Appendix B

A. Proof of Proposition 1

We first show that the function (18) solves the Bellman equation

$$\overline{V}(\overline{M}_{\ell-1}, d_{\ell-1}, \Delta\overline{e}_{\ell-1}, \overline{e}_{\ell-1}, s_{\ell-1}) =$$

$$\sup_{\overline{c}_{\ell-1}, \overline{x}_\ell(p_\ell)} \left\{ -exp(-\overline{\alpha}\,\overline{c}_{\ell-1})h + \overline{E}_{\ell-1}\overline{V}(\overline{M}_\ell, d_\ell, \Delta\overline{e}_\ell, \overline{e}_\ell, s_\