.

What characteristics of a stock determine its covariance with the flow portfolio, and hence its sensitivity to fund flows? A stock's relative weight in the active and index fund influences the sign of the covariance. Indeed, stocks that the active fund overweights are likely to covary positively with the flow portfolio because they receive positive weight in that portfolio, while stocks that the active fund underweights are likely to covary negatively.

A stock's idiosyncratic risk influences the magnitude of the covariance: stocks with high idiosyncratic risk have higher covariance with the flow portfolio in absolute value, and are thus more sensitive to fund flows. The intuition is that changes in the cost C_t of investing in the active fund cause the investor to rebalance between the active and the index fund, but do not affect her overall

exposure to the index. Indeed, since investing in the index fund is costless, the investor's willingness to bear risk that correlates perfectly with the index remains constant, and only her willingness to bear orthogonal risk changes. Therefore, changes in C_t, and the fund flows they trigger, do not affect the expected return and price of the index, and of stocks that correlate perfectly with the index.[11] They affect stocks that carry orthogonal, i.e., idiosyncratic, risk.

Since changes in C_t, and the fund flows they trigger, affect prices, they contribute to comovement between stocks. Corollary 3.4 decomposes the covariance matrix of stock returns into a fundamental covariance, driven purely by cashflows, and a non-fundamental covariance, introduced by fund flows.

Corollary 3.4 (Comovement) *The covariance matrix of stock returns is*

$$Cov_t(dR_t, dR_t') = \left(\Sigma + s^2\gamma_1^2 \Sigma p_f' p_f \Sigma\right) dt, \tag{3.16}$$

the sum of a fundamental covariance Σdt, driven purely by cashflows, and a non-fundamental covariance $s^2\gamma_1^2 \Sigma p_f' p_f \Sigma dt$, introduced by fund flows. The non-fundamental covariance is positive for stock pairs whose covariance with the flow portfolio has the same sign, and is negative otherwise.

The non-fundamental covariance between a pair of stocks is proportional to the product of the covariances between each stock in the pair and the flow portfolio. It is thus large in absolute value when the stocks have high idiosyncratic risk, because they are more affected by changes in C_t. Moreover, it can be positive or negative: positive for stock pairs whose covariance with the flow portfolio has the same sign, and negative otherwise. Intuitively, two stocks move in the same direction in response to fund flows if they are both overweighted or both underweighted by the active fund, but move in opposite directions if one is overweighted and the other underweighted.

Corollary 3.5 derives the cross section of stocks' expected returns. A stock's expected return is determined by the stock's covariance with two risk factors: the market index and the flow portfolio. The covariance with the index is driven purely by cashflows. The covariance with the flow portfolio is instead influenced by the stock's relative weight in the active and index fund, and by the stock's idiosyncratic risk. It determines both the stock's sensitivity to fund flows (Corollary 3.3) and the stock's expected return (Corollary 3.5). Our model implies that stock characteristics such as relative weight across investment funds and idiosyncratic risk should be considered alongside cashflow risk in empirical studies of expected returns.

[11] Stocks that correlate perfectly with the index, and hence have no idiosyncratic risk, have zero covariance with the flow portfolio, as the last equality in Corollary 3.3 confirms.

Corollary 3.5 (Expected Returns) *Stocks' expected returns are given by the two-factor model*

$$E_t(dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}Cov_t(dR_t, \eta dR_t) + \Lambda_t Cov_t(dR_t, p_f dR_t), \tag{3.17}$$

with the factors being the market index and the flow portfolio. The factor risk premium Λ_t associated to the flow portfolio is

$$\Lambda_t = r\bar{\alpha} + \frac{1}{1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}}\left(\gamma_1^R C_t + \gamma_2^R y_t - \gamma_1 s^2 \bar{q}_1\right), \tag{3.18}$$

where (γ_1^R, γ_2^R) are constants. For small s, the constant γ_1^R is negative if

$$\lambda < \lambda^R \equiv \frac{\bar{\alpha}}{2(\alpha+\bar{\alpha}) + \frac{\psi\eta\Sigma\eta'}{2\Delta}\left[r + (r+2\kappa)\sqrt{1+\frac{4(\alpha+\bar{\alpha})\Delta}{r\psi\eta\Sigma\eta'}}\right]}, \tag{3.19}$$

and is positive otherwise, and the constant γ_2^R is negative.

The factor risk premium of the index is constant over time. The factor risk premium Λ_t of the flow portfolio, however, is time-varying and depends on fund flows. For example, outflows from the active fund raise Λ_t, hence raising the expected returns of stocks that covary positively with the flow portfolio. The time-variation of Λ_t is closely related to momentum and reversal.

Consider an increase in the cost C_t of investing in the active fund. Corollary 3.3 shows that the prices of stocks that covary positively with the flow portfolio decrease at time t, in anticipation of the future outflows from the active fund. Corollary 3.5 implies that when $\gamma_1^R < 0$, the expected returns of these stocks also decrease at time t. The simultaneous decrease in prices and expected returns can appear puzzling since a decrease in expected returns is typically accompanied by a price increase. The explanation is that while expected returns decrease in the short run, they increase in the long run, in response to the gradual outflows triggered by the increase in C_t. It is the latter increase that causes the price decrease at time t.

The short-run decrease in expected returns is puzzling. Indeed, in absorbing the investor's outflows, the manager (who as we point out in Section 2.3 can be viewed as an aggregate of all smart-money agents) buys stocks that the active fund overweights. Why is he willing to buy these stocks, even though their expected return has decreased? The intuition is that the manager prefers to guarantee a "bird in the hand." Indeed, the anticipation of future outflows causes active

overweights to become underpriced and offer an attractive return over a long horizon. The manager could earn an even more attractive return, on average, by buying these stocks after the outflows occur. This, however, exposes him to the risk that the outflows might not occur, in which case the stocks would cease to be underpriced. Thus, the manager might prefer to guarantee an attractive long-horizon return (bird in the hand), and pass up on the opportunity to exploit an uncertain short-run price drop (two birds in the bush). Note that in seeking to guarantee the long-horizon return, the manager is, in effect, causing the short-run drop. This is because his buying pressure prevents the price in the short run from dropping to a level that fully reflects the future outflows, i.e., from which a short-run drop is not expected.

The bird-in-the-hand effect can be seen formally in the manager's first-order condition (3.7). Following an increase in C_t, the expected return of a stock that covaries positively with the flow portfolio decreases, lowering the left-hand side of (3.7). Therefore, the manager remains willing to hold the stock only if its risk, described by the right-hand side of (3.7), also decreases. The decrease in risk is not caused by a lower covariance between the stock and the manager's portfolio $\hat{z}_t$ (first term in the right-hand side). Indeed, because the adjustment cost prevents the investor from adjusting instantaneously, $\hat{z}_t$ remains constant immediately following the increase in C_t. The decrease in risk is instead driven by the manager's hedging demand (second term in the right-hand side), which means that a stock covarying positively with the flow portfolio becomes a better hedge for the manager when C_t increases. The intuition is that when C_t increases, mispricing becomes severe, and the manager has attractive investment opportunities. Hedging against a reduction in these opportunities requires holding stocks that perform well when C_t decreases, and these are the stocks covarying positively with the flow portfolio. Holding such stocks guarantees the manager an attractive long-horizon return—the bird-in-the-hand effect.

The response of expected returns to changes in C_t causes returns to be predictable based on past returns. We characterize this predictability in Corollary 3.6. As in the rest of our analysis, we evaluate returns over an infinitesimal time period dt. We compute the covariance between the vector of returns at time t, i.e., between t and $t + dt$, and the same vector at time $t' > t$, i.e., between t' and $t' + dt$.

Corollary 3.6 (Return Predictability) *The covariance between stock returns at time t and those at time $t' > t$ is*

$$Cov_t(dR_t, dR'_{t'}) = \left[\chi_1 e^{-\kappa(t'-t)} + \chi_2 e^{-b_2(t'-t)}\right] \Sigma p'_f p_f \Sigma dt dt', \tag{3.20}$$

where (χ_1, χ_2) *are constants. For small* s*, the term in the square bracket of (3.20) is positive if* $t' - t < \hat{u}$ *and negative if* $t' - t > \hat{u}$*, for a threshold* $\hat{u}$ *which is positive if* $\lambda < \lambda^R$ *and zero if* $\lambda > \lambda^R$*. A stock's return predicts positively the stock's subsequent return for* $t' - t < \hat{u}$ *(short-run momentum) and negatively for* $t' - t > \hat{u}$ *(long-run reversal). It predicts in the same manner the subsequent return of another stock when the covariance between each stock in the pair and the flow portfolio has the same sign, and in the opposite manner otherwise.*

This autocovariance matrix of stock returns is equal to the non-fundamental (contemporaneous) covariance matrix times a scalar, which is negative for long lags but can be positive for short lags. Thus, stocks can exhibit positive autocovariance for short lags and negative for long lags, i.e., short-run momentum and long-run reversal. This is because changes in C_t can move prices and short-run expected returns in the same direction, but long-run expected returns in the opposite direction.

The non-diagonal elements of the autocovariance matrix characterize lead-lag effects, i.e., whether the past return of one stock predicts the future return of another. Lead-lag effects have the same sign as autocovariance for stock pairs whose covariance with the flow portfolio has the same sign. This is because changes in C_t influence both stocks in the same manner.

Momentum arises when the parameter λ that characterizes the manager's career concerns is not too large ($\lambda < \lambda^R$). Corollary 3.7 derives more generally the effects of career concerns on stock prices.

Corollary 3.7 (Career Concerns) *For small* s*, an increase in the parameter* λ *that characterizes the manager's career concerns*

- *Raises* γ_1*, and thus increases the non-fundamental volatility and covariance of stock returns.*
- *Lowers* $\chi_1 + \chi_2$ *and* $\hat{u}$*, and thus reduces the size of momentum and the horizon over which it occurs.*

A career-concerned manager seeks to hedge against outflows. Hedging requires holding a portfolio close to the market index since outflows do not affect the index price. Moreover, the manager's demand to hedge against outflows becomes stronger following outflows. This is because outflows are triggered by increases in the cost C_t, which raise the manager's perk and hence his marginal benefit from investor participation.

The increase in the manager's hedging demand following outflows exacerbates stocks' non-fundamental volatility and covariance, and reduces momentum. For example, an increase in C_t causes stocks that the active fund overweights to drop because of the anticipation of future outflows (Corollary 3.3). The drop is exacerbated because the increase in C_t makes the manager more willing to hedge against outflows, and hence less willing to deviate from the market index and overweight some stocks. This "career-concern" effect also works against the bird-in-the-hand effect, which causes momentum, because the latter makes it attractive for the manager to increase his holdings of active overweights following increases in C_t. When λ exceeds the threshold λ^R, the career-concern effect dominates and momentum does not arise.

4 Asymmetric Information

In this section we study the case of asymmetric information, where the investor does not observe the cost C_t and seeks to infer it from the returns and share prices of the active and index funds. To prevent share prices from fully revealing C_t, as they would under (3.1), we introduce time-variation in stocks' expected dividends. We replace (2.1) by

$$dD_t = F_t dt + \sigma dB_t^D, \tag{4.1}$$

and assume that F_t follows the process

$$dF_t = \kappa(\bar{F} - F_t)dt + \phi\sigma dB_t^F, \tag{4.2}$$

where the mean-reversion parameter κ is the same as for C_t for simplicity, $\bar{F}$ is a long-run mean, ϕ is a positive scalar, and B_t^F is a d-dimensional Brownian motion independent of (B_t^D, B_t^C). The diffusion matrices σ of D_t and $\phi\sigma$ of F_t are proportional for simplicity. We assume that only the manager observes F_t, so F_t is reflected in prices and prevents the investor from inferring C_t. We set $f \equiv 1 + \phi^2/(r+\kappa)^2$.

We look for an equilibrium with the following characteristics. The investor's conditional distribution of C_t is normal with mean $\hat{C}_t$. The variance of the conditional distribution is, in general, a deterministic function of time, but we focus on the steady state reached for $t \to \infty$, where it is constant. Stock prices take the form

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r+\kappa} - (a_0 + a_1\hat{C}_t + a_2 C_t + a_3 y_t), \tag{4.3}$$

where (a_0, a_1, a_2, a_3) are constant vectors. The first two terms are the present value of expected dividends discounted at the riskless rate r, and the third term is a risk discount linear in $(\hat{C}_t, C_t, y_t)$. We conjecture that the effects of $(\hat{C}_t, C_t, y_t)$ depend on the covariance with the flow portfolio, as is the case for (C_t, y_t) under symmetric information. That is, there exist constants $(\gamma_1, \gamma_2, \gamma_3)$ such that for $i = 1, 2, 3$,

$$a_i = \gamma_i \Sigma p_f'. \tag{4.4}$$

The rate $v_t \equiv dy_t/dt$ at which the investor changes her active-fund holdings in our conjectured equilibrium is

$$v_t = b_0 - b_1 \hat{C}_t - b_2 y_t, \tag{4.5}$$

where (b_0, b_1, b_2) are constants. We refer to an equilibrium satisfying (4.3)-(4.5) as linear.

4.1 Investor's Inference

The investor seeks to infer the cost C_t from fund returns and share prices. She observes the net-of-cost return $z_t dR_t - C_t dt$ of the active fund, the return ηdR_t of the index fund, the price $z_t S_t$ of the active fund, and the price ηS_t of the index fund. Because she observes prices, she also observes capital gains, and therefore can deduce net dividends (i.e., dividends minus C_t).

In equilibrium, the active fund's portfolio z_t is equal to $\theta - x_t \eta$. Since the investor knows x_t, observing the price and net dividends of the active and index funds is informationally equivalent to observing the price and net dividends of the index fund and of a hypothetical fund holding the true market portfolio θ. Therefore, we can take the investor's information to be the net dividends of the true market portfolio $\theta dD_t - C_t dt$, the dividends of the index fund ηdD_t, the price of the true market portfolio θS_t, and the price of the index fund ηS_t.[12] We solve the investor's inference problem using recursive (Kalman) filtering.

Proposition 4.1 *The mean $\hat{C}_t$ of the investor's conditional distribution of C_t evolves according to*

[12] We are assuming that the investor's information is the same in and out of equilibrium, i.e., the manager cannot manipulate the investor's beliefs by deviating from his equilibrium strategy and choosing a portfolio $z_t \neq \theta - x_t \eta$. This is consistent with the assumption of a competitive manager. Indeed, one interpretation of this assumption is that there exists a continuum of managers, each with the same C_t. A deviation by one manager would then not affect the investors' beliefs about C_t because these would depend on averages across managers.

the process

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left\{ p_f \left[dD_t - E_t(dD_t) \right] - (C_t - \hat{C}_t)dt \right\}$$
$$- \beta_2 p_f \left[dS_t + a_1 d\hat{C}_t + a_3 dy_t - E_t(dS_t + a_1 d\hat{C}_t + a_3 dy_t) \right], \tag{4.6}$$

where

$$\beta_1 \equiv T \left[1 - (r+k)\frac{\gamma_2 \Delta}{\eta \Sigma \eta'} \right] \frac{\eta \Sigma \eta'}{\Delta}, \tag{4.7}$$

$$\beta_2 \equiv \frac{s^2 \gamma_2}{\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\eta \Sigma \eta'}}, \tag{4.8}$$

and T denotes the distribution's steady-state variance. The variance T is the unique positive solution of the quadratic equation

$$T^2 \left[1 - (r+\kappa)\frac{\gamma_2 \Delta}{\eta \Sigma \eta'} \right]^2 \frac{\eta \Sigma \eta'}{\Delta} + 2\kappa T - \frac{\frac{s^2 \phi^2}{(r+\kappa)^2}}{\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\eta \Sigma \eta'}} = 0. \tag{4.9}$$

The term in β_1 in (4.6) represents the investor's learning from net dividends. Recalling the definition (3.12) of the flow portfolio, we can write this term as

$$-\beta_1 \left\{ \theta dD_t - C_t dt - E_t \left(\theta dD_t - C_t dt \right) - \frac{\eta \Sigma \theta'}{\eta \Sigma \eta'} \left[\eta dD_t - E_t(\eta dD_t) \right] \right\}. \tag{4.10}$$

The investor lowers her estimate of the cost C_t if the net dividends of the true market portfolio $\theta dD_t - C_t dt$ are above expectations. Of course, net dividends can be high not only because C_t is low, but also because gross dividends are high. The investor adjusts for this by comparing with the dividends ηD_t of the index fund. The adjustment is made by computing the regression residual of $\theta dD_t - C_t dt$ on ηD_t, which is the term in curly brackets in (4.10).

The term in β_2 in (4.6) represents the investor's learning from prices. The investor lowers her estimate of C_t if the price of the true market portfolio is above expectations. Indeed, the price can be high because the manager knows privately that C_t is low, and anticipates that the investor will increase her participation in the fund, causing the price to rise, as she learns about C_t. As with dividends, the investor needs to account for the fact that the price of the true market portfolio can be high not only because C_t is low, but also because the manager expects future dividends to be high (F_t small). She adjusts for this by comparing with the price of the index fund. Note that if

F_t is constant ($\phi = 0$), learning from prices is perfect: (4.9) implies that the conditional variance T is zero.

Because the investor compares the performance of the true market portfolio, and hence of the active fund, to that of the index fund, she is effectively using the index as a benchmark. Note that benchmarking is not part of an explicit contract tying the manager's compensation to the index. Compensation is tied to the index only implicitly: if the active fund outperforms the index, the investor infers that C_t is low and increases her participation in the fund.

4.2 Optimization

The manager chooses controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (2.3) subject to the budget constraint (3.3), the normalization (3.4), and the investor's holding policy (4.5). Since stock prices depend on $(\hat{C}_t, C_t, y_t)$, the same is true for the manager's value function. We conjecture that the value function is

$$\bar{V}(W_t, \bar{X}_t) \equiv -\exp\left[-\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2, \bar{q}_3)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right)\right], \tag{4.11}$$

where $\bar{X}_t \equiv (\hat{C}_t, C_t, y_t)'$, $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)$ are constants, and $\bar{Q}$ is a constant symmetric 3×3 matrix.

Proposition 4.2 *The value function (4.11) satisfies the Bellman equation (3.6) if* $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q})$ *satisfy a system of ten scalar equations.*

The investor chooses controls (c_t, x_t, v_t) to maximize the expected utility (2.2) subject to the budget constraint (3.8) and the manager's portfolio policy $z_t = \theta - x_t\eta$. We study this optimization problem in two steps: first optimize over (c_t, x_t), assuming that v_t is given by (4.5), and then derive conditions under which (4.5) is optimal. We solve the first problem using dynamic programming, and conjecture the value function (3.9), where $X_t \equiv (\hat{C}_t, y_t)'$, (q_0, q_1, q_2) are constants, and Q is a constant symmetric 2×2 matrix.

Proposition 4.3 *The value function (3.9) satisfies the Bellman equation (3.10) if* (q_0, q_1, q_2, Q) *satisfy a system of six scalar equations. The control* v_t *given by (4.5) is optimal if* (b_0, b_1, b_2) *satisfy a system of three scalar equations.*

4.3 Equilibrium

Proposition 4.4 shows that a unique linear equilibrium exists when the diffusion coefficient s of C_t is small. Our numerical solutions for general values of s in Section 5 seem to also generate a unique linear equilibrium, with properties similar to those derived in the rest of this section for small s.[13]

Proposition 4.4 *For small s, there exists a unique linear equilibrium. The constants $(b_1, b_2, \gamma_1, \gamma_2)$ are positive and the constant γ_3 is negative.*

Asymmetric information gives rise to amplification: cashflow shocks trigger fund flows, which amplify the effects of cashflow shocks on stock returns. Suppose, for example, that a stock experiences a negative cashflow shock. If the stock is overweighted by the active fund, then the shock lowers the return of the active fund more than of the index fund. As a consequence, the investor infers that C_t has increased, and flows out of the active and into the index fund. Since the active fund overweights the stock, the investor's flows cause the stock to be sold and push its price down. Conversely, if the stock is underweighted, then the investor infers that C_t has decreased, and flows out of the index and into the active fund. Since the active fund underweights the stock, the investor's flows cause again the stock to be sold and push its price down.

Amplification is related to comovement. Recall that under symmetric information fund flows generate comovement between a pair of stocks because they affect the expected return of each stock in the pair. This channel of comovement, to which we refer as ER/ER (where ER stands for expected return) is also present under asymmetric information. Asymmetric information introduces an additional channel involving fund flows, to which we refer as CF/ER (where CF stands for cashflow). This is that cashflow news of one stock in a pair trigger fund flows which affect the expected return of the other stock. The CF/ER channel is the one related to amplification.

While the ER/ER and CF/ER channels are conceptually distinct, they have similar effects: the covariance matrix generated by CF/ER is equal to that generated by ER/ER times a positive scalar. Thus, if ER/ER generates positive covariance between a pair of stocks, so does CF/ER, and if the former covariance is large, so is the latter. Consider, for example, two stocks that the active fund overweights. Since outflows from the active fund (triggered by, e.g., a cashflow shock to a third stock) push down the prices of both stocks, ER/ER generates positive covariance. Moreover, since

[13] This applies to $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 < 0$, and to Corollaries 4.1-4.4. The only exception is that for large values of s, the short-run momentum derived in Corollary 4.4 for all values of λ arises only when λ is not too large.

a negative cashflow shock to one stock triggers outflows from the active fund, which push down the price of the other stock, CF/ER also generates positive covariance. The former covariance is large if the two stocks have high idiosyncratic risk since this makes them more sensitive to fund flows. But high idiosyncratic risk also renders the latter covariance large: cashflow shocks to stocks having low correlation with the index generate a large discrepancy between the active and the index return, hence triggering large fund flows.

The proportionality of the covariance matrices generated by ER/ER and CF/ER implies also proportionality of the non-fundamental covariance matrix under asymmetric information (ER/ER and CF/ER) to that under symmetric information (ER/ER). The proportionality coefficient in the latter relationship is larger than one for small s. Thus, the non-fundamental volatility of each stock is larger under asymmetric information, and so is the absolute value of the non-fundamental covariance between any pair of stocks. This is because of the amplification channel CF/ER, which is present only under asymmetric information.

Corollary 4.1 (Comovement and Amplification) *The covariance matrix of stock returns is*

$$Cov_t(dR_t, dR_t') = \left(f\Sigma + k\Sigma p_f' p_f \Sigma\right) dt, \tag{4.12}$$

where k is a positive constant. The fundamental covariance $f\Sigma dt$ is driven by purely cashflows and is equal to its symmetric-information counterpart.[14] *The non-fundamental covariance $k\Sigma p_f' p_f \Sigma dt$ is introduced by fund flows and is equal to its symmetric-information counterpart times a positive scalar, which is larger than one for small s.*

Stocks' expected returns are determined by the covariance with the market index and the flow portfolio, as under symmetric information. Moreover, the risk premium Λ_t of the flow portfolio is time-varying, and its variation is closely related to momentum and reversal. Consider, for example, a cashflow shock that is negative and hits a stock that the active fund overweights. The shock raises $\hat{C}_t$, the investor's estimate of C_t. This lowers the time-t prices and expected returns of stocks covarying positively with the flow portfolio ($\gamma_1 > 0$ and $\gamma_1^R < 0$), and raises their expected returns in the long run.

Corollary 4.2 (Expected Returns) *Stocks' expected returns are given by the two-factor model (3.17), with the factors being the market index and the flow portfolio. The factor risk premium Λ_t*

[14] The fundamental covariance Σdt in Corollary 3.4 needs to be multiplied by f to account for the effects of F_t. See the proof of the corollary.

associated to the flow portfolio is

$$\Lambda_t = r\bar{\alpha} + \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \left(\gamma_1^R \hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - k_1 \bar{q}_1 - k_2 \bar{q}_2 \right), \tag{4.13}$$

where $(\gamma_1^R, \gamma_2^R, \gamma_3^R, k_1, k_2)$ *are constants. For small* s*, the constants* (γ_1^R, γ_3^R) *are negative and the constant* γ_2^R *is positive.*

The response of expected returns to cashflow shocks causes returns to be predictable based on these shocks. We characterize this predictability in Corollary 4.3, where we compute the covariance between the vectors (dD_t, dF_t) of cashflow shocks at time t and the vector of returns at time $t' > t$. Both covariance matrices are equal to the non-fundamental covariance matrix times a scalar which is positive for short lags and negative for long lags. Thus, cashflow shocks generate short-run momentum and long-run reversal in returns. Note that predictability based on cashflows arises only under asymmetric information because only then cashflow shocks trigger fund flows.

Corollary 4.3 (Return Predictability Based on Cashflows) *The covariance between cashflow shocks* (dD_t, dF_t) *at time* t *and stock returns at time* $t' > t$ *is given by*

$$Cov_t(dD_t, dR'_{t'}) = \frac{\beta_1(r+\kappa)Cov_t(dF_t, dR'_{t'})}{\beta_2 \phi^2} = \left[\chi_1^D e^{-(\kappa+\rho)(t'-t)} + \chi_2^D e^{-b_2(t'-t)} \right] \Sigma p'_f p_f \Sigma dt dt', \tag{4.14}$$

where $(\chi_1^D, \chi_2^D, \rho)$ *are constants. For small* s*, the term in the square bracket of (4.14) is positive if* $t' - t < \hat{u}^D$ *and negative if* $t' - t > \hat{u}^D$*, for a threshold* $\hat{u}^D > 0$*. A stock's cashflow shocks predict positively the stock's subsequent return for* $t' - t < \hat{u}^D$ *(short-run momentum) and negatively for* $t' - t > \hat{u}^D$ *(long-run reversal). They predict in the same manner the subsequent return of another stock when the covariance between each stock in the pair and the flow portfolio has the same sign, and in the opposite manner otherwise.*

We finally examine predictability based on past returns rather than cashflows. This predictability is driven both by cashflow shocks and by shocks to C_t, and has the same form as under symmetric information.[15]

[15]The only difference is that short-run momentum arises for all values of λ. This result, however, relies on the assumption that s is small; for large values of s, short-run momentum arises only when λ is not too large, as under symmetric information.

Corollary 4.4 (Return Predictability) *The covariance between stock returns at time t and those at time $t' > t$ is*

$$Cov_t(dR_t, dR_{t'}') = \left[\chi_1 e^{-(\kappa+\rho)(t'-t)} + \chi_2 e^{-\kappa(t'-t)} + \chi_3 e^{-b_2(t'-t)}\right] \Sigma p_f' p_f \Sigma dt dt', \tag{4.15}$$

where (χ_1, χ_2, χ_3) are constants. For small s, the term in the square bracket of (4.15) is positive if $t' - t < \hat{u}$ and negative if $t' - t > \hat{u}$, for a threshold $\hat{u} > 0$. Given $\hat{u}$, predictability is as in Corollary 3.6.

5 Momentum and Value Strategies

In this section we compute the magnitude of momentum and value effects that our model generates. The magnitude of these effects is typically measured through the performance of trading strategies. We first construct a measure of performance and then compute it in a calibration of our model.

5.1 Performance Measure

Consider a trading strategy consisting of a vector of weights $w_t \equiv (w_{1t}, .., w_{Nt})$, where w_{nt} is the number of shares invested in stock n at time t. Part of the strategy's expected return is compensation for bearing risk that correlates with the market index. We focus on the remainder by index-adjusting the strategy, i.e., combining it with a position in the index such that the covariance between the overall position and the index is zero. The index-adjusted strategy is

$$\hat{w}_t \equiv w_t - \frac{Cov_t(w_t dR_t, \eta dR_t)}{Var_t(\eta dR_t)}\eta. \tag{5.1}$$

Note that the position in the index can be time-varying, reflecting possible time-variation in the covariance between the strategy and the index. We measure the performance of the strategy w_t by the Sharpe ratio of its index-adjusted version $\hat{w}_t$.[16] The Sharpe ratio is the ratio of expected return to standard deviation. We also divide by $\sqrt{dt}$ to express the Sharpe ratio in annualized terms, given that returns are evaluated over an infinitesimal period dt. The Sharpe ratio corresponding

[16]Empirical studies that compute Sharpe ratios of momentum and value strategies typically consider long-short portfolios with zero initial investment, i.e., require the dollar weights to sum to zero. Our index-adjustment is in a similar spirit: the weights that sum to zero are the number of shares times the covariance between one share and the index rather than times the dollar value of one share. We define weights differently because this preserves linearity and simplifies the algebra.

to the strategy w_t thus is

$$SR_w \equiv \frac{E(\hat{w}_t dR_t)}{\sqrt{Var(\hat{w}_t dR_t)dt}}. \tag{5.2}$$

Proposition 5.1 computes the Sharpe ratio under the prices in the asymmetric-information equilibrium of Section 4 and in the steady state reached for $t \to \infty$. It also determines the strategy maximizing the Sharpe ratio.

Proposition 5.1 *The Sharpe ratio corresponding to the strategy w_t is*

$$SR_w = \frac{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) E\left(\Lambda_t w_t \Sigma p_f'\right)}{\sqrt{f\left[E(w_t\Sigma w_t') - \frac{E[(w_t\Sigma\eta')^2]}{\eta\Sigma\eta'}\right] + kE[(w_t\Sigma p_f')^2]}}, \tag{5.3}$$

and is maximized for $w_t = \Lambda_t p_f$.

The intuition for the optimal strategy can be derived from the two-factor model of Corollary 4.2. A strategy's expected return consists of a compensation for bearing risk that correlates with the index, and a compensation for bearing risk that correlates with the flow portfolio. Index adjustment isolates the latter component. Maximizing that component per unit of risk requires holding the flow portfolio since this eliminates uncompensated risk. Moreover, investment in the flow portfolio should be larger when the premium Λ_t associated to that risk factor is high. Since time-variation in Λ_t is caused by fund flows, past and anticipated, the optimal strategy effectively exploits mispricing generated by flows.

Momentum and value strategies exploit aspects of the flow-generated mispricing, and are imperfect approximations of the optimal strategy. We consider the following simple implementation of cross-sectional momentum and value strategies:

$$\left(w_t^M\right)' \equiv \int_{t-\tau}^{t} dR_u, \tag{5.4}$$

$$\left(w_t^V\right)' \equiv \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r+\kappa} - S_t, \tag{5.5}$$

respectively. A stock's momentum weight increases linearly in the stock's cumulative past return over the window $[t-\tau, t]$ for some $\tau > 0$. A stock's value weight increases linearly in the difference between the present value of the stock's expected dividends discounted at the riskless rate, and the stock's price.

Substituting (5.4) and (5.5) into (5.3) yields closed-form solutions for the Sharpe ratios of momentum and value strategies. We omit these calculations because of space constraints. The calculations are in Vayanos and Woolley (VW 2012), who also compute closed-form Sharpe ratios achieved by alternative implementations of momentum and value strategies; by optimal combinations of these strategies; and for general investment horizons.

5.2 Calibration of Model Parameters

We set the riskless rate r to 4%. We assume that there are $N = 10$ stocks, which we interpret as industry sectors. We assume that the market index η includes one share of each stock, i.e., $\eta = \mathbf{1}$, where $\mathbf{1} \equiv (1, .., 1)$, and that the true market portfolio includes one share of each stock on average, i.e., $\bar{\theta} \equiv \sum_{n=1}^{N} \theta_n / N = 1$. These are normalizations because we can redefine one share of each stock and of the index, leaving Sharpe ratios unchanged. We assume that stocks are symmetric in the sense that they all have the same standard deviation of dividends and the same pairwise correlations. (Our closed-form solutions for Sharpe ratios, however, do not require any symmetry.) Hence, the covariance matrix of dividends is $\Sigma = \hat{\sigma}^2(I + \omega \mathbf{1}'\mathbf{1})$, where I is the identity matrix and $(\hat{\sigma}, \omega)$ are scalars. We calibrate $\hat{\sigma}$ using the Sharpe ratio SR_η of the market index η. Closed-form solutions for SR_η and for all other quantities used in the calibration are in VW. We express SR_η in annualized terms, and set it to 30%. This is equal to the Sharpe ratio of the S&P500 index, assuming an annual expected excess return of 4.5% and a standard deviation of 15%. The implied value of $\hat{\sigma}$ is 0.22.[17] We calibrate ω using the correlation between industry sectors and the market. Ang and Chen (2002) find that the average correlation between the returns of an industry sector and of a broad market index is 87% across the 13 sectors that they consider. The implied value of ω is seven. We set ϕ to 0.3. This parameter determines the size of shocks to the expected dividend rate F_t relative to dividends D_t, and has small effects on our calibration results.

VW show that the only characteristic of the true market portfolio θ that affects Sharpe ratios when stocks are symmetric is $\sigma(\theta) \equiv \sqrt{\sum_{n=1}^{N} (\theta_n - \bar{\theta})^2}$. This is the standard deviation across stocks of the number of shares included in θ, and must be strictly positive so that θ differs from the market index η. We calibrate $\sigma(\theta)$ using the average deviation between the weight that an active fund gives to an industry sector and the sector's weight in a broad market index. Kacperczyk, Sialm and Zheng (KSZ1 2005) find that the sum of squared deviations across the ten sectors that they

[17] Note that $\hat{\sigma}$ is a volatility per share rather than per dollar because this is how returns are expressed in our model. We calibrate using Sharpe ratios because these are comparable for per-share and per-dollar returns.

consider is 4.36% for the median fund, implying an average deviation of 6.6% ($10 \times 6.6\%^2 = 4.36\%$). To map this into a value for $\sigma(\theta)$, we adjust for the fact that θ is the sum of active- and index-fund holdings. The holdings of active funds are about ten times those of index funds in KSZ1's sample period, so the average deviation for a combined active and index fund (which is what θ represents) is 6%. The implied value of $\sigma(\theta)$ is 0.6.

To calibrate the diffusion coefficient s of the cost C_t, we recall the cost's interpretation as minus the return gap. Kacperczyk, Sialm and Zheng (KSZ 2008) find that the top decile of mutual funds in terms of lagged one-year return gap earn a monthly CAPM alpha of 0.273%, while the bottom decile earn -0.431%.[18] Since in our model there is only one active fund, we interpret the differential between deciles in a time-series rather than a cross-sectional sense. The implied value of s is 1.6. We set the persistence parameter κ of the cost to 0.3. This is consistent with KSZ's finding that shocks to the return gap shrink to about one-third of their size within four years ($\log(3)/0.3 = 3.7$). We set the long-run mean $\bar{C}$ of the cost to zero, consistent with KSZ's finding that the average return gap in the cross-section is zero. With $\bar{C} = 0$, negative values of the cost are equally likely as positive values, which means that the cost cannot be interpreted solely as a managerial perk. Hence, we emphasize again the managerial-ability interpretation, and for consistency set the parameter λ to zero.

We calibrate the adjustment-cost parameter ψ using the empirical response of fund flows to performance. Coval and Stafford (2007) find that a positive shock to a fund's return generates flows into the fund in each of the next four quarters, with the effect dying off in the fifth. We set $\psi = 1.2$, which ensures that following a positive shock to the active fund's return, the investor's holdings y_t in the fund increase in the next four quarters and start decreasing afterwards.

We set the investor's coefficient of absolute risk aversion α to one. This is a normalization because we can redefine the units of the consumption good, leaving Sharpe ratios unchanged. To calibrate the risk aversion of the manager, we recall that he can be interpreted as an aggregate of all smart-money agents with the expertise to exploit mispricings. We are interested in the capital that these experts own, rather than in the capital they might manage on behalf of outsiders, since only the former can be used to exploit mispricings generated by outsiders' flows. Since most of the financial expertise lies within the financial industry, the capital of experts can be linked to that industry's GDP share. Philippon (2008) reports that the GDP share of the Finance and Insurance industry was 5.5% on average during 1960-2007 in the US. We view this as an upper bound since

[18] KSZ derive two different sets of estimates; we focus on those derived using a back-testing procedure that reduces estimation noise.

only part of that industry concerns asset markets, and set the manager's coefficient of absolute risk aversion $\bar{\alpha}$ to 30. This means that the manager accounts for 3.2% (=1/(30+1)) of aggregate risk tolerance.[19]

As an independent check for our choices of s and $\bar{\alpha}$, we compute two additional quantities: the turnover and the return variance generated by fund flows. Lou (2012) finds that the standard deviation of a stock's quarterly turnover generated by fund flows is 0.7%. Since the funds in Lou's sample account for about 10% of market capitalization, the standard deviation of flow-generated volume is 7% of assets managed by these funds; we find 7.8%. Greenwood and Thesmar (GT 2011) find that fund flows explain 8% of stock-return variance; we find 16%. GT's sample, however, includes less than half of all professionally-managed wealth. Accounting for that, and for possible measurement noise, could well produce a number even larger than 16%. Raising s and $\bar{\alpha}$ to match such a larger number would raise the Sharpe ratios of momentum and value strategies that we find in the next section.

5.3 Calibration Results

The maximum Sharpe ratio across all strategies (Proposition 5.1) is 61%. The value strategy (5.5) achieves a Sharpe ratio of 26%. Figure 1 plots the Sharpe ratio of the momentum strategy (5.4) as a function of the length τ of the window over which past returns are calculated. This Sharpe ratio is positive for windows of less than three years, and then turns negative. Thus, a strategy based on short-run momentum is profitable, and so is one based on long-run reversal. The highest Sharpe ratio of momentum is achieved using a window of four months, and is 40%. Moreover, windows from one to 11 months yield Sharpe ratios larger than 30%, the ratio of the market index. The Sharpe ratio of momentum converges to zero as the window length goes to zero because very recent performance is a very noisy signal of future fund flows.

VW decompose momentum profits into three sources. The first is the positive short-run response of expected returns to shocks (Corollaries 3.6 and 4.4): shocks hit by positive shocks receive high weight in the momentum strategy, and are expected to do well in the short run. The second is the time-series variation of expected returns (regardless of how these respond to shocks): stocks whose conditional expected returns are higher than their unconditional averages receive high expected weight in the momentum strategy, and are expected do well in the short run because

[19]Risk tolerance in our model is independent of capital because of exponential utility. Our choice of $\bar{\alpha}$ is based on the notion that risk tolerance is proportional to capital, which is true under power utility.

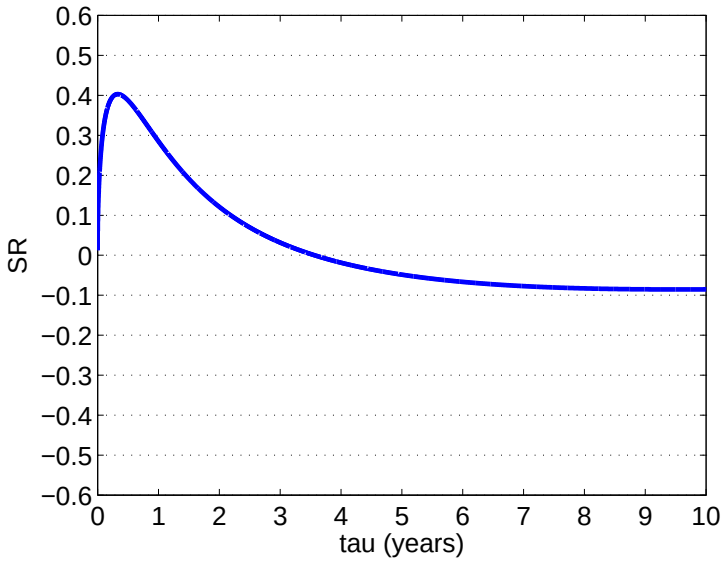

Figure 1: Sharpe ratio of the momentum strategy (5.4) as a function of the length τ of the window over which past returns are calculated. The window length is measured in years. The parameters for which the figure is drawn are described in Section 5.2.

conditional expected returns are persistent. The third, emphasized by Conrad and Kaul (1998), is the cross-sectional variation of unconditional expected returns: stocks with high unconditional expected returns receive high expected weight in the momentum strategy, and are expected to do well going forward. The first source of profits is dominant in our calibration: for example, 62% of the maximum Sharpe ratio in Figure 1 is generated by the first source, 36% by the second, and 2% by the third.

Asness, Moskowitz and Pedersen (AMP 2012) present evidence on Sharpe ratios of momentum and value strategies across a variety of markets and asset classes. Strategies exploiting momentum of individual stocks within a market yield an average Sharpe ratio of 70% across the four markets that AMP consider (US, UK, Continental Europe, Japan), and strategies exploiting value yield 36%. Strategies exploiting momentum of country-level stock indices yield 34%, and so do strategies exploiting value. Our Sharpe ratios are somewhat smaller on average. This is not surprising since we focus only on flows between investment funds and ignore other types of flows, e.g., those generated by individuals holding stocks directly. Such flows would likely increase the Sharpe ratios relative to our calibration. Our calibration shows, however, that even by focusing only on institutional flows and restricting parameters based on evidence from the mutual-fund literature, we can generate sizeable Sharpe ratios.

6 Conclusion

We propose a theory of momentum and reversal based on delegated portfolio management. Flows between investment funds are triggered by changes in fund managers' efficiency, which investors either observe directly or infer from past performance. Momentum arises if fund flows exhibit inertia, and because rational prices do not fully adjust to reflect future flows. Reversal arises because flows push prices away from fundamental values. Besides momentum and reversal, flows generate comovement, lead-lag effects and amplification, with these being larger for high-idiosyncratic-risk assets. While our model focuses on institutional flows, the mechanisms that we uncover are broader and can apply to other types of flows, e.g., those generated by individuals holding stocks directly.

Our model's implications for return predictability map naturally into implications for active portfolio management. We sketch some of these implications in Section 5, where we compute the Sharpe ratios of momentum and value strategies. Much more can be done, however, and in a tractable closed-form manner. For example, while we evaluate momentum and value strategies in isolation, these can be combined. Asness, Moskowitz and Pedersen (2012) find empirically a negative correlation between these strategies, and hence large gains in combining them. Our model also yields a negative correlation, as shown in Vayanos and Woolley (VW 2012). Perhaps more intriguingly, the Sharpe ratio of the optimal momentum-value combination is significantly smaller than the maximum possible Sharpe ratio. Thus, momentum and value strategies can be improved, possibly by using information on fund flows. Finally, while momentum delivers a higher Sharpe ratio than value over a short horizon, VW show that the comparison reverses over a long horizon. Hence, the portfolio allocation of a long-horizon investor between momentum and value can be very different than of a short-horizon investor.

References

Asness, C., J. Liew and R. Stevens, 1997, "Parallels Between the Cross-Sectional Predictability of Stock and Country Returns," *Journal of Portfolio Management,* 23, 79-87.

Asness C., T. Moskowitz and L. Pedersen, 2012, "Value and Momentum Everywhere," *Journal of Finance,* forthcoming.

Albuquerque R. and J. Miao, 2012, "Advance Information and Asset Prices," *Journal of Economic Theory,* forthcoming.

Ang A. and J. Chen, 2002, "Asymmetric Correlations of Equity Portfolios," *Journal of Financial Economics,* 63, 443-494.

Anton, M. and C. Polk, 2012, "Connected Stocks," *Journal of Finance,* forthcoming.

Barberis, N. and A. Shleifer, 2003, "Style Investing," *Journal of Financial Economics,* 68, 161-199.

Barberis, N., A. Shleifer and R. Vishny, 1998, "A Model of Investor Sentiment," *Journal of Financial Economics,* 49, 307-343.

Basak, S. and A. Pavlova, 2012, "Asset Prices and Institutional Investors," *American Economic Review,* forthcoming.

Berk, J. and R. Green, 2004, "Mutual Fund Flows and Performance in Rational Markets," *Journal of Political Economy,* 112, 1269-1295.

Berk, J., R. Green and V. Naik, 1999, "Optimal Investment, Growth Options, and Security Returns," *Journal of Finance,* 54, 1553-1607.

Bhojraj, S. and B. Swaminathan, 2006, "Macromomentum: Returns Predictability in International Equity Indices," *Journal of Business,* 79, 429-451.

Brennan, M., 1993, "Agency and Asset Pricing," working paper, University of California at Los Angeles.

Brunnermeier, M. and Y. Sannikov, 2012, "A Macroeconomic Model with a Financial Sector," working paper, Princeton University.

Cespa, G. and X. Vives, 2012, "Expectations, Liquidity, and Short-term Trading," working paper, Cass Business School.

Conrad, J. and G. Kaul, 1998, "An Anatomy of Trading Strategies," *Review of Financial Studies,* 11, 489-520.

Coval, J. and E. Stafford, 2007, "Asset Fire Sales (and Purchases) in Equity Markets," *Journal of Financial Economics,* 86, 479-512.

Cuoco, D. and R. Kaniel, 2011, "Equilibrium Prices in the Presence of Delegated Portfolio Management," *Journal of Financial Economics,* 101, 264-296.

Daniel, K., D. Hirshleifer and A. Subrahmanyam, 1998, "A Theory of Overconfidence, Self-Attribution, and Security Market Under and Over-Reactions," *Journal of Finance,* 53, 1839-1885.

Dasgupta, A. and A. Prat, 2008, "Information Aggregation in Financial Markets with Career Concerns," *Journal of Economic Theory,* 143, 83-113.

Dasgupta, A., A. Prat and M. Verardo, 2011, "The Price Impact of Institutional Herding," *Review of Financial Studies,* 24, 892-925.

DeBondt, W. and R. Thaler, 1985, "Does the Stock Market Overreact?," *Journal of Finance,* 40, 793-805.

Fama, E. and K. French, 1992, "The Cross-Section of Expected Stock Returns," *Journal of Finance,* 47, 427-465.

Fama, E. and K. French, 1998, "Value versus Growth: The International Evidence," *Journal of Finance,* 53, 1975-1999.

Gorton, G., F. Hayashi and G. Rouwenhorst, 2012, "The Fundamentals of Commodity Futures Returns," *Review of Finance,* forthcoming.

Greenwood, R. and D. Thesmar, 2011, "Stock Price Fragility," *Journal of Financial Economics,* 102, 471-490.

Grinblatt, M. and T. Moskowitz, 1999, "Do Industries Explain Momentum?," *Journal of Finance,* 54, 1249-1290.

Grinblatt, M. and S. Titman, 1989, "Mutual Fund Performance: An Analysis of Quarterly Portfolio Holdings," *Journal of Business,* 62, 393-416.

Gromb, D. and D. Vayanos, 2010, "Limits of Arbitrage," *Annual Review of Financial Economics,* 2, 251-275.

Guerreri, V. and P. Kondor, 2012, "Fund Managers, Career Concerns and Asset Price Volatility," *American Economic Review,* 102, 1986-2017.

He, Z. and A. Krishnamurthy, 2012a, "A Model of Capital and Crises," *Review of Economic Studies,* 79, 735-777.

He, Z. and A. Krishnamurthy, 2012b, "Intermediary Asset Pricing," *American Economic Review,* forthcoming.

Hong, H. and J. Stein, 1999, "A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets," *Journal of Finance,* 54, 2143-2184.

Jegadeesh, N. and S. Titman, 1993, "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance,* 48, 65-91.

Johnson, T., 2002, "Rational Momentum Effects," *Journal of Finance,* 57, 585-608.

Jotikasthira, P., C. Lundblad and T. Ramadorai, 2012, "Asset Fire Sales and Purchases and the International Transmission of Funding Shocks," *Journal of Finance,* 67, 2015-2050.

Kacperczyk, M., C. Sialm and L. Zheng, 2005, "Industry Concentration of Actively Managed Equity Mutual Funds," *Journal of Finance,* 60, 1983-2011.

Kacperczyk, M., C. Sialm and L. Zheng, 2008, "Unobserved Actions of Mutual Funds," *Review of Financial Studies,* 21, 2379-2416.

Kaniel, R. and P. Kondor, 2012, "The Delegated Lucas Tree," *Review of Financial Studies,* forthcoming.

Liptser, R. and A. Shiryaev, 2000, *Statistics of Random Processes I,* second edition, Springer-Verlag, New York.

Lou, D., 2012, "A Flow-Based Explanation for Return Predictability," *Review of Financial Studies,* 25, 3457-3489.

Malliaris, S. and H. Yan, 2011, "Reputation Concerns and Slow-Moving Capital," working paper, Yale University.

Petajisto, A., 2009, "Why Do Demand Curves for Stocks Slope Down?," *Journal of Financial and Quantitative Analysis,* 44, 1013-1044.

Philippon, T., 2008, "The Evolution of the US Financial Industry from 1860 to 2007: Theory and Evidence," working paper, New York University.

Rouwenhorst, G., 1998, "International Momentum Strategies," *Journal of Finance,* 53, 267-284.

Shin, H., 2006, "Disclosure Risk and Price Drift," *Journal of Accounting Research,* 44, 351-379.

Shleifer, A. and R. Vishny, 1997, "The Limits of Arbitrage," *Journal of Finance,* 52, 35-55.

Vayanos, D., 2004, "Flight to Quality, Flight to Liquidity, and the Pricing of Risk," working paper, London School of Economics.

Vayanos, D. and P. Woolley, 2012, "A Theoretical Analysis of Momentum and Value Strategies," working paper, London School of Economics.

Vayanos, D. and P. Woolley, 2013, "The Bird-in-the-Hand Effect and Asset Momentum," working paper, London School of Economics.

Wermers, R., 2000, "Mutual Fund Performance: An Empirical Decomposition into Stock-Picking Talent, Style, Transaction Costs, and Expenses," *Journal of Finance,* 55, 1655-1703.

Appendix

A Symmetric Information

Proof of Proposition 3.1: Eqs. (2.1), (2.4), (3.1) and (3.2) imply that the vector of returns is

$$\begin{aligned} dR_t =& dD_t + dS_t - rS_t dt \\ =& \left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) dt + \sigma dB_t^D - s a_1 dB_t^C, \end{aligned} \tag{A.1}$$

where

$$a_1^R \equiv (r+\kappa)a_1 + b_1 a_2,$$

$$a_2^R \equiv (r+b_2)a_2.$$

Eqs. (2.4), (3.2), (3.3) and (A.1) imply that

$$d\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right) = \bar{G}dt + r\bar{\alpha}\hat{z}_t\sigma dB_t^D - s\left[r\bar{\alpha}\hat{z}_t a_1 - \bar{f}_1(\bar{X}_t)\right] dB_t^C, \tag{A.2}$$

where

$$\begin{aligned} \bar{G} \equiv& r\bar{\alpha}\left[rW_t + \hat{z}_t\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) + \lambda C_t y_t - \bar{c}_t\right] \\ &+ \bar{f}_1(\bar{X}_t)\kappa(\bar{C} - C_t) + \bar{f}_2(\bar{X}_t)v_t + \frac{1}{2}s^2\bar{q}_{11}, \end{aligned}$$

$$\bar{f}_1(\bar{X}_t) \equiv \bar{q}_1 + \bar{q}_{11}C_t + \bar{q}_{12}y_t,$$

$$\bar{f}_2(\bar{X}_t) \equiv \bar{q}_2 + \bar{q}_{12}C_t + \bar{q}_{22}y_t,$$

and $\bar{q}_{ij}$ denotes the (i,j)'th element of $\bar{Q}$. Eqs. (3.5) and (A.2) imply that

$$\mathcal{D}\bar{V} = -\bar{V}\left\{\bar{G} - \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t\Sigma\hat{z}_t' - \frac{1}{2}s^2\left[r\bar{\alpha}\hat{z}_t a_1 - \bar{f}_1(\bar{X}_t)\right]^2\right\}. \tag{A.3}$$

Substituting (A.3) into (3.6), we can write the first-order conditions with respect to $\bar{c}_t$ and $\hat{z}_t$ as

$$\bar{\alpha}\exp(-\bar{\alpha}\bar{c}_t) + r\bar{\alpha}\bar{V} = 0, \tag{A.4}$$

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(\Sigma + s^2 a_1 a_1')\hat{z}_t', \tag{A.5}$$

respectively, where

$$\bar{h}(\bar{X}_t) \equiv ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2 + s^2 a_1 \bar{f}_1(\bar{X}_t). \tag{A.6}$$

Eq. (A.5) is equivalent to (3.7) because of (2.4), (A.1) and

$$Cov_t(dR_t, dR_t') = \left(\Sigma + s^2 a_1 a_1'\right) dt \tag{A.7}$$

which follows from (A.1). Using (A.3) and (A.4), we can simplify (3.6) to

$$\bar{G} - \frac{1}{2}(r\bar{\alpha})^2 \hat{z}_t(\Sigma + s^2 a_1 a_1')\hat{z}_t' + r\bar{\alpha}s^2 \hat{z}_t a_1 \bar{f}_1(\bar{X}_t) - \frac{1}{2}s^2 \bar{f}_1(\bar{X}_t)^2 + \bar{\beta} - r = 0. \tag{A.8}$$

Eqs. (3.5) and (A.4) imply that

$$\bar{c}_t = rW_t + \frac{1}{\bar{\alpha}}\left[\bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t - \log(r)\right]. \tag{A.9}$$

Substituting (A.9) into (A.8) the terms in W_t cancel, and we are left with

$$\begin{aligned}
&r\bar{\alpha}\hat{z}_t\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) + r\bar{\alpha}\lambda C_t y_t - r\left[\bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right] \\
&+ \bar{f}_1(\bar{X}_t)\kappa(\bar{C} - C_t) + \bar{f}_2(\bar{X}_t)v_t + \frac{1}{2}s^2\bar{q}_{11} + \bar{\beta} - r + r\log(r) \\
&- \frac{1}{2}(r\bar{\alpha})^2 \hat{z}_t(\Sigma + s^2 a_1 a_1')\hat{z}_t' + r\bar{\alpha}s^2 \hat{z}_t a_1 \bar{f}_1(\bar{X}_t) - \frac{1}{2}s^2 \bar{f}_1(\bar{X}_t)^2 = 0.
\end{aligned} \tag{A.10}$$

The terms in (A.10) that involve $\hat{z}_t$ can be written as

$$\begin{aligned}
&r\bar{\alpha}\hat{z}_t\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) - \frac{1}{2}(r\bar{\alpha})^2 \hat{z}_t(\Sigma + s^2 a_1 a_1')\hat{z}_t' + r\bar{\alpha}s^2 \hat{z}_t a_1 \bar{f}_1(\bar{X}_t) \\
&= r\bar{\alpha}\hat{z}_t \bar{h}(\bar{X}_t) - \frac{1}{2}(r\bar{\alpha})^2 \hat{z}_t(\Sigma + s^2 a_1 a_1')\hat{z}_t' \\
&= \frac{1}{2}r\bar{\alpha}\hat{z}_t \bar{h}(\bar{X}_t) \\
&= \frac{1}{2}\bar{h}(\bar{X}_t)'(\Sigma + s^2 a_1 a_1')^{-1}\bar{h}(\bar{X}_t),
\end{aligned} \tag{A.11}$$

where the first step follows from (A.6) and the last two from (A.5). Substituting (A.11) into (A.10), we find

$$\begin{aligned}
&\frac{1}{2}\bar{h}(\bar{X}_t)'(\Sigma + s^2 a_1 a_1')^{-1}\bar{h}(\bar{X}_t) + r\bar{\alpha}\lambda C_t y_t - r\left[\bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right] \\
&+ \bar{f}_1(\bar{X}_t)\kappa(\bar{C} - C_t) + \bar{f}_2(\bar{X}_t)v_t + \frac{1}{2}s^2\left[\bar{q}_{11} - \bar{f}_1(\bar{X}_t)^2\right] + \bar{\beta} - r + r\log(r) = 0.
\end{aligned} \tag{A.12}$$

Eq. (A.12) is quadratic in $\bar{X}_t$. Identifying quadratic, linear and constant terms yields six scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{Q})$. We defer the derivation of these equations until the proof of Proposition 3.3 (see (A.48)-(A.50)). ∎

Proof of Proposition 3.2: Suppose that the investor optimizes over (c_t, x_t) but follows the control v_t given by (3.2). Eqs. (2.4), (3.2), (3.8), (3.9) and (A.1) imply that

$$d\left(r\alpha W_t + q_0 + q_1 C_t + \frac{1}{2}q_{11}C_t^2\right) = Gdt + r\alpha(x_t\eta + y_t z_t)\sigma dB_t^D - s\left[r\alpha(x_t\eta + y_t z_t)a_1 - f_1(X_t)\right]dB_t^C, \tag{A.13}$$

where

$$\begin{aligned} G \equiv & r\alpha\left[rW_t + (x_t\eta + y_t z_t)\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1\bar{C} - b_0 a_2\right) - y_t C_t - \frac{1}{2}\psi v_t^2 - c_t\right] \\ & + f_1(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2 q_{11}, \end{aligned}$$

$$f_1(X_t) \equiv q_1 + q_{11}C_t + q_{12}y_t,$$

$$f_2(X_t) \equiv q_2 + q_{12}C_t + q_{22}y_t,$$

and q_{ij} denotes the (i,j)'th element of Q. Eqs. (3.9) and (A.13) imply that

$$\mathcal{D}V = -V\left\{G - \frac{1}{2}(r\alpha)^2(x_t\eta + y_t z_t)\Sigma(x_t\eta + y_t z_t)' - \frac{1}{2}s^2\left[r\alpha(x_t\eta + y_t z_t)a_1 - f_1(X_t)\right]^2\right\}. \tag{A.14}$$

Substituting (A.14) into (3.10), we can write the first-order conditions with respect to c_t and x_t as

$$\alpha\exp(-\alpha c_t) + r\alpha V = 0, \tag{A.15}$$

$$\eta h(X_t) = r\alpha\eta(\Sigma + s^2 a_1 a_1')(x_t\eta + y_t z_t)', \tag{A.16}$$

respectively, where

$$h(X_t) \equiv ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1\bar{C} - b_0 a_2 + s^2 a_1 f_1(X_t). \tag{A.17}$$

Solving for c_t, and proceeding as in the proof of Proposition 3.1, we can simplify (3.10) to

$$\begin{aligned} & r\alpha(x_t\eta + y_t z_t)\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1\bar{C} - b_0 a_2\right) - r\alpha y_t C_t - \frac{1}{2}r\alpha\psi v_t^2 \\ & - r\left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t\right] + f_1(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2 q_{11} + \beta - r + r\log(r) \\ & - \frac{1}{2}(r\alpha)^2(x_t\eta + y_t z_t)(\Sigma + s^2 a_1 a_1')(x_t\eta + y_t z_t)' + r\alpha s^2(x_t\eta + y_t z_t)a_1 f_1(X_t) - \frac{1}{2}s^2 f_1(X_t)^2 = 0. \end{aligned} \tag{A.18}$$

The terms in (A.18) that involve $x_t\eta + y_t z_t$ can be written as

$$
\begin{aligned}
&r\alpha(x_t\eta + y_t z_t)\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) \\
&- \frac{1}{2}(r\alpha)^2(x_t\eta + y_t z_t)(\Sigma + s^2 a_1 a_1')(x_t\eta + y_t z_t)' + r\alpha s^2(x_t\eta + y_t z_t)a_1 f_1(X_t) \\
&= r\alpha(x_t\eta + y_t z_t)h(X_t) - \frac{1}{2}(r\alpha)^2(x_t\eta + y_t z_t)(\Sigma + s^2 a_1 a_1')(x_t\eta + y_t z_t)' \\
&= r\alpha y_t\theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2\theta(\Sigma + s^2 a_1 a_1')\theta' \\
&+ r\alpha x_t(1-y_t)\left\{\eta h(X_t) - \frac{1}{2}r\alpha\eta(\Sigma + s^2 a_1 a_1')\left[x_t(1-y_t)\eta + 2y_t\theta\right]'\right\}', \qquad \text{(A.19)}
\end{aligned}
$$

where the first step follows from (A.17) and the second from the equilibrium condition $z_t = \theta - x_t\eta$. Using $z_t = \theta - x_t\eta$, we can write (A.16) as

$$
\begin{aligned}
&\eta h(X_t) = r\alpha\eta(\Sigma + s^2 a_1 a_1')\left[x_t(1-y_t)\eta + y_t\theta\right]' \\
\Rightarrow &x_t(1-y_t) = \frac{\eta h(X_t) - r\alpha y_t\eta(\Sigma + s^2 a_1 a_1')\theta'}{r\alpha\eta(\Sigma + s^2 a_1 a_1')\eta'}. \qquad \text{(A.20)}
\end{aligned}
$$

Eqs. (A.20) implies that

$$
\begin{aligned}
&r\alpha x_t(1-y_t)\left\{\eta h(X_t) - \frac{1}{2}r\alpha\eta(\Sigma + s^2 a_1 a_1')\left[x_t(1-y_t)\eta + 2y_t\theta\right]'\right\}' \\
&= \frac{1}{2}[r\alpha x_t(1-y_t)]^2\eta(\Sigma + s^2 a_1 a_1')\eta' \\
&= \frac{1}{2}\frac{\left[\eta h(X_t) - r\alpha y_t\eta(\Sigma + s^2 a_1 a_1')\theta'\right]^2}{\eta(\Sigma + s^2 a_1 a_1')\eta'}. \qquad \text{(A.21)}
\end{aligned}
$$

Substituting (A.19) and (A.21) into (A.18), we find

$$
\begin{aligned}
&r\alpha y_t\theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2\theta(\Sigma + s^2 a_1 a_1')\theta' + \frac{1}{2}\frac{\left[\eta h(X_t) - r\alpha y_t\eta(\Sigma + s^2 a_1 a_1')\theta'\right]^2}{\eta(\Sigma + s^2 a_1 a_1')\eta'} - r\alpha y_t C_t - \frac{1}{2}r\alpha\psi v_t^2 \\
&- r\left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t\right] + f_1(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2\left[q_{11} - f_1(X_t)^2\right] \\
&+ \beta - r + r\log(r) = 0. \qquad \text{(A.22)}
\end{aligned}
$$

Since v_t in (3.2) is linear in X_t, (A.22) is quadratic in X_t. Identifying quadratic, linear and constant terms yields six scalar equations in (q_0, q_1, q_2, Q). We defer the derivation of these equations until the proof of Proposition 3.3 (see (A.52)-(A.54)).

We next study optimization over v_t, and derive a first-order condition under which the control (3.2) is optimal. We use a perturbation argument, which consists in assuming that the investor follows the control (3.2) except for an infinitesimal deviation over an infinitesimal internal.[20] Suppose that the investor adds $\omega d\epsilon$ to the control (3.2) over the interval $[t, t+d\epsilon]$ and subtracts $\omega d\epsilon$ over the interval $[t+dt-d\epsilon, t+dt]$, where the infinitesimal $d\epsilon > 0$ is $o(dt)$. The increase in adjustment cost over the first interval is $\psi v_t \omega (d\epsilon)^2$ and over the second interval is $-\psi v_{t+dt}\omega(d\epsilon)^2$. These changes reduce the investor's wealth at time $t+dt$ by

$$\begin{aligned}
&\psi v_t \omega (d\epsilon)^2 (1+rdt) - \psi v_{t+dt}\omega(d\epsilon)^2 \\
&= \psi\omega(d\epsilon)^2(rv_t dt - dv_t) \\
&= \psi\omega(d\epsilon)^2(rv_t dt + b_1 dC_t + b_2 dy_t) \\
&= \psi\omega(d\epsilon)^2 \left\{(r+b_2)v_t dt + b_1\left[\kappa(\bar{C}-C_t)dt + sdB_t^C\right]\right\}, \qquad \text{(A.23)}
\end{aligned}$$

where the second step follows from (3.2) and the third from (2.4). The change in the investor's wealth between t and $t+dt$ is derived from (3.8) and (A.1), by subtracting (A.23) and replacing y_t by $y_t + \omega(d\epsilon)^2$:

$$dW_t = G_\omega dt - \psi\omega(d\epsilon)^2 b_1\left[\kappa(\bar{C}-C_t)dt + sdB_t^C\right] + \left\{x_t\eta + \left[y_t+\omega(d\epsilon)^2\right]z_t\right\}\left[\sigma dB_t^D - sa_1 dB_t^C\right], \qquad \text{(A.24)}$$

where

$$\begin{aligned}
G_\omega \equiv & rW_t + \left\{x_t\eta + \left[y_t+\omega(d\epsilon)^2\right]z_t\right\}\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1\bar{C} - b_0 a_2\right) - \left[y_t+\omega(d\epsilon)^2\right]C_t \\
& - \frac{\psi v_t^2}{2} - c_t - \psi\omega(d\epsilon)^2(r+b_2)v_t.
\end{aligned}$$

The investor's position in the active fund at $t+dt$ is the same under the deviation as under no deviation. Therefore, the investor's expected utility at $t+dt$ is given by the value function (3.9) with the wealth W_{t+dt} determined by (A.24). The drift $\mathcal{D}V$ corresponding to the change in the value function between t and $t+dt$ is given by the following counterpart of (A.14):

$$\begin{aligned}
\mathcal{D}V = & -V\left\{G - \frac{1}{2}(r\alpha)^2\left\{x_t\eta + \left[y_t+\omega(d\epsilon)^2\right]z_t\right\}\Sigma\left\{x_t\eta + \left[y_t+\omega(d\epsilon)^2\right]z_t\right\}'\right. \\
& \left. -\frac{1}{2}s^2\left[r\alpha\left\{x_t\eta + \left[y_t+\omega(d\epsilon)^2\right]z_t\right\}a_1 - f_{1\omega}(X_t)\right]^2\right\}, \qquad \text{(A.25)}
\end{aligned}$$

[20] The perturbation argument is simpler than the dynamic programming approach, which assumes that the investor can follow any control v_t over the entire history. Indeed, under the dynamic programming approach, the state variable y_t which describes the investor's holdings in the active fund must be replaced by two state variables: the holdings out of equilibrium, and the holdings in equilibrium. This is because the latter affect the equilibrium price, which the investor takes as given.

where

$$G \equiv r\alpha G_\omega + f_{1\omega}(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2 q_{11},$$

$$f_{1\omega}(X_t) \equiv f_1(X_t) - r\alpha\psi\omega(d\epsilon)^2 b_1.$$

The drift is maximum for $\omega = 0$, and this yields the first-order condition

$$z_t h(X_t) - r\alpha\psi b_1 s^2(x_t\eta + y_t z_t)a_1 - C_t = r\alpha z_t(\Sigma + s^2 a_1 a_1')(x_t\eta + y_t z_t)' + \psi h_\psi(X_t), \qquad \text{(A.26)}$$

where

$$h_\psi(X_t) \equiv (r + b_2)v_t + b_1\kappa(\bar{C} - C_t) - b_1 s^2 f_1(X_t).$$

Using (A.16) and the equilibrium condition $z_t = \theta - x_t\eta$, we can write (A.26) as

$$\theta h(X_t) - r\alpha\psi b_1 s^2 \left[x_t(1-y_t)\eta + y_t\theta\right] a_1 - C_t = r\alpha\theta(\Sigma + s^2 a_1 a_1')\left[x_t(1-y_t)\eta + y_t\theta\right]' + \psi h_\psi(X_t). \quad \text{(A.27)}$$

Using (A.20), we can write (A.27) as

$$\theta\left[h(X_t) - r\alpha\psi b_1 s^2 y_t a_1\right] - r\alpha\psi b_1 s^2 \frac{\eta h(X_t) - r\alpha y_t \eta(\Sigma + s^2 a_1 a_1')\theta'}{r\alpha\eta(\Sigma + s^2 a_1 a_1')\eta'}\eta a_1 - C_t$$

$$= r\alpha\theta(\Sigma + s^2 a_1 a_1')\left[y_t\theta + \frac{\eta h(X_t) - r\alpha y_t\eta(\Sigma + s^2 a_1 a_1')\theta'}{r\alpha\eta(\Sigma + s^2 a_1 a_1')\eta'}\eta\right]' + \psi h_\psi(X_t). \qquad \text{(A.28)}$$

Eq. (A.28) is linear in X_t. Identifying linear and constant terms, yields three scalar equations in (b_0, b_1, b_2). We defer the derivation of these equations until the proof of Proposition 3.3 (see (A.43)-(A.45)). ∎

Proof of Proposition 3.3: We first impose market clearing and derive the constants $(a_0, a_1, a_2, b_0, b_1, b_2)$ as functions of $(\bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_2, Q)$. For these derivations, as well as for many later proofs, we use the following important properties of the flow portfolio:

$$\eta\Sigma p_f' = 0,$$

$$\theta\Sigma p_f' = p_f\Sigma p_f' = \frac{\Delta}{\eta\Sigma\eta'}.$$

Setting $z_t = \theta - x_t\eta$ and $\bar{y}_t = 1 - y_t$, we can write (A.5) as

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(\Sigma + s^2 a_1 a_1')(1 - y_t)(\theta - x_t\eta)'. \qquad \text{(A.29)}$$

Premultiplying (A.29) by η, dividing by $r\bar{\alpha}$, and adding to (A.20) divided by $r\alpha$, we find

$$\eta\left[\frac{h(X_t)}{r\alpha}+\frac{\bar{h}(\bar{X}_t)}{r\bar{\alpha}}\right]=\eta(\Sigma+s^2a_1a_1')\theta'. \tag{A.30}$$

Eq. (A.30) is linear in (C_t, y_t). Identifying terms in C_t and y_t, we find

$$\left(\frac{r+\kappa+s^2q_{11}}{r\alpha}+\frac{r+\kappa+s^2\bar{q}_{11}}{r\bar{\alpha}}\right)\eta a_1+\frac{b_1(\alpha+\bar{\alpha})}{r\alpha\bar{\alpha}}\eta a_2=0, \tag{A.31}$$

$$\left(\frac{s^2q_{12}}{r\alpha}+\frac{s^2\bar{q}_{12}}{r\bar{\alpha}}\right)\eta a_1+\frac{(r+b_2)(\alpha+\bar{\alpha})}{r\alpha\bar{\alpha}}\eta a_2=0, \tag{A.32}$$

respectively. Eqs. (A.31) and (A.32) imply that

$$\eta a_1=\eta a_2=0. \tag{A.33}$$

Identifying constant terms in (A.30), and using (A.33), we find

$$\eta a_0=\frac{\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\eta\Sigma\theta'. \tag{A.34}$$

Substituting (A.34) and (A.33) into (A.20), we find

$$\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\eta\Sigma\theta'=r\alpha\eta\Sigma\left[x_t(1-y_t)\eta+y_t\theta\right]'\Rightarrow x_t=\frac{\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}-y_t}{1-y_t}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}. \tag{A.35}$$

Substituting (A.35) into (A.29), we find

$$\bar{h}(\bar{X}_t)=r\bar{\alpha}(\Sigma+s^2a_1a_1')\left[\frac{\alpha}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta+(1-y_t)p_f\right]'. \tag{A.36}$$

Eq. (A.36) is linear in $\bar{X}_t$. Identifying terms in C_t and y_t, we find

$$(r+\kappa+s^2\bar{q}_{11})a_1+b_1a_2=0, \tag{A.37}$$

$$s^2\bar{q}_{12}a_1+(r+b_2)a_2=-r\bar{\alpha}\left(\Sigma p_f'+s^2a_1'p_f'a_1\right), \tag{A.38}$$

respectively. Therefore, (a_1, a_2) are collinear to the vector $\Sigma p_f'$, as in (3.11). Substituting (3.11) into (A.37) and (A.38), we find

$$(r+\kappa+s^2\bar{q}_{11})\gamma_1+b_1\gamma_2=0, \tag{A.39}$$

$$s^2\bar{q}_{12}\gamma_1+(r+b_2)\gamma_2=-r\bar{\alpha}\left(1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right), \tag{A.40}$$

respectively. Identifying constant terms in (A.36), and using (3.11), we find

$$a_0 = \frac{\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \left[\frac{\gamma_1(\kappa\bar{C} - s^2\bar{q}_1) + b_0\gamma_2}{r} + \bar{\alpha}\left(1 + \frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\right]\Sigma p_f'. \tag{A.41}$$

Using (A.35), we can write (A.27) as

$$\begin{aligned}
&\theta h(X_t) - r\alpha\psi b_1 s^2\left(\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta + y_t p_f\right)a_1 - C_t \\
&= r\alpha\theta(\Sigma + s^2 a_1 a_1')\left(\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta + y_t p_f\right)' + \psi h_\psi(X_t) \\
&\Rightarrow \theta h(X_t) - r\alpha\psi b_1 s^2\gamma_1\frac{\Delta}{\eta\Sigma\eta'}y_t - C_t = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{(\eta\Sigma\theta')^2}{\eta\Sigma\eta'} + r\alpha\left(1 + \frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}y_t + \psi h_\psi(X_t),
\end{aligned} \tag{A.42}$$

where the second step follows from (3.11). Eq. (A.42) is linear in (C_t, y_t). Identifying terms in C_t and y_t, and using (3.2) and (3.11), we find

$$\left[(r + \kappa + s^2 q_{11})\gamma_1 + b_1\gamma_2\right]\frac{\Delta}{\eta\Sigma\eta'} - 1 = -\psi b_1(r + \kappa + b_2 + s^2 q_{11}), \tag{A.43}$$

$$\left[(r+b_2)\gamma_2 + (q_{12} - r\alpha\psi b_1)s^2\gamma_1\right]\frac{\Delta}{\eta\Sigma\eta'} = r\alpha\left(1 + \frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'} - \psi\left[(r+b_2)b_2 + b_1 s^2 q_{12}\right], \tag{A.44}$$

respectively. Identifying constant terms, and using (3.2), (3.11) and (A.41), we find

$$\left[s^2\gamma_1(q_1 - \bar{q}_1) + r\bar{\alpha}\left(1 + \frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\right]\frac{\Delta}{\eta\Sigma\eta'} = \psi\left[(r+b_2)b_0 + b_1(\kappa\bar{C} - s^2 q_1)\right]. \tag{A.45}$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, a_1, a_2, b_0, b_1, b_2, \gamma_1, \gamma_2, \bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{Q}, q_0, q_1, q_2, Q)$. The equations linking them are (3.11), (A.39)-(A.41), (A.43)-(A.45), the six equations derived from (A.12) by identifying quadratic, linear and constant terms, and the six equations derived from (A.22) through the same procedure. To simplify the system, we note that the variables $(\bar{q}_0, q_0)$ enter only in the equations derived from (A.12) and (A.22) by identifying constants. Therefore they can be determined separately, and we need to consider only the equations derived from (A.12) and (A.22) by identifying linear and quadratic terms. We next simplify these equations, using implications of market clearing.

Using (A.36), we find

$$\frac{1}{2}\bar{h}(\bar{X}_t)'(\Sigma + s^2 a_1 a_1')^{-1}\bar{h}(\bar{X}_t) = \frac{r^2\alpha^2\bar{\alpha}^2(\eta\Sigma\theta')^2}{2(\alpha+\bar{\alpha})^2\eta\Sigma\eta'} + \frac{1}{2}r^2\bar{\alpha}^2(1-y_t)^2\left(1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}. \quad \text{(A.46)}$$

We next substitute (A.46) into (A.8), and identify terms. Identifying terms in C_t^2, $C_t y_t$ and y_t^2, we find

$$\frac{1}{2}\bar{X}_t'\left(\bar{Q}\bar{\mathcal{R}}_2\bar{Q} + \bar{Q}\bar{\mathcal{R}}_1 + \bar{\mathcal{R}}_1'\bar{Q} - \bar{\mathcal{R}}_0\right)\bar{X}_t = 0, \quad \text{(A.47)}$$

where

$$\bar{\mathcal{R}}_2 \equiv \begin{pmatrix} s^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\bar{\mathcal{R}}_1 \equiv \begin{pmatrix} \frac{r}{2}+\kappa & 0 \\ b_1 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\bar{\mathcal{R}}_0 \equiv \begin{pmatrix} 0 & r\bar{\alpha}\lambda \\ r\bar{\alpha}\lambda & r^2\bar{\alpha}^2\left(1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'} \end{pmatrix}.$$

Eq. (A.47) must hold for all $\bar{X}_t$. Since the square matrix in (A.47) is symmetric, it must equal zero, and this yields the algebraic Riccati equation

$$\bar{Q}\bar{\mathcal{R}}_2\bar{Q} + \bar{Q}\bar{\mathcal{R}}_1 + \bar{\mathcal{R}}_1'\bar{Q} - \bar{\mathcal{R}}_0 = 0. \quad \text{(A.48)}$$

We next identify terms in C_t and y_t, which yield

$$(r+\kappa+s^2\bar{q}_{11})\bar{q}_1 + b_1\bar{q}_2 - \kappa\bar{C}\bar{q}_{11} - b_0\bar{q}_{12} = 0, \quad \text{(A.49)}$$

$$(r+b_2)\bar{q}_2 + s^2\bar{q}_1\bar{q}_{12} + r^2\bar{\alpha}^2\left(1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'} - \kappa\bar{C}\bar{q}_{12} - b_0\bar{q}_{22} = 0, \quad \text{(A.50)}$$

respectively. Using (3.11) and (A.35), we can write (A.20) as

$$\eta h(X_t) = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\eta\Sigma\theta'. \quad \text{(A.51)}$$

Using (3.2), (3.11), (A.42) and (A.51), we find that the equation derived from (A.22) by identifying terms in C_t^2, $C_t y_t$ and y_t^2 is

$$Q\mathcal{R}_2 Q + Q\mathcal{R}_1 + \mathcal{R}_1' Q - \mathcal{R}_0 = 0, \quad \text{(A.52)}$$

where

$$\mathcal{R}_2 \equiv \begin{pmatrix} s^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{R}_1 \equiv \begin{pmatrix} \frac{r}{2}+\kappa & r\alpha\psi b_1 s^2 \\ b_1 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\mathcal{R}_0 \equiv \begin{pmatrix} -r\alpha\psi b_1^2 & -r\alpha\psi b_1(r+\kappa+2b_2) \\ -r\alpha\psi b_1(r+\kappa+2b_2) & r^2\alpha^2\left(1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}+2r^2\alpha^2\psi b_1 s^2\gamma_1\frac{\Delta}{\eta\Sigma\eta'}-r\alpha\psi b_2(2r+3b_2) \end{pmatrix},$$

and the equations derived by identifying terms in C_t and y_t are

$$(r+\kappa+s^2q_{11})q_1+b_1q_2-r\alpha\psi b_0b_1-\kappa\bar{C}q_{11}-b_0q_{12}=0, \tag{A.53}$$

$$(r+b_2)q_2+s^2(q_{12}+r\alpha\psi b_1)q_1-r\alpha\psi\left[(r+2b_2)b_0+b_1\kappa\bar{C}\right]-\kappa\bar{C}q_{12}-b_0q_{22}=0, \tag{A.54}$$

respectively.

Solving for equilibrium amounts to solving the system of (3.11), (A.39)-(A.41), (A.43)-(A.45), (A.48)-(A.50) and (A.52)-(A.54) in the unknowns $(a_0, a_1, a_2, b_0, b_1, b_2, \gamma_1, \gamma_2, \bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_2, Q)$. This reduces to solving the system of (A.39), (A.40), (A.43), (A.44), (A.48) and (A.52) in the unknowns $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q)$: given $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q)$, (a_1, a_2) can be determined from (3.11), $(b_0, \bar{q}_1, \bar{q}_2, q_1, q_2)$ from the linear system of (A.45), (A.49), (A.50), (A.53) and (A.54), and a_0 from (A.41). We replace the system of (A.39), (A.40), (A.43), (A.44), (A.48) and (A.52) by the equivalent system of (A.39), (A.40), (A.48), (A.52),

$$\psi b_1(r+\kappa+b_2+s^2q_{11})=1+s^2\gamma_1(\bar{q}_{11}-q_{11})\frac{\Delta}{\eta\Sigma\eta'}, \tag{A.55}$$

$$\psi\left[(r+b_2)b_2+b_1s^2q_{12}\right]-r\alpha\psi b_1s^2\gamma_1\frac{\Delta}{\eta\Sigma\eta'}=r(\alpha+\bar{\alpha})\left(1+\frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}+s^2\gamma_1(\bar{q}_{12}-q_{12})\frac{\Delta}{\eta\Sigma\eta'}. \tag{A.56}$$

Eqs. (A.55) and (A.56) follow by substituting (A.39) and (A.40) into (A.43) and (A.44).

For $s = 0$, (A.39), (A.40), (A.48), (A.52), (A.55) and (A.56) become

$$(r + \kappa)\gamma_1 + b_1\gamma_2 = 0, \tag{A.57}$$

$$(r + b_2)\gamma_2 = -r\bar{\alpha}, \tag{A.58}$$

$$\bar{Q}\bar{\mathcal{R}}_1^0 + \bar{\mathcal{R}}_1^{0\prime}\bar{Q} - \bar{\mathcal{R}}_0^0 = 0, \tag{A.59}$$

$$Q\mathcal{R}_1^0 + \mathcal{R}_1^{0\prime}Q - \mathcal{R}_0^0 = 0, \tag{A.60}$$

$$\psi b_1(r + \kappa + b_2) = 1, \tag{A.61}$$

$$\psi(r + b_2)b_2 = r(\alpha + \bar{\alpha})\frac{\Delta}{\eta\Sigma\eta'}, \tag{A.62}$$

respectively, where

$$\bar{\mathcal{R}}_1^0 = \mathcal{R}_1^0 \equiv \begin{pmatrix} \frac{r}{2} + \kappa & 0 \\ b_1 & \frac{r}{2} + b_2 \end{pmatrix},$$

$$\bar{\mathcal{R}}_0^0 \equiv \begin{pmatrix} 0 & r\bar{\alpha}\lambda \\ r\bar{\alpha}\lambda & r^2\bar{\alpha}^2\frac{\Delta}{\eta\Sigma\eta'} \end{pmatrix},$$

$$\mathcal{R}_0^0 \equiv \begin{pmatrix} -r\alpha\psi b_1^2 & -r\alpha\psi b_1(r + \kappa + 2b_2) \\ -r\alpha\psi b_1(r + \kappa + 2b_2) & r^2\alpha^2\frac{\Delta}{\eta\Sigma\eta'} - r\alpha\psi b_2(2r + 3b_2) \end{pmatrix}.$$

Eq. (A.62) is quadratic and has a unique positive solution b_2.[21] Given b_2, b_1 is determined uniquely from (A.61), γ_2 from (A.58), γ_1 from (A.57), $\bar{Q}$ from (A.59) (which is linear in $\bar{Q}$), and Q from (A.60) (which is linear in Q). We denote this solution by $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \bar{Q}^0, Q^0)$.

To show that the system of (A.39), (A.40), (A.48), (A.52), (A.55) and (A.56) has a solution for small s, we apply the implicit function theorem. We move all terms in each equation to the left-hand side, and stack all left-hand sides into a vector $\mathcal{F}$, in the order (A.56), (A.55), (A.40), (A.39), (A.48), (A.52). Treated as a function of $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q, s)$, $\mathcal{F}$ is continuously differentiable around the point $\mathcal{A} \equiv (b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \bar{Q}^0, Q^0, 0)$ and is equal to zero at $\mathcal{A}$. To show that the Jacobian matrix of $\mathcal{F}$ with respect to $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q)$ has non-zero determinant at $\mathcal{A}$, we note that $\mathcal{F}$ has a triangular structure for $s = 0$: $\mathcal{F}_1$ depends only on b_2, $\mathcal{F}_2$ only on (b_1, b_2), $\mathcal{F}_3$ only on (b_2, γ_2), $\mathcal{F}_4$ only on $(b_1, \gamma_1, \gamma_2)$, $\mathcal{F}_5$ only on $(b_1, b_2, \bar{Q})$, and $\mathcal{F}_6$ only on (b_1, b_2, Q). Therefore, the Jacobian

[21] The positive solution is the relevant one. Indeed, since the negative solution satisfies $r + 2b_2 < 0$, (A.59) implies that $\bar{q}_{22} < 0$. Therefore, the manager's certainty equivalent would converge to $-\infty$ at the rate y_t^2 when $|y_t|$ goes to ∞ and C_t is held constant. The manager can, however, achieve higher certainty equivalent by not investing in the active fund.

matrix of $\mathcal{F}$ has non-zero determinant at $\mathcal{A}$ if the derivatives of $\mathcal{F}_1$ with respect to b_2, $\mathcal{F}_2$ with respect to b_1, $\mathcal{F}_3$ with respect to γ_2, and $\mathcal{F}_4$ with respect to γ_1 are non-zero, and the Jacobian matrices of $\mathcal{F}_5$ with respect to $\bar{Q}$ and $\mathcal{F}_6$ with respect to Q have non-zero determinants. These results follow from (A.57)-(A.62) and the positivity of (b_1^0, b_2^0). Therefore, the implicit function theorem applies, and the system of (A.39), (A.40), (A.48), (A.52), (A.55) and (A.56) has a solution for small s. This solution is unique in a neighborhood of $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \bar{Q}^0, Q^0)$, which corresponds to the unique equilibrium for $s = 0$. Since $b_1^0 > 0$, $b_2^0 > 0$, $\gamma_1^0 > 0$, $\gamma_2^0 < 0$, continuity implies that $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 < 0$ for small s. ∎

Proof of Corollary 3.1: Eq. (3.2) implies that when $C_t = 0$ for all t, $\lim_{t\to\infty} y_t = b_0/b_2$. Moreover, when $\bar{C} = s = 0$,

$$\frac{b_0}{b_2} = \frac{r\bar{\alpha}\frac{\Delta}{\eta\Sigma\eta'}}{\psi(r+b_2)b_2} = \frac{\bar{\alpha}}{(\alpha+\bar{\alpha})},$$

where the first step follows from (A.45) and the second from (A.62). Eqs. (A.35) and $\lim_{t\to\infty} y_t = \bar{\alpha}/(\alpha+\bar{\alpha})$ imply that $\lim_{t\to\infty} x_t = 0$. The first equality in (3.13) follows from (A.1), (A.5), (A.6) and

$$\lim_{t\to\infty} \hat{z}_t = \lim_{t\to\infty} \bar{y}_t \lim_{t\to\infty} z_t = (1 - \lim_{t\to\infty} y_t)(\theta - \lim_{t\to\infty} x_t\eta) = \frac{\alpha}{\alpha+\bar{\alpha}}\theta,$$

and the second equality follows from (A.7). ∎

Proof of Corollary 3.2: The investor's stock holdings are

$$\begin{aligned} x_t\eta + y_t z_t &= x_t\eta + y_t(\theta - x_t\eta) \\ &= y_t p_f + \frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}, \end{aligned}$$

where the second step follows from (A.35). Therefore,

$$\frac{\partial E_t(x_{t'}\eta + y_{t'}z_{t'})}{\partial C_t} = \frac{\partial E_t(y_{t'})}{\partial C_t} p_f. \tag{A.63}$$

To determine how a shock to C_t affects $(E_t(C_{t'}), E_t(y_{t'}))$, we use the linearity of (2.4) and (3.2), and solve the "impulse-response" dynamics

$$dC_t = -\kappa C_t dt, \tag{A.64}$$

$$dy_t = -(b_1 C_t + b_2 y_t)dt, \tag{A.65}$$

with the initial conditions

$$C_t = 1,$$

$$y_t = 0.$$

The solution is

$$C_{t'} = e^{-\kappa(t'-t)}, \tag{A.66}$$

$$y_{t'} = -\frac{b_1\left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right]}{b_2 - \kappa}. \tag{A.67}$$

The derivative $\partial E_t(y_{t'})/\partial C_t$ is equal to (A.67). Substituting into (A.63), we find (3.14). Since $b_1 > 0$ for small s, the coefficient of p_f in (3.14) is negative. ∎

Proof of Corollary 3.3: The first equality in (3.15) follows from (3.1) and (3.11). The second equality follows from (3.11) and (A.7). To derive the third equality, we note from (A.7) that

$$Cov_t(\eta dR_t, p_f dR_t) = 0.$$

Therefore, if β denotes the regression coefficient of dR_t on ηdR_t, then

$$\begin{aligned} Cov_t(dR_t, p_f dR_t) &= Cov_t\left(dR_t - \beta\eta dR_t, p_f dR_t\right) \\ &= Cov_t\left(d\epsilon_t, p_f dR_t\right) \\ &= Cov_t\left[d\epsilon_t, p_f(dR_t - \beta\eta dR_t)\right] \\ &= Cov_t\left(d\epsilon_t, p_f d\epsilon_t\right), \end{aligned}$$

where the second and fourth steps follow from the definition of $d\epsilon_t$, and the third step follows because $d\epsilon_t$ is independent of ηdR_t. ∎

Proof of Corollary 3.4: The corollary follows by substituting (3.11) into (A.7). ∎

Proof of Corollary 3.5: Stocks' expected returns are

$$\begin{aligned} E_t(dR_t) &= \left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) dt \\ &= \left\{\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \left[\gamma_1^R C_t + \gamma_2^R y_t + r\bar{\alpha}\left(1 + \frac{s^2\gamma_1^2\Delta}{\eta\Sigma\eta'}\right) - \gamma_1 s^2 \bar{q}_1\right]\Sigma p_f'\right\} dt \\ &= \left[\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}(\Sigma + s^2 a_1 a_1')\eta' + \Lambda_t(\Sigma + s^2 a_1 a_1')p_f'\right] dt, \end{aligned} \tag{A.68}$$

where

$$\gamma_1^R \equiv (r+\kappa)\gamma_1 + b_1\gamma_2,$$

$$\gamma_2^R \equiv (r+b_2)\gamma_2.$$

The first step in (A.68) follows from (A.1), the second from (3.11) and (A.41), and the third from (3.11) and (3.18). Eq. (A.68) is equivalent to (3.17) because of (A.7).

Eq. (A.39) implies that γ_1^R has the opposite sign of $\gamma_1\bar{q}_{11}$. For small s, $\gamma_1 > 0$ and $\bar{q}_{11}$ has the same sign as its value $\bar{q}_{11}^0$ for $s = 0$. Eq. (A.59) implies that

$$\begin{aligned}
\bar{q}_{11}^0 &= -\frac{2b_1^0\bar{q}_{12}^0}{r+2\kappa} \\
&= -\frac{2b_1^0}{(r+2\kappa)(r+\kappa+b_2^0)}\left(r\bar{\alpha}\lambda - b_1^0\bar{q}_{22}^0\right), \\
&= -\frac{2r\bar{\alpha}b_1^0}{(r+2\kappa)(r+\kappa+b_2^0)}\left[\lambda - \frac{r\bar{\alpha}b_1^0\Delta}{(r+2b_2^0)\eta\Sigma\eta'}\right], \\
&= -\frac{2r\bar{\alpha}b_1^0}{(r+2\kappa)(r+\kappa+b_2^0)}\left[\lambda - \frac{r\bar{\alpha}\Delta}{\psi(r+\kappa+b_2^0)(r+2b_2^0)\eta\Sigma\eta'}\right],
\end{aligned} \tag{A.69}$$

where the last step follows from (A.61). Using (A.62), we find

$$\begin{aligned}
\psi(r+\kappa+b_2^0)(r+2b_2^0) &= 2r(\alpha+\bar{\alpha})\frac{\Delta}{\eta\Sigma\eta'} + \psi\left[(r+2\kappa)b_2^0 + r(r+\kappa)\right] \\
&= 2r(\alpha+\bar{\alpha})\frac{\Delta}{\eta\Sigma\eta'} + \frac{\psi r}{2}\left[r + (r+2\kappa)\sqrt{1+\frac{4(\alpha+\bar{\alpha})f\Delta}{r\psi\eta\Sigma\eta'}}\right].
\end{aligned} \tag{A.70}$$

Eqs. (A.69) and (A.70) imply that $\bar{q}_{11}^0$ is positive if (3.19) holds, and is negative otherwise. Therefore, for small s, γ_1^R is negative if (3.19) holds, and is positive otherwise. Moreover, $\gamma_2^R < 0$ since $b_2 > 0$ and $\gamma_2 < 0$. ∎

Proof of Corollary 3.6: The autocovariance matrix is

$$\begin{aligned}
&Cov_t(dR_t, dR_{t'}') \\
&= Cov_t\left\{\sigma dB_t^D - sa_1 dB_t^C, \left[\left(a_1^R C_{t'} + a_2^R y_{t'}\right)dt' + \sigma dB_{t'}^D - sa_1 dB_{t'}^C\right]'\right\} \\
&= Cov_t\left[\sigma dB_t^D - sa_1 dB_t^C, \left(a_1^R C_{t'} + a_2^R y_{t'}\right)' dt'\right] \\
&= Cov_t\left[-sa_1 dB_t^C, \left(a_1^R C_{t'} + a_2^R y_{t'}\right)' dt'\right] \\
&= -s\gamma_1 Cov_t\left(dB_t^C, \gamma_1^R C_{t'} + \gamma_2^R y_{t'}\right)\Sigma p_f' p_f \Sigma dt',
\end{aligned} \tag{A.71}$$

where the first step follows by using (A.1) and omitting quantities known at time t, the second step follows because the increments $(dB^D_{t'}, dB^C_{t'})$ are independent of information up to time t', the third step follows because of (2.4), (3.2) and the independence of (B^D_t, B^C_t), and the fourth step follows from (3.11). Using (2.4) and (3.2), we can express $(C_{t'}, y_{t'})$ as a function of their time t values and the Brownian shocks dB^C_u for $u \in [t, t']$. The covariance (A.71) depends only on how the Brownian shock dB^C_t impacts $(C_{t'}, y_{t'})$. To compute this impact, we solve the impulse-response dynamics (A.64) and (A.65) with the initial conditions

$$C_t = sdB^C_t,$$

$$y_t = 0.$$

The solution is

$$C_{t'} = e^{-\kappa(t'-t)} sdB^C_t, \tag{A.72}$$

$$y_{t'} = -\frac{b_1\left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right]}{b_2 - \kappa} sdB^C_t. \tag{A.73}$$

Substituting into (A.71), we find (3.20) with

$$\chi_1 \equiv s^2\gamma_1\left(\frac{b_1\gamma_2^R}{b_2-\kappa} - \gamma_1^R\right) = s^2(r+\kappa)\gamma_1\left(\frac{b_1\gamma_2}{b_2-\kappa} - \gamma_1\right), \tag{A.74}$$

$$\chi_2 \equiv -\frac{s^2 b_1\gamma_1\gamma_2^R}{b_2-\kappa} = -\frac{s^2(r+b_2)b_1\gamma_1\gamma_2}{b_2-\kappa}. \tag{A.75}$$

The function $\chi(u) \equiv \chi_1 e^{-\kappa u} + \chi_2 e^{-b_2 u}$ can change sign only once, is equal to $-s^2\gamma_1\gamma_1^R$ when $u = 0$, and has the sign of χ_1 if $b_2 > \kappa$ and of χ_2 if $b_2 < \kappa$ when u goes to ∞. For small s, γ_1^R is negative if (3.19) holds, and is positive otherwise. The opposite is true for $\chi(0)$ since $\gamma_1 > 0$. Since, in addition, $b_1 > 0$, $b_2 > 0$ and $\gamma_2 < 0$, (A.74) and (A.75) imply that $\chi_1 < 0$ if $b_2 > \kappa$ and $\chi_2 < 0$ if $b_2 < \kappa$. Therefore, there exists a threshold $\hat{u} \geq 0$, which is positive if (3.19) holds and is zero otherwise, such that $\chi(u) > 0$ for $0 < u < \hat{u}$ and $\chi(u) < 0$ for $u > \hat{u}$. ∎

Proof of Corollary 3.7: Eqs. (A.57), (A.58), (A.61) and (A.62) imply that $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0)$, the terms in $(b_1, b_2, \gamma_1, \gamma_2)$ of order zero in s^2, are independent of λ. Hence, to determine how λ affects

$(b_1, b_2, \gamma_1, \gamma_2)$, we must consider the terms of order one. We set

$$b_1 = b_1^0 + s^2 b_1^1 + o(s^2), \tag{A.76}$$

$$b_2 = b_2^0 + s^2 b_2^1 + o(s^2), \tag{A.77}$$

$$\gamma_1 = \gamma_1^0 + s^2 \gamma_1^1 + o(s^2), \tag{A.78}$$

$$\gamma_2 = \gamma_2^0 + s^2 \gamma_2^1 + o(s^2). \tag{A.79}$$

Substituting (A.76)-(A.79) into (A.39), (A.40), (A.55) and (A.56), and using (A.57), (A.58), (A.61) and (A.62) to eliminate terms of order zero, we find

$$(r+\kappa)\gamma_1^1 + \gamma_2^0 b_1^1 + b_1^0 \gamma_2^1 = -\gamma_1^0 \bar{q}_{11}^0, \tag{A.80}$$

$$(r+b_2^0)\gamma_2^1 + \gamma_2^0 b_2^1 = -\gamma_1^0 \bar{q}_{12}^0 - r\bar{\alpha}(\gamma_1^0)^2 \frac{\Delta}{\eta\Sigma\eta'}, \tag{A.81}$$

$$\psi\left[b_1^0 b_2^1 + (r+\kappa+b_2^0)b_1^1\right] = \gamma_1^0(\bar{q}_{11}^0 - q_{11}^0)\frac{\Delta}{\eta\Sigma\eta'} - \psi b_1^0 q_{11}^0, \tag{A.82}$$

$$\psi(r+2b_2^0)b_2^1 = r(\alpha+\bar{\alpha})\left(\frac{\gamma_1^0\Delta}{\eta\Sigma\eta'}\right)^2 + \gamma_1^0(\bar{q}_{12}^0 - q_{12}^0)\frac{\Delta}{\eta\Sigma\eta'} - \psi b_1^0 q_{12}^0 + r\alpha\psi b_1^0\gamma_1^0\frac{\Delta}{\eta\Sigma\eta'}. \tag{A.83}$$

Differentiating (A.80)-(A.83) with respect to λ, and noting that $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, q_{11}^0, q_{12}^0)$ are independent of λ, we find

$$\frac{\partial\gamma_1^1}{\partial\lambda} = \gamma_1^0\left\{\frac{b_1^0}{r+b_2^0}\left[1 + \frac{(2r+2\kappa+b_2^0)\gamma_2^0\frac{\Delta}{\eta\Sigma\eta'}}{\psi(r+\kappa+b_2^0)(r+2b_2^0)}\right]\frac{\partial\bar{q}_{12}^0}{\partial\lambda} - \left[1 + \gamma_2^0\frac{\frac{\Delta}{\eta\Sigma\eta'}}{\psi(r+\kappa+b_2^0)}\right]\frac{\partial\bar{q}_{11}^0}{\partial\lambda}\right\}, \tag{A.84}$$

$$\frac{\partial\gamma_2^1}{\partial\lambda} = -\frac{\gamma_1^0}{r+b_2^0}\left[1 + \frac{\gamma_2^0\frac{\Delta}{\eta\Sigma\eta'}}{\psi(r+2b_2^0)}\right]\frac{\partial\bar{q}_{12}^0}{\partial\lambda}, \tag{A.85}$$

$$\frac{\partial b_1^1}{\partial\lambda} = \frac{\frac{\gamma_1^0\Delta}{\eta\Sigma\eta'}\left(\frac{\partial\bar{q}_{11}^0}{\partial\lambda} - \frac{b_1^0}{r+2b_2^0}\frac{\partial\bar{q}_{12}^0}{\partial\lambda}\right)}{\psi(r+\kappa+b_2^0)}, \tag{A.86}$$

$$\frac{\partial b_2^1}{\partial\lambda} = \frac{\frac{\gamma_1^0\Delta}{\eta\Sigma\eta'}\frac{\partial\bar{q}_{12}^0}{\partial\lambda}}{\psi(r+2b_2^0)}. \tag{A.87}$$

Substituting γ_2^0 from (A.58), and using (A.62), we find that the terms in square brackets in (A.84) are positive. Moreover, $(b_1^0, b_2^0, \gamma_1^0)$ are positive, and (A.69) implies that $\partial\bar{q}_{11}^0/\partial\lambda < 0$ and $\partial\bar{q}_{12}^0/\partial\lambda > 0$. Therefore, $\partial\gamma_1^1/\partial\lambda > 0$, and so $\partial\gamma_1/\partial\lambda > 0$. Eqs. (A.74) and (A.75) imply that

$$\chi_1 + \chi_2 = -s^2\gamma_1\gamma_1^R = s^4\gamma_1^2\bar{q}_{11} = s^4(\gamma_1^0)^2\bar{q}_{11}^0 + o(s^4),$$

where the second step follows from (A.39). Since $\partial\bar{q}_{11}^0/\partial\lambda < 0$, $\partial(\chi_1+\chi_2)/\partial\lambda < 0$. The value $\hat{u}$ for the function $\chi(u)$ is equal to zero is

$$\hat{u} = -\frac{1}{b_2-\kappa}\log\left[1-\frac{(b_2-\kappa)\gamma_1^R}{b_1\gamma_2^R}\right] = -\frac{1}{b_2-\kappa}\log\left[1+\frac{s^2(b_2-\kappa)\gamma_1\bar{q}_{11}}{b_1(r+b_2)\gamma_2}\right] = -\frac{s^2\gamma_1^0\bar{q}_{11}^0}{b_1^0(r+b_2^0)\gamma_2^0}+o(s^2),$$

where the second step follows from (A.39). Since $\gamma_2^0 < 0$ and $\partial\bar{q}_{11}^0/\partial\lambda < 0$, $\partial\hat{u}/\partial\lambda < 0$. ∎

B Asymmetric Information

Proof of Proposition 4.1: We use Theorem 10.3 of Liptser and Shiryaev (LS 2000). The investor learns about C_t, which follows the process (2.4). She observes the following information:

- The net dividends of the true market portfolio $\theta D_t - C_t dt$. This corresponds to the process $\xi_{1t} \equiv \theta D_t - \int_0^t C_s ds$.
- The dividends of the index fund ηdD_t. This corresponds to the process $\xi_{2t} \equiv \eta D_t$.
- The price of the true market portfolio θS_t. Given the conjecture (4.3) for stock prices, this is equivalent to observing the process $\xi_{3t} \equiv \theta(S_t + a_1\hat{C}_t + a_3 y_t)$.
- The price of the index portfolio ηS_t. This is equivalent to observing the process $\xi_{4t} \equiv \eta(S_t + a_1\hat{C}_t + a_3 y_t)$.

The dynamics of ξ_{1t} are

$$\begin{aligned} d\xi_{1t} &= \theta(F_t dt + \sigma dB_t^D) - C_t dt \\ &= \left[(r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r+\kappa)\xi_{3t} + (r+\kappa)\theta a_2 C_t - C_t\right] dt + \theta\sigma dB_t^D \\ &= \left[(r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r+\kappa)\xi_{3t} - \left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\eta\Sigma\eta'}\right) C_t\right] dt + \theta\sigma dB_t^D, \end{aligned} \tag{B.1}$$

where the first step follows from (4.1), the second from (4.3), and the third from (4.4). Likewise, the dynamics of ξ_{2t} are

$$d\xi_{2t} = \left[(r+\kappa)\eta a_0 - \frac{\kappa\eta\bar{F}}{r} + (r+\kappa)\xi_{4t}\right] dt + \eta\sigma dB_t^D. \tag{B.2}$$

The dynamics of ξ_{3t} are

$$\begin{aligned} d\xi_{3t} &= d\left\{\theta\left[\frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r+\kappa} - (a_0 + a_2 C_t)\right]\right\} \\ &= \theta\left[\frac{\kappa(\bar{F} - F_t)dt + \phi\sigma dB_t^F}{r+\kappa} - a_2\left[\kappa(\bar{C} - C_t)dt + s dB_t^C\right]\right] \\ &= \kappa\left[\theta\left(\frac{\bar{F}}{r} - a_0 - a_2\bar{C}\right) - \xi_{3t}\right]dt + \frac{\phi\theta\sigma dB_t^F}{r+\kappa} - s\theta a_2 dB_t^C \\ &= \kappa\left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\eta\Sigma\eta'} - \xi_{3t}\right)dt + \frac{\phi\theta\sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2\Delta dB_t^C}{\eta\Sigma\eta'}, \end{aligned} \tag{B.3}$$

where the first step follows from (4.3), the second from (2.4) and (4.2), and the fourth from (4.4). Likewise, the dynamics of ξ_{4t} are

$$d\xi_{4t} = \kappa\left(\frac{\eta\bar{F}}{r} - \eta a_0 - \xi_{4t}\right)dt + \frac{\phi\eta\sigma dB_t^F}{r+\kappa}. \tag{B.4}$$

The dynamics (2.4) and (B.1)-(B.4) map into the dynamics (10.62) and (10.63) of LS by setting $\theta_t \equiv C_t$, $\xi_t \equiv (\xi_{1t}, \xi_{2t}, \xi_{3t}, \xi_{4t})'$, $W_{1t} \equiv \begin{pmatrix} B_t^D \\ B_t^F \end{pmatrix}$, $W_{2t} \equiv B_t^C$, $a_0(t) \equiv \kappa\bar{C}$, $a_1(t) \equiv -\kappa$, $a_2(t) \equiv 0$, $b_1(t) \equiv 0$, $b_2(t) \equiv s$, $\gamma_t \equiv T$,

$$A_0(t) \equiv \begin{pmatrix} (r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} \\ (r+\kappa)\eta a_0 - \frac{\kappa\eta\bar{F}}{r} \\ \kappa\left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\eta\Sigma\eta'}\right) \\ \kappa\left(\frac{\eta\bar{F}}{r} - \eta a_0\right) \end{pmatrix},$$

$$A_1(t) \equiv -\begin{pmatrix} 1 - \frac{(r+\kappa)\gamma_2\Delta}{\eta\Sigma\eta'} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A_2(t) \equiv \begin{pmatrix} 0 & 0 & r+\kappa & 0 \\ 0 & 0 & 0 & r+\kappa \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & -\kappa \end{pmatrix},$$

$$B_1(t) \equiv \begin{pmatrix} \theta\sigma & 0 \\ \eta\sigma & 0 \\ 0 & \frac{\phi\theta\sigma}{r+\kappa} \\ 0 & \frac{\phi\eta\sigma}{r+\kappa} \end{pmatrix},$$

$$B_2(t) \equiv -\begin{pmatrix} 0 \\ 0 \\ \frac{s\gamma_2\Delta}{\eta\Sigma\eta'} \\ 0 \end{pmatrix}.$$

The quantities $(b \circ b)(t)$, $(b \circ B)(t)$, and $(B \circ B)(t)$, defined in LS (10.80) are

$$(b \circ b)(t) = s^2,$$

$$(b \circ B)(t) = -\begin{pmatrix} 0 & 0 & \frac{s^2\gamma_2\Delta}{\eta\Sigma\eta'} & 0 \end{pmatrix},$$

$$(B \circ B)(t) = \begin{pmatrix} \theta\Sigma\theta' & \eta\Sigma\theta' & 0 & 0 \\ \eta\Sigma\theta' & \eta\Sigma\eta' & 0 & 0 \\ 0 & 0 & \frac{\phi^2\theta\Sigma\theta'}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta^2}{(\eta\Sigma\eta')^2} & \frac{\phi^2\eta\Sigma\theta'}{(r+\kappa)^2} \\ 0 & 0 & \frac{\phi^2\eta\Sigma\theta'}{(r+\kappa)^2} & \frac{\phi^2\eta\Sigma\eta'}{(r+\kappa)^2} \end{pmatrix}.$$

Theorem 10.3 of LS (first subequation of (10.81)) implies that

$$\begin{aligned} d\hat{C}_t =& \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left\{ d\xi_{1t} - \left[(r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r+\kappa)\xi_{3t} - \left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\eta\Sigma\eta'}\right)\hat{C}_t \right] dt \right. \\ & \left. - \frac{\eta\Sigma\theta'}{\eta\Sigma\eta'} \left[d\xi_{2t} - \left[(r+\kappa)\eta a_0 - \frac{\kappa\eta\bar{F}}{r} + (r+\kappa)\xi_{4t} \right] dt \right] \right\} \\ & - \beta_2 \left\{ d\xi_{3t} - \kappa \left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\eta\Sigma\eta'} - \xi_{3t} \right) dt \right. \\ & \left. - \frac{\eta\Sigma\theta'}{\eta\Sigma\eta'} \left[d\xi_{4t} - \kappa \left(\frac{\eta\bar{F}}{r} - \eta a_0 - \xi_{4t} \right) dt \right] \right\}. \end{aligned} \tag{B.5}$$

Eq. (4.6) follows from (B.5) by noting that the term in dt after each $d\xi_{it}$, $i = 1, 2, 3, 4$, is $E_t(d\xi_{it})$. In subsequent proofs we use a different form of (4.6), where we replace each $d\xi_{it}$, $i = 1, 2, 3, 4$, by its value in (B.1)-(B.4):

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left[p_f \sigma dB_t^D - \left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\eta\Sigma\eta'}\right)(C_t - \hat{C}_t)dt \right] - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2\Delta dB_t^C}{\eta\Sigma\eta'} \right). \tag{B.6}$$

Eq. (4.9) follows from Theorem 10.3 of LS (second subequation of (10.81)). ∎

Proof of Proposition 4.2: Eqs. (2.4), (4.1)-(4.5) and (B.6) imply that the vector of returns is

$$\begin{aligned} dR_t =& \left\{ ra_0 + \left[\gamma_1^R \hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0\gamma_3 \right] \Sigma p_f' \right\} dt + \left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f \sigma \right) dB_t^D \\ & + \frac{\phi}{r+\kappa} \left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma \right) dB_t^F - s\gamma_2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'} \right) \Sigma p_f' dB_t^C, \end{aligned} \tag{B.7}$$

where

$$\gamma_1^R \equiv (r+\kappa+\rho)\gamma_1 + b_1\gamma_3,$$

$$\gamma_2^R \equiv (r+\kappa)\gamma_2 - \rho\gamma_1,$$

$$\gamma_3^R \equiv (r+b_2)\gamma_3,$$

and

$$\rho \equiv \beta_1\left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\eta\Sigma\eta'}\right). \tag{B.8}$$

Eqs. (2.4), (3.3), (4.5), (B.6) and (B.7) imply that

$$\begin{aligned}
&d\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1,\bar{q}_2,\bar{q}_3)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right)\\
&= \bar{G}dt + \left[r\bar{\alpha}\hat{z}_t\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f\sigma\right) - \beta_1\bar{f}_1(\bar{X}_t)p_f\sigma\right]dB_t^D\\
&+ \frac{\phi}{r+\kappa}\left[r\bar{\alpha}\hat{z}_t\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f\sigma\right) - \beta_2\bar{f}_1(\bar{X}_t)p_f\sigma\right]dB_t^F\\
&- s\left[r\bar{\alpha}\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\eta\Sigma\eta'} - \bar{f}_2(\bar{X}_t)\right]dB_t^C,
\end{aligned} \tag{B.9}$$

where

$$\begin{aligned}
\bar{G} \equiv& r\bar{\alpha}\left(rW_t + \hat{z}_t\left\{ra_0 + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right]\Sigma p_f'\right\} + \lambda C_t y_t - \bar{c}_t\right)\\
&+ \bar{f}_1(\bar{X}_t)\left[\kappa(\bar{C}-\hat{C}_t) + \rho(C_t - \hat{C}_t)\right] + \bar{f}_2(\bar{X}_t)\kappa(\bar{C}-C_t) + \bar{f}_3(\bar{X}_t)v_t\\
&+ \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta\bar{q}_{11}}{\eta\Sigma\eta'} + \frac{s^2\beta_2\gamma_2\Delta\bar{q}_{12}}{\eta\Sigma\eta'} + \frac{1}{2}s^2\bar{q}_{22},
\end{aligned}$$

$$\bar{f}_1(\bar{X}_t) \equiv \bar{q}_1 + \bar{q}_{11}\hat{C}_t + \bar{q}_{12}C_t + \bar{q}_{13}y_t,$$

$$\bar{f}_2(\bar{X}_t) \equiv \bar{q}_2 + \bar{q}_{12}\hat{C}_t + \bar{q}_{22}C_t + \bar{q}_{23}y_t,$$

$$\bar{f}_3(\bar{X}_t) \equiv \bar{q}_3 + \bar{q}_{13}\hat{C}_t + \bar{q}_{23}C_t + \bar{q}_{33}y_t.$$

Eqs. (4.11) and (B.9) imply that

$$\begin{aligned}\mathcal{D}\bar{V} = &-\bar{V}\Bigg\{\bar{G} - \frac{1}{2}(r\bar{\alpha})^2 f\hat{z}_t\Sigma\hat{z}_t' \\ &- \frac{1}{2}\beta_1\left[r\bar{\alpha}\gamma_1\hat{z}_t\Sigma p_f' - \bar{f}_1(\bar{X}_t)\right]\left[r\bar{\alpha}\left(2+\frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_1\Delta\bar{f}_1(\bar{X}_t)}{\eta\Sigma\eta'}\right] \\ &- \frac{1}{2}\frac{\phi^2\beta_2}{(r+\kappa)^2}\left[r\bar{\alpha}\gamma_1\hat{z}_t\Sigma p_f' - \bar{f}_1(\bar{X}_t)\right]\left[r\bar{\alpha}\left(2+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_2\Delta\bar{f}_1(\bar{X}_t)}{\eta\Sigma\eta'}\right] \\ &- \frac{1}{2}s^2\left[r\bar{\alpha}\gamma_2\left(1+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\eta\Sigma\eta'} - \bar{f}_2(\bar{X}_t)\right]^2\Bigg\}. \end{aligned} \tag{B.10}$$

Substituting (B.10) into (3.6), we can write the first-order conditions with respect to $\bar{c}_t$ and $\hat{z}_t$ as (A.4) and

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p_f' p_f\Sigma)\hat{z}_t', \tag{B.11}$$

respectively, where

$$\bar{h}(\bar{X}_t) \equiv ra_0 + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3 + k_1\bar{f}_1(\bar{X}_t) + k_2\bar{f}_2(\bar{X}_t)\right]\Sigma p_f', \tag{B.12}$$

$$k \equiv \beta_1\gamma_1\left(2+\frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right) + \frac{\phi^2\beta_2\gamma_1}{(r+\kappa)^2}\left(2+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right) + s^2\gamma_2^2\left(1+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)^2, \tag{B.13}$$

$$k_1 \equiv \beta_1\left(1+\frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right) + \frac{\phi^2\beta_2}{(r+\kappa)^2}\left(1+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right) + \frac{s^2\beta_2\gamma_2^2\Delta}{\eta\Sigma\eta'}\left(1+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right), \tag{B.14}$$

$$k_2 \equiv s^2\gamma_2\left(1+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right). \tag{B.15}$$

Proceeding as in the proof of Proposition 3.1, we find the following counterpart of (A.12):

$$\begin{aligned}&\frac{1}{2}\bar{h}(\bar{X}_t)'(f\Sigma + k\Sigma p_f' p_f\Sigma)^{-1}\bar{h}(\bar{X}_t) + r\bar{\alpha}\lambda C_t y_t - r\left[\bar{q}_0 + (\bar{q}_1,\bar{q}_2,\bar{q}_3)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right] \\ &+ \bar{f}_1(\bar{X}_t)\left[\kappa(\bar{C}-\hat{C}_t) + \rho(C_t-\hat{C}_t)\right] + \bar{f}_2(\bar{X}_t)\kappa(\bar{C}-C_t) + \bar{f}_3(\bar{X}_t)v_t \\ &+ \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta\bar{q}_{11}}{\eta\Sigma\eta'} + \frac{s^2\beta_2\gamma_2\Delta\bar{q}_{12}}{\eta\Sigma\eta'} + \frac{1}{2}s^2\bar{q}_{22} + \bar{\beta} - r + r\log(r) \\ &- \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{f}_1(\bar{X}_t)^2}{\eta\Sigma\eta'} - \frac{1}{2}s^2\left[\frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\eta\Sigma\eta'} + \bar{f}_2(\bar{X}_t)\right]^2 = 0. \end{aligned} \tag{B.16}$$

Eq. (B.16) is quadratic in $\bar{X}_t$. Identifying quadratic, linear and constant terms yields ten scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q})$. We defer the derivation of these equations until the proof of Proposition 4.4 (see (B.40)-(B.43)). ∎

Proof of Proposition 4.3: Dynamics under the investor's filtration can be deduced from those under the manager's by replacing C_t by the investor's expectation $\hat{C}_t$. Eq. (B.6) implies that the dynamics of $\hat{C}_t$ are

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 p_f \sigma d\hat{B}_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2 \Delta dB_t^C}{\eta \Sigma \eta'} \right), \tag{B.17}$$

where $\hat{B}_t^D$ is a Brownian motion under the investor's filtration. Eq. (B.7) implies that the net-of-cost return of the active fund is

$$\begin{aligned} z_t dR_t - C_t dt &= z_t \left\{ ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3 \right] \Sigma p_f' \right\} dt - \hat{C}_t dt \\ &+ z_t \left(\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma \right) d\hat{B}_t^D + z_t \frac{\phi}{r+\kappa} \left(\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma \right) dB_t^F - s\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma \eta'} \right) z_t \Sigma p_f' dB_t^C, \end{aligned} \tag{B.18}$$

and the return of the index fund is

$$\begin{aligned} \eta dR_t &= \eta \left\{ ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3 \right] \Sigma p_f' \right\} dt \\ &+ \eta \left(\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma \right) d\hat{B}_t^D + \eta \frac{\phi}{r+\kappa} \left(\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma \right) dB_t^F - s\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma \eta'} \right) \eta \Sigma p_f' dB_t^C. \end{aligned} \tag{B.19}$$

Suppose that the investor optimizes over (c_t, x_t) but follows the control v_t given by (4.5). Eqs. (3.8), (4.5), (B.17), (B.18) and (B.19) imply that

$$\begin{aligned} &d\left(r\alpha W_t + q_0 + (q_1, q_2)X_t + \frac{1}{2} X_t' Q X_t \right) \\ &= G dt + \left[r\alpha(x_t\eta + y_t z_t)\left(\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma \right) - \beta_1 f_1(X_t) p_f \sigma \right] d\hat{B}_t^D \\ &+ \frac{\phi}{r+\kappa} \left[r\alpha(x_t\eta + y_t z_t)\left(\sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma \right) - \beta_2 f_1(X_t) p_f \sigma \right] dB_t^F \\ &- s \left[r\bar{\alpha}\gamma_2 \left(1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma \eta'} \right)(x_t\eta + y_t z_t)\Sigma p_f' - \frac{\beta_2 \gamma_2 \Delta f_1(X_t)}{\eta \Sigma \eta'} \right] dB_t^C, \end{aligned} \tag{B.20}$$

where

$$G \equiv r\alpha \left[rW_t + (x_t\eta + y_t z_t) \left\{ ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0\gamma_3 \right] \Sigma p_f' \right\} - y_t \hat{C}_t \right.$$
$$\left. - \frac{\psi v_t^2}{2} - c_t \right] + f_1(X_t)\kappa(\bar{C} - \hat{C}_t) + f_2(X_t)v_t + \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'} \right] \frac{\Delta q_{11}}{\eta\Sigma\eta'},$$

$$f_1(X_t) \equiv q_1 + q_{11}\hat{C}_t + q_{12}y_t,$$

$$f_2(X_t) \equiv q_2 + q_{12}\hat{C}_t + q_{22}y_t.$$

Eqs. (3.9) and (B.20) imply that

$$\begin{aligned}
\mathcal{D}V = &- V \left\{ G - \frac{1}{2}(r\alpha)^2 f (x_t\eta + y_t z_t)\Sigma(x_t\eta + y_t z_t)' \right. \\
&- \frac{1}{2}\beta_1 \left[r\alpha\gamma_1(x_t\eta + y_t z_t)\Sigma p_f' - f_1(X_t) \right] \left[r\alpha \left(2 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'} \right)(x_t\eta + y_t z_t)\Sigma p_f' - \frac{\beta_1\Delta f_1(X_t)}{\eta\Sigma\eta'} \right] \\
&- \frac{1}{2}\frac{\phi^2\beta_2}{(r+\kappa)^2} \left[r\alpha\gamma_1(x_t\eta + y_t z_t)\Sigma p_f' - f_1(X_t) \right] \left[r\alpha \left(2 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'} \right)(x_t\eta + y_t z_t)\Sigma p_f' - \frac{\beta_2\Delta f_1(X_t)}{\eta\Sigma\eta'} \right] \\
&\left. - \frac{1}{2}s^2 \left[r\alpha\gamma_2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'} \right)(x_t\eta + y_t z_t)\Sigma p_f' - \frac{\beta_2\gamma_2\Delta f_1(X_t)}{\eta\Sigma\eta'} \right]^2 \right\}. \qquad \text{(B.21)}
\end{aligned}$$

Substituting (B.21) into (3.10), we can write the first-order conditions with respect to c_t and x_t as (A.15) and

$$\eta h(X_t) = r\alpha\eta(f\Sigma + k\Sigma p_f' p_f \Sigma)(x_t\eta + y_t z_t)', \qquad \text{(B.22)}$$

respectively, where

$$h(X_t) \equiv ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0\gamma_3 + k_1 f_1(X_t) \right] \Sigma p_f'. \qquad \text{(B.23)}$$

Proceeding as in the proof of Proposition 3.2, we find the following counterpart of (A.22):

$$\begin{aligned}
&r\alpha y_t\theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2 \theta(f\Sigma + k\Sigma p_f' p_f\Sigma)\theta' + \frac{[\eta h(X_t) - r\alpha f y_t \eta\Sigma\theta']^2}{2f\eta\Sigma\eta'} - r\alpha y_t\hat{C}_t - \frac{1}{2}r\alpha\psi v_t^2 \\
&- r\left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t \right] + f_1(X_t)\kappa(\bar{C} - \hat{C}_t) + f_2(X_t)v_t \\
&+ \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'} \right] \frac{\Delta[q_{11} - f_1(X_t)^2]}{\eta\Sigma\eta'} + \beta - r + r\log(r) = 0. \qquad \text{(B.24)}
\end{aligned}$$

Since v_t in (3.2) is linear in X_t, (B.24) is quadratic in X_t. Identifying quadratic, linear and constant terms yields six scalar equations in (q_0, q_1, q_2, Q). We defer the derivation of these equations until the proof of Proposition 4.4 (see (B.44)-(B.46)).

We next study optimization over v_t, using the same perturbation argument as in the proof of Proposition 3.2. The counterparts of (A.27) and (A.28) are

$$\theta\left[h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p_f'\right] - \hat{C}_t = r\alpha\theta(f\Sigma + k\Sigma p_f' p_f \Sigma)\left[x_t(1-y_t)\eta + y_t\theta\right]' + \psi h_\psi(X_t), \tag{B.25}$$

$$\theta\left[h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p_f'\right] - \hat{C}_t = r\alpha\theta(f\Sigma + k\Sigma p_f' p_f \Sigma)\left[y_t\theta + \frac{\eta h(X_t) - r\alpha y_t f\eta\Sigma\theta'}{r\alpha f\eta\Sigma\eta'}\eta\right]' + \psi h_\psi(X_t), \tag{B.26}$$

respectively, where

$$h_\psi(X_t) \equiv (r+b_2)v_t + b_1\kappa(\bar{C} - \hat{C}_t) - b_1\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta f_1(X_t)}{\eta\Sigma\eta'}.$$

Eq. (B.26) is linear in X_t. Identifying linear and constant terms, yields three scalar equations in (b_0, b_1, b_2). We defer the derivation of these equations until the proof of Proposition 4.4 (see (B.36)-(B.38)). ∎

Proof of Proposition 4.4: We first impose market clearing and derive the constants $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3)$ as functions of $(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. Setting $z_t = \theta - x_t\eta$ and $\bar{y}_t = 1 - y_t$, we can write (B.11) and (B.22) as

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p_f' p_f\Sigma)(1-y_t)(\theta - x_t\eta)', \tag{B.27}$$

$$\eta h(X_t) = r\alpha\eta(f\Sigma + k\Sigma p_f' p_f\Sigma)\left[x_t(1-y_t)\eta + y_t\theta\right]', \tag{B.28}$$

respectively. Premultiplying (B.27) by η, dividing by $r\bar{\alpha}$, and adding to (B.28) divided by $r\alpha$, we find

$$\eta\left[\frac{h(X_t)}{r\alpha} + \frac{\bar{h}(\bar{X}_t)}{r\bar{\alpha}}\right] = \eta(f\Sigma + k\Sigma p_f' p_f\Sigma)\theta'. \tag{B.29}$$

Eq. (B.29) is linear in $(\hat{C}_t, C_t, y_t)$. The terms in $\hat{C}_t$, C_t and y_t are zero because $\eta\Sigma p_f' = 0$. Identifying constant terms, we find (A.34). Substituting (A.34) into (B.28), we find (A.35).

Substituting (A.35) into (B.27), we find

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p_f' p_f\Sigma)\left[\frac{\alpha}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta + (1-y_t)p_f\right]'. \tag{B.30}$$

Eq. (B.30) is linear in $\bar{X}_t$. Identifying terms in $\hat{C}_t$, C_t and y_t, we find

$$(r+\kappa+\rho)\gamma_1 + b_1\gamma_3 + k_1\bar{q}_{11} + k_2\bar{q}_{12} = 0, \tag{B.31}$$

$$(r+\kappa)\gamma_2 - \rho\gamma_1 + k_1\bar{q}_{12} + k_2\bar{q}_{22} = 0, \tag{B.32}$$

$$(r+b_2)\gamma_3 + k_1\bar{q}_{13} + k_2\bar{q}_{23} = -r\bar{\alpha}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right), \tag{B.33}$$

respectively. Identifying constant terms, we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \left[\frac{\kappa(\gamma_1+\gamma_2)\bar{C} + b_0\gamma_3 - k_1\bar{q}_1 - k_2\bar{q}_2}{r} + \bar{\alpha}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\right]\Sigma p_f'. \tag{B.34}$$

Using (A.35), we can write (B.26) as

$$\begin{aligned}
&\theta h(X_t) - r\alpha\psi b_1 k_1 \frac{\Delta}{\eta\Sigma\eta'} y_t - \hat{C}_t = r\alpha\theta(f\Sigma + k\Sigma p_f' p_f \Sigma)\left(\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta + y_t p_f\right)' + \psi h_\psi(X_t) \\
&\Rightarrow \theta h(X_t) - r\alpha\psi b_1 k_1 \frac{\Delta}{\eta\Sigma\eta'} y_t - \hat{C}_t = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{(\eta\Sigma\theta')^2}{\eta\Sigma\eta'} + r\alpha\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'} y_t + \psi h_\psi(X_t).
\end{aligned} \tag{B.35}$$

Eq. (B.35) is linear in $(\hat{C}_t, y_t)$. Identifying terms in $\hat{C}_t$ and y_t, and using (4.5), we find

$$\begin{aligned}
&\left[(r+\kappa)(\gamma_1+\gamma_2) + b_1\gamma_3 + k_1 q_{11}\right]\frac{\Delta}{\eta\Sigma\eta'} - 1 \\
&= -\psi b_1\left\{r + \kappa + b_2 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta q_{11}}{\eta\Sigma\eta'}\right\},
\end{aligned} \tag{B.36}$$

$$\begin{aligned}
&\left[(r+b_2)\gamma_3 + (q_{12} - r\alpha\psi b_1)k_1\right]\frac{\Delta}{\eta\Sigma\eta'} \\
&= r\alpha\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'} - \psi\left\{(r+b_2)b_2 + b_1\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta q_{12}}{\eta\Sigma\eta'}\right\},
\end{aligned} \tag{B.37}$$

respectively. Identifying constant terms, and using (4.5) and (B.34), we find

$$\begin{aligned}
&\left[k_1(q_1 - \bar{q}_1) - k_2\bar{q}_2 + r\bar{\alpha}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\right]\frac{\Delta}{\eta\Sigma\eta'} \\
&= \psi\left\{(r+b_2)b_0 + b_1\kappa\bar{C} - b_1\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta q_1}{\eta\Sigma\eta'}\right\}.
\end{aligned} \tag{B.38}$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. (As in Proposition 3.3, we can drop $(\bar{q}_0, q_0)$.)

The equations linking them are (4.7)-(4.9), (B.31)-(B.34), (B.36)-(B.38), the nine equations derived from (B.16) by identifying linear and quadratic terms, and the five equations derived from (B.24) by identifying linear and quadratic terms. We next simplify the latter two sets of equations, using implications of market clearing.

Using (B.30), we find

$$\frac{1}{2}\bar{h}(\bar{X}_t)'(f\Sigma+k\Sigma p_f p_f'\Sigma)^{-1}\bar{h}(\bar{X}_t)=\frac{r^2\alpha^2\bar{\alpha}^2 f(\eta\Sigma\theta')^2}{2(\alpha+\bar{\alpha})^2\eta\Sigma\eta'}+\frac{1}{2}r^2\bar{\alpha}^2(1-y_t)^2\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}. \quad \text{(B.39)}$$

We next substitute (B.39) into (B.16), and identify terms. Quadratic terms yield the algebraic Riccati equation

$$\bar{Q}\bar{\mathcal{R}}_2\bar{Q}+\bar{Q}\bar{\mathcal{R}}_1+\bar{\mathcal{R}}_1'\bar{Q}-\bar{\mathcal{R}}_0=0, \quad \text{(B.40)}$$

where

$$\bar{\mathcal{R}}_2\equiv\begin{pmatrix}\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta}{\eta\Sigma\eta'} & \frac{s^2\beta_2\gamma_2\Delta}{\eta\Sigma\eta'} & 0\\ \frac{s^2\beta_2\gamma_2\Delta}{\eta\Sigma\eta'} & s^2 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\bar{\mathcal{R}}_1\equiv\begin{pmatrix}\frac{r}{2}+\kappa+\rho & -\rho & 0\\ 0 & \frac{r}{2}+\kappa & 0\\ b_1 & 0 & \frac{r}{2}+b_2\end{pmatrix},$$

$$\bar{\mathcal{R}}_0\equiv\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & r\bar{\alpha}\lambda\\ 0 & r\bar{\alpha}\lambda & r^2\bar{\alpha}^2\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}\end{pmatrix}.$$

Terms in $\hat{C}_t$, C_t and y_t yield

$$(r+\kappa+\rho)\,\bar{q}_1+b_1\bar{q}_3+\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{q}_1\bar{q}_{11}}{\eta\Sigma\eta'}+s^2\left(\frac{\beta_2\gamma_2\Delta\bar{q}_1}{\eta\Sigma\eta'}+\bar{q}_2\right)\left(\frac{\beta_2\gamma_2\Delta\bar{q}_{11}}{\eta\Sigma\eta'}+\bar{q}_{12}\right)$$
$$-\kappa\bar{C}(\bar{q}_{11}+\bar{q}_{12})-b_0\bar{q}_{13}=0, \quad \text{(B.41)}$$

$$(r+\kappa)\bar{q}_2-\rho\bar{q}_1+\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{q}_1\bar{q}_{12}}{\eta\Sigma\eta'}+s^2\left(\frac{\beta_2\gamma_2\Delta\bar{q}_1}{\eta\Sigma\eta'}+\bar{q}_2\right)\left(\frac{\beta_2\gamma_2\Delta\bar{q}_{12}}{\eta\Sigma\eta'}+\bar{q}_{22}\right)$$
$$-\kappa\bar{C}(\bar{q}_{12}+\bar{q}_{22})-b_0\bar{q}_{23}=0, \quad \text{(B.42)}$$

$$(r+b_2)\bar{q}_3+\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{q}_1\bar{q}_{13}}{\eta\Sigma\eta'}+s^2\left(\frac{\beta_2\gamma_2\Delta\bar{q}_1}{\eta\Sigma\eta'}+\bar{q}_2\right)\left(\frac{\beta_2\gamma_2\Delta\bar{q}_{13}}{\eta\Sigma\eta'}+\bar{q}_{23}\right)$$
$$+r^2\bar{\alpha}^2\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}-\kappa\bar{C}(\bar{q}_{13}+\bar{q}_{23})-b_0\bar{q}_{33}=0, \quad \text{(B.43)}$$

respectively. Using (A.35), we can write (B.28) as (A.51). Using (4.5), (A.51) and (B.35), we find that the equation derived from (B.24) by identifying quadratic terms is

$$Q\mathcal{R}_2 Q + Q\mathcal{R}_1 + \mathcal{R}_1' Q - \mathcal{R}_0 = 0, \tag{B.44}$$

where

$$\mathcal{R}_2 \equiv \begin{pmatrix} \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta}{\eta\Sigma\eta'} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{R}_1 \equiv \begin{pmatrix} \frac{r}{2}+\kappa & r\alpha\psi b_1\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta}{\eta\Sigma\eta'} \\ b_1 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\mathcal{R}_0 \equiv \begin{pmatrix} -r\alpha\psi b_1^2 & -r\alpha\psi b_1(r+\kappa+2b_2) \\ -r\alpha\psi b_1(r+\kappa+2b_2) & r^2\alpha^2\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'} + 2r^2\alpha^2\psi b_1 k_1 \frac{\Delta}{\eta\Sigma\eta'} - r\alpha\psi b_2(2r+3b_2) \end{pmatrix},$$

and the equations derived by identifying terms $\hat{C}_t$ and y_t are

$$(r+\kappa)q_1 + b_1 q_2 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta q_1 q_{11}}{\eta\Sigma\eta'} - \kappa\bar{C}q_{11} - b_0 q_{12} - r\alpha\psi b_0 b_1 = 0, \tag{B.45}$$

$$(r+b_2)q_2 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta(q_{12}+r\alpha\psi b_1)q_1}{\eta\Sigma\eta'} - \kappa\bar{C}q_{12} - b_0 q_{22}$$
$$- r\alpha\psi\left[b_0(r+2b_2) + b_1\kappa\bar{C}\right] = 0, \tag{B.46}$$

respectively.

Solving for equilibrium amounts to solving the system of (4.7)-(4.9), (B.31)-(B.34), (B.36)-(B.38), (B.40)-(B.43), (B.44)-(B.46) in the unknowns $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. This reduces to solving the system of (4.7)-(4.9), (B.31)-(B.33), (B.36), (B.37), (B.40), (B.44) in the unknowns $(b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{Q}, Q)$: given $(b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{Q}, Q)$, $(b_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, q_1, q_2)$ can be determined from the linear system of (B.38), (B.41)-(B.43), (B.45), (B.46), and a_0 from (B.34). We replace the system of (4.7)-(4.9), (B.31)-(B.33), (B.36), (B.37), (B.40), (B.44) by the

equivalent system of (4.7)-(4.9), (B.31)-(B.33), (B.40), (B.44),

$$\psi b_1 \left\{ r+\kappa+b_2+\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta q_{11}}{\eta\Sigma\eta'}\right\}$$

$$= 1+\left[k_1(\bar{q}_{11}+\bar{q}_{12})+k_2(\bar{q}_{12}+\bar{q}_{22})-k_1 q_{11}\right]\frac{\Delta}{\eta\Sigma\eta'}, \tag{B.47}$$

$$\psi\left\{(r+b_2)b_2+b_1\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta q_{12}}{\eta\Sigma\eta'}\right\}-r\alpha\psi b_1 k_1\frac{\Delta}{\eta\Sigma\eta'}$$

$$= r(\alpha+\bar{\alpha})\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}+\left(k_1\bar{q}_{13}+k_2\bar{q}_{23}-k_1 q_{12}\right)\frac{\Delta}{\eta\Sigma\eta'}. \tag{B.48}$$

For $s=0$, the unique non-negative solution of (4.9) is $T=0$. Eqs. (4.7), (4.8), (B.8) and (B.13)-(B.15) imply that $\beta_1=\beta_2=\rho=k=k_1=k_2=0$. Eqs. (B.31)-(B.33), (B.40), (B.44), (B.47) and (B.48) become

$$(r+\kappa)\gamma_1+b_1\gamma_3=0, \tag{B.49}$$

$$(r+\kappa)\gamma_2=0, \tag{B.50}$$

$$(r+b_2)\gamma_3=-r\bar{\alpha}f, \tag{B.51}$$

(A.59), (A.60), (A.61) and (A.62), respectively, where

$$\bar{\mathcal{R}}_1^0\equiv\begin{pmatrix}\frac{r}{2}+\kappa & 0 & 0\\ 0 & \frac{r}{2}+\kappa & 0\\ b_1 & 0 & \frac{r}{2}+b_2\end{pmatrix},$$

$$\bar{\mathcal{R}}_0\equiv\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & r\bar{\alpha}\lambda\\ 0 & r\bar{\alpha}\lambda & r^2\bar{\alpha}^2 f\frac{\Delta}{\eta\Sigma\eta'}\end{pmatrix},$$

and $(\mathcal{R}_1^0,\mathcal{R}_0^0)$ are as under symmetric information (Proposition 3.3). Given the unique positive solution b_2 of (A.62), $(b_1,\gamma_3,\gamma_1,\bar{Q},Q)$ are determined uniquely from (A.61), (B.51), (B.49), (A.59) and (A.60), respectively, and (B.50) implies that $\gamma_2=0$. We denote the solution for $s=0$ by $(b_1^0,b_2^0,\gamma_1^0,\gamma_2^0,\gamma_3^0,\beta_1^0,\beta_2^0,T^0,\bar{Q}^0,Q^0)$. The variables $(b_1^0,b_2^0,\gamma_1^0,\gamma_3^0,Q^0)$ coincide with $(b_1^0,b_2^0,\gamma_1^0,\gamma_2^0,Q^0)$ under symmetric information. Proceeding as in the proof of Proposition 3.3, we can apply the implicit function theorem and show that the system of (4.7)-(4.9), (B.31)-(B.33), (B.40), (B.44), (B.47), (B.48) has a solution for small s. Moreover, this solution is unique in a neighborhood of $(b_1^0,b_2^0,\gamma_1^0,\gamma_2^0,\gamma_3^0,\beta_1^0,\beta_2^0,T^0,\bar{Q}^0,Q^0)$, which corresponds to the unique equilibrium for $s=0$. Since $b_1^0>0$, $b_2^0>0$, $\gamma_1^0>0$, $\gamma_3^0<0$, continuity implies that $b_1>0$, $b_2>0$, $\gamma_1>0$, $\gamma_3<0$ for small

s. Since $\gamma_2^0 = 0$, continuity does not establish the sign of γ_2 for small s, so we need to study the asymptotic behavior of the solution. Eqs. (4.9), (4.7) and (4.8) imply that

$$T = \frac{s^2}{2\kappa} + o(s^2), \tag{B.52}$$

$$\beta_1 = \frac{\eta\Sigma\eta'}{2\kappa\Delta}s^2 + o(s^2) \equiv \hat{\beta}_1^0 s^2 + o(s^2), \tag{B.53}$$

$$\beta_2 = o(s^2), \tag{B.54}$$

respectively, where $\frac{o(s^2)}{s^2}$ converges to zero when s goes to zero. Eqs. (B.8) and (B.13)-(B.15) imply that

$$\rho = \hat{\beta}_1^0 s^2 + o(s^2), \tag{B.55}$$

$$k = 2\hat{\beta}_1^0\gamma_1^0 s^2 + o(s^2), \tag{B.56}$$

$$k_1 = \hat{\beta}_1^0 s^2 + o(s^2), \tag{B.57}$$

$$k_2 = o(s^2), \tag{B.58}$$

respectively, and (B.40) implies that

$$\bar{Q}^0 = \begin{pmatrix} \frac{2r^2\bar{\alpha}^2(b_1^0)^2 f\Delta}{(r+2\kappa)(r+\kappa+b_2^0)(r+2b_2^0)\eta\Sigma\eta'} & -\frac{r\bar{\alpha}b_1^0\lambda}{(r+2\kappa)(r+\kappa+b_2^0)} & -\frac{r^2\bar{\alpha}^2 b_1^0 f\Delta}{(r+\kappa+b_2^0)(r+2b_2^0)\eta\Sigma\eta'} \\ -\frac{r\bar{\alpha}b_1^0\lambda}{(r+2\kappa)(r+\kappa+b_2^0)} & 0 & \frac{r\bar{\alpha}\lambda}{r+\kappa+b_2^0} \\ -\frac{r^2\bar{\alpha}^2 b_1^0 f\Delta}{(r+\kappa+b_2^0)(r+2b_2^0)\eta\Sigma\eta'} & \frac{r\bar{\alpha}\lambda}{r+\kappa+b_2^0} & \frac{r^2\bar{\alpha}^2 f\Delta}{(r+2b_2^0)\eta\Sigma\eta'} \end{pmatrix}. \tag{B.59}$$

Eqs. (B.32), (B.55), (B.57), (B.58) and (B.59) imply that

$$\gamma_2 = \frac{\hat{\beta}_1^0}{r+\kappa}\left[\gamma_1^0 + \frac{r\bar{\alpha}b_1^0\lambda}{(r+2\kappa)(r+\kappa+b_2^0)}\right]s^2 + o(s^2).$$

Therefore, $\gamma_2 > 0$. ∎

Proof of Corollary 4.1: Eq. (B.7) implies that the covariance matrix of stock returns is

$$\begin{aligned} Cov_t(dR_t, dR_t') = &\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f\sigma\right)\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f\sigma\right)' \\ &+ \frac{\phi^2}{(r+\kappa)^2}\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f\sigma\right)\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f\sigma\right)' \\ &+ s^2\gamma_2^2\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)^2\Sigma p_f' p_f\Sigma, \end{aligned}$$

which is equal to (4.12) because of (B.13). To compare with symmetric information, we need to also introduce F_t in that case. Replacing (2.1) by (4.1) and (4.2), we find that the vector of returns under symmetric information changes from (A.1) to

$$dR_t = \left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) dt + \sigma \left(dB_t^D + \frac{\phi dB_t^F}{r+\kappa}\right) - s a_1 dB_t^C. \tag{B.60}$$

Hence, the covariance matrix under symmetric information changes from (3.16) to

$$Cov_t(dR_t, dR_t') = (f\Sigma + s^2\gamma_1^2 \Sigma p_f' p_f \Sigma)dt. \tag{B.61}$$

Eqs. (4.12) and (B.61) imply that the fundamental covariance under asymmetric information is equal to that under symmetric information, and the non-fundamental covariance is proportional. Moreover, the proportionality coefficient is larger than one if $k > s^2\gamma_{1sym}^2$, where γ_{1sym} denotes the value of γ_1 under symmetric information. Rearranging (B.13), we find

$$k = 2\left\{\beta_1 + \beta_2\left[\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\right\}\gamma_1 + s^2\gamma_2^2 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\gamma_1^2\Delta}{\eta\Sigma\eta'}. \tag{B.62}$$

Rearranging (4.8), we find

$$\beta_2\left[\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right] = s^2\gamma_2, \tag{B.63}$$

and rearranging (4.9), we find

$$\begin{aligned} &T^2\left[1 - (r+k)\frac{\gamma_2\Delta}{\eta\Sigma\eta'}\right]^2 \frac{\eta\Sigma\eta'}{\Delta} + \frac{\frac{s^4\gamma_2^2\Delta}{\eta\Sigma\eta'}}{\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta}{\eta\Sigma\eta'}} = s^2 - 2\kappa T \\ &\Rightarrow \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta}{\eta\Sigma\eta'} = s^2 - 2\kappa T, \end{aligned} \tag{B.64}$$

where the second step follows from (4.7) and (4.8). Substituting (B.63) and (B.64) into (B.62), we find

$$\begin{aligned} k &= 2\beta_1\gamma_1 + s^2(\gamma_1+\gamma_2)^2 - 2\kappa T\gamma_1^2 \\ &= s^2(\gamma_1+\gamma_2)^2 + 2T\gamma_1\left[\frac{\eta\Sigma\eta'}{\Delta} - \kappa\gamma_1 - (r+\kappa)\gamma_2\right], \end{aligned} \tag{B.65}$$

where the second step follows from (4.7).

Eqs. (B.52), (B.65) and $\gamma_2^0 = 0$ imply that for small s,

$$k = s^2(\gamma_1^0)^2 + \frac{s^2\gamma_1^0}{\kappa}\left(\frac{\eta\Sigma\eta'}{\Delta} - \kappa\gamma_1^0\right) + o(s^2). \tag{B.66}$$

The variables (b_1^0, b_2^0) are identical under symmetric and asymmetric information. Moreover, (A.57), (A.58), (B.49) and (B.51) imply that the same is true for γ_1^0. Therefore, $k > s^2\gamma_{1sym}^2$ for small s if

$$\begin{aligned}
&\frac{\eta\Sigma\eta'}{\Delta} - \kappa\gamma_1^0 > 0 \\
&\Leftrightarrow \frac{\eta\Sigma\eta'}{\Delta} - \frac{\kappa r\bar{\alpha}fb_1^0}{(r+\kappa)(r+b_2^0)} > 0 \\
&\Leftrightarrow \frac{\eta\Sigma\eta'}{\Delta}\left[1 - \frac{\kappa\bar{\alpha}b_2^0}{(r+\kappa)(\alpha+\bar{\alpha})(r+\kappa+b_2^0)}\right] > 0,
\end{aligned} \tag{B.67}$$

where the second step follows from (B.49) and (B.51), and the third from (A.61) and (A.62). Since $b_2^0 > 0$, (B.67) holds. ∎

Proof of Corollary 4.2: Stocks' expected returns are

$$\begin{aligned}
E_t(dR_t) &= \left\{ra_0 + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right]\Sigma p_f'\right\}dt \\
&= \left\{\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t + r\bar{\alpha}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) - k_1\bar{q}_1 - k_2\bar{q}_2\right]\Sigma p_f'\right\}dt \\
&= \left[\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\left(f\Sigma + k\Sigma p_f' p_f\Sigma\right)\eta' + \Lambda_t\left(f\Sigma + k\Sigma p_f' p_f\Sigma\right)p_f'\right]dt,
\end{aligned} \tag{B.68}$$

where the first step follows from (B.7), the second from (B.34), and the third from (4.13). Eq. (B.68) is equivalent to (3.17) because of (4.12).

Eqs. (B.31) and (B.32) imply that γ_1^R and γ_2^R have the opposite sign of $k_1\bar{q}_{11} + k_2\bar{q}_{12}$ and $k_1\bar{q}_{12} + k_2\bar{q}_{22}$, respectively. Eqs. (B.57) and (B.58) imply that for small s, the latter variables have the same sign as $\bar{q}_{11}^0$ and $\bar{q}_{12}^0$, respectively. Since $b_1^0 > 0$ and $b_2^0 > 0$, (B.59) implies that $\bar{q}_{11}^0 > 0$ and $\bar{q}_{12}^0 < 0$. Therefore, for small s, $\gamma_1^R < 0$ and $\gamma_2^R > 0$. Moreover, $\gamma_3^R < 0$ since $b_2 > 0$ and $\gamma_3 < 0$. ∎

Proof of Corollary 4.3: Using (B.7) and proceeding as in the derivation of (A.71), we find

$$Cov_t(dD_t, dR_{t'}') = \sigma Cov_t\left(dB_t^D, \gamma_1^R\hat{C}_{t'} + \gamma_2^R C_{t'} + \gamma_3^R y_{t'}\right)p_f\Sigma dt', \tag{B.69}$$

$$Cov_t(dF_t, dR_{t'}') = \phi\sigma Cov_t\left(dB_t^F, \gamma_1^R\hat{C}_{t'} + \gamma_2^R C_{t'} + \gamma_3^R y_{t'}\right)p_f\Sigma dt'. \tag{B.70}$$

The covariances (B.69) and (B.70) depend only on how the Brownian shocks dB_t^D and dB_t^F, respectively, impact $(\hat{C}_{t'}, C_{t'}, y_{t'})$. To compute the impact of these shocks, as well as of dB_t^C for the next corollary, we solve the impulse-response dynamics

$$dC_t = -\kappa C_t dt, \tag{B.71}$$

$$d\hat{C}_t = \left[-\kappa \hat{C}_t + \rho(C_t - \hat{C}_t)\right] dt, \tag{B.72}$$

$$dy_t = -\left(b_1 \hat{C}_t + b_2 y_t\right) dt, \tag{B.73}$$

with the initial conditions

$$C_t = s dB_t^C,$$

$$\hat{C}_t = -\beta_1 p_f \sigma dB_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2 \Delta dB_t^C}{\eta \Sigma \eta'}\right),$$

$$y_t = 0.$$

The solution is (A.72),

$$\hat{C}_{t'} = e^{-\kappa(t'-t)} s dB_t^C - e^{-(\kappa+\rho)(t'-t)} \left[\beta_1 p_f \sigma dB_t^D + \frac{\phi \beta_2 p_f \sigma dB_t^F}{r+\kappa} + s\left(1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma \eta'}\right) dB_t^C\right], \tag{B.74}$$

$$\begin{aligned} y_{t'} = &-\frac{b_1}{b_2 - \kappa}\left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right] s dB_t^C \\ &+ \frac{b_1}{b_2 - \kappa - \rho}\left[e^{-(\kappa+\rho)(t'-t)} - e^{-b_2(t'-t)}\right]\left[\beta_1 p_f \sigma dB_t^D + \frac{\phi \beta_2 p_f \sigma dB_t^F}{r+\kappa} + s\left(1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma \eta'}\right) dB_t^C\right]. \end{aligned} \tag{B.75}$$

Substituting (A.72), (B.74) and (B.75) into (B.69) and (B.70), and using the mutual independence of (dB_t^D, dB_t^F, dB_t^C), we find (4.14) with

$$\chi_1^D \equiv \beta_1 \left(\frac{b_1 \gamma_3^R}{b_2 - \kappa - \rho} - \gamma_1^R\right) = (r + \kappa + \rho)\beta_1 \left(\frac{b_1 \gamma_3}{b_2 - \kappa - \rho} - \gamma_1\right), \tag{B.76}$$

$$\chi_2^D \equiv -\frac{b_1 \beta_1 \gamma_3^R}{b_2 - \kappa - \rho} = -\frac{(r + b_2) b_1 \beta_1 \gamma_3}{b_2 - \kappa - \rho}. \tag{B.77}$$

The function $\chi^D(u) \equiv \chi_1^D e^{-(\kappa+\rho)u} + \chi_2^D e^{-b_2 u}$ can change sign only once, is equal to $-\beta_1 \gamma_1^R$ when $u = 0$, and has the sign of χ_1 if $b_2 > \kappa + \rho$ and of χ_2 if $b_2 < \kappa + \rho$ when u goes to ∞. For small s, $\chi(0) > 0$ since $\gamma_1^R < 0$. Since, in addition, $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_3 < 0$ and $\rho > 0$, (B.76) and

(B.77) imply that $\chi_1 < 0$ if $b_2 > \kappa + \rho$ and $\chi_2 < 0$ if $b_2 < \kappa + \rho$. Therefore, there exists a threshold $\hat{u}^D > 0$ such that $\chi(u) > 0$ for $0 < u < \hat{u}^D$ and $\chi(u) < 0$ for $u > \hat{u}^D$. ∎

Proof of Corollary 4.4: Using (B.7) and proceeding as in the derivation of (A.71), we find

$$\begin{aligned} Cov_t(dR_t, dR'_{t'}) = &\left(\sigma + \beta_1\gamma_1\Sigma p'_f p_f \sigma\right) Cov_t\left(dB^D_t, \gamma^R_1\hat{C}_{t'} + \gamma^R_2 C_{t'} + \gamma^R_3 y_{t'}\right) p_f \Sigma dt' \\ &+ \frac{\phi}{r+\kappa}\left(\sigma + \beta_2\gamma_1\Sigma p'_f p_f \sigma\right) Cov_t\left(dB^F_t, \gamma^R_1\hat{C}_{t'} + \gamma^R_2 C_{t'} + \gamma^R_3 y_{t'}\right) p_f \Sigma dt' \\ &- s\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right) Cov_t\left(dB^C_t, \gamma^R_1\hat{C}_{t'} + \gamma^R_2 C_{t'} + \gamma^R_3 y_{t'}\right) \Sigma p'_f p_f \Sigma dt'. \end{aligned} \tag{B.78}$$

Substituting (A.72), (B.74) and (B.75) into (B.78), and using (4.8) and the mutual independence of (dB^D_t, dB^F_t, dB^C_t), we find (4.15) with

$$\chi_1 \equiv \chi^D_1\left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right), \tag{B.79}$$

$$\begin{aligned} \chi_2 &\equiv s^2\gamma_2\left(\frac{b_1\gamma^R_3}{b_2-\kappa} - \gamma^R_1 - \gamma^R_2\right)\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right) \\ &= s^2(r+\kappa)\gamma_2\left(\frac{b_1\gamma_3}{b_2-\kappa} - \gamma_1 - \gamma_2\right)\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right), \end{aligned} \tag{B.80}$$

$$\begin{aligned} \chi_3 &\equiv -b_1\gamma^R_3\left[\frac{\beta_1}{b_2-\kappa-\rho}\left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right) + \frac{s^2\gamma_2}{b_2-\kappa}\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\right] \\ &= -(r+b_2)b_1\gamma_3\left[\frac{\beta_1}{b_2-\kappa-\rho}\left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right) + \frac{s^2\gamma_2}{b_2-\kappa}\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\right]. \end{aligned} \tag{B.81}$$

The function $\chi(u) \equiv \chi_1 e^{-(\kappa+\rho)u} + \chi_2 e^{-\kappa u} + \chi_3 e^{-b_2 u}$ has the same sign as $\hat{\chi}(u) \equiv \chi_1 e^{-\rho u} + \chi_2 + \chi_3 e^{-(b_2-\kappa)u}$. The latter function is equal to

$$-\beta_1\gamma^R_1\left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right) - s^2\gamma_2(\gamma^R_1 + \gamma^R_2)\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)$$

when $u = 0$, and has the sign of χ_2 if $b_2 > \kappa$ and $\rho > 0$ and of χ_3 if $b_2 < \kappa$ and $\rho > 0$ when u goes to ∞. Moreover, its derivative $\hat{\chi}'(u) = -\chi_1\rho e^{-\rho u} - \chi_3(b_2-\kappa)e^{-(b_2-\kappa)u}$ is equal to

$$-\chi_1\rho - \chi_3(b_2-\kappa) = \beta_1(\rho\gamma^R_1 + b_1\gamma^R_3)\left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right) + s^2 b_1\gamma_2\gamma^R_3\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right) \tag{B.82}$$

when $u = 0$. For small s, $\chi(0) > 0$ since $\gamma^R_1 < 0$ and $s^2\gamma_2/\beta_1 = o(1)$. Since, in addition, $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 < 0$, $\gamma^R_3 < 0$ and $\rho > 0$, (B.80) and (B.81) imply that $\chi_2 < 0$ if $b_2 > \kappa$

and $\chi_3 < 0$ if $b_2 < \kappa$, and (B.82) implies that $\hat{\chi}'(0) < 0$. Since $\hat{\chi}'(u)$ can change sign only once, it is either negative or negative and then positive. Therefore, $\hat{\chi}(u)$ is positive and then negative. The same is true for $\chi(u)$, which means that there exists a threshold $\hat{u} > 0$ such that $\chi(u) > 0$ for $0 < u < \hat{u}$ and $\chi(u) < 0$ for $u > \hat{u}$. ∎

C Momentum and Value Strategies

Proof of Proposition 5.1: Using (4.12), we can write (5.1) as

$$\hat{w}_t \equiv w_t - \frac{w_t \Sigma \eta'}{\eta \Sigma \eta'} \eta. \tag{C.1}$$

The expected return of the index-adjusted strategy is

$$\begin{aligned}
E(\hat{w}_t dR_t) =& E\left[E_t(\hat{w}_t dR_t)\right] \\
=& E\left[\hat{w}_t E_t(dR_t)\right] \\
=& E\left\{\left(w_t - \frac{w_t \Sigma \eta'}{\eta \Sigma \eta'}\eta\right)\left[\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\left(f\Sigma + k\Sigma p_f' p_f \Sigma\right)\eta' + \Lambda_t\left(f\Sigma + k\Sigma p_f' p_f \Sigma\right)p_f'\right] dt\right\} \\
=& E\left\{\left(w_t - \frac{w_t \Sigma \eta'}{\eta \Sigma \eta'}\eta\right)\left[\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'} f\Sigma\eta' + \Lambda_t\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\Sigma p_f'\right] dt\right\} \\
=& \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) E\left(\Lambda_t w_t \Sigma p_f'\right) dt,
\end{aligned} \tag{C.2}$$

where the third step follows from (B.68) and (C.1). The variance of the index-adjusted strategy is

$$\begin{aligned}
Var(\hat{w}_t dR_t) =& E[Var_t(\hat{w}_t dR_t)] + Var[E_t(\hat{w}_t dR_t)] \\
=& E[Var_t(\hat{w}_t dR_t)] \\
=& E\left[\left(w_t - \frac{w_t \Sigma \eta'}{\eta \Sigma \eta'}\eta\right)\left(f\Sigma + k\Sigma p_f' p_f \Sigma\right)\left(w_t - \frac{w_t \Sigma \eta'}{\eta \Sigma \eta'}\eta\right)'\right] dt \\
=& \left\{f\left[E(w_t \Sigma w_t') - \frac{E\left[(w_t\Sigma\eta')^2\right]}{\eta\Sigma\eta'}\right] + kE[(w_t\Sigma p_f')^2]\right\} dt,
\end{aligned} \tag{C.3}$$

where the second step follows because $E[Var_t(w_t dR_t)]$ is of order dt and $Var[E_t(w_t dR_t)]$ of order $(dt)^2$, and the third step follows from (4.12) and (C.1). Eq. (5.3) follows from (5.2), (C.2) and (C.3).

To show that the Sharpe ratio is maximized for $w_t = \Lambda_t p_f$, we write, for any given t, the strategy w_t as a linear combination of the market index, the flow portfolio, and an orthogonal component, i.e.,

$$w_t = \lambda_{1t}\eta + \lambda_{2t}p_f + \check{w}_t, \tag{C.4}$$

where $\eta\Sigma\check{w}_t = p_f\Sigma\check{w}_t = 0$. Substituting w_t from (C.4), we can write (5.3) as

$$SR_w = \frac{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}E(\Lambda_t\lambda_{2t})}{\sqrt{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}E(\lambda_{2t}^2) + fE(\check{w}_t\Sigma\check{w}_t')}}. \tag{C.5}$$

The Sharpe ratio is maximized for $\check{w}_t = 0$. Substituting into (C.5), we find

$$SR_w = \frac{E(\Lambda_t\lambda_{2t})}{\sqrt{E(\lambda_{2t}^2)}}\sqrt{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}}. \tag{C.6}$$

The Cauchy-Schwarz inequality implies that the term

$$\frac{E(\Lambda_t\lambda_{2t})}{\sqrt{E(\lambda_{2t}^2)}}$$

is maximized when λ_{2t} is proportional to Λ_t. Therefore, the Sharpe ratio is maximized by the strategy $\Lambda_t p_f$. ∎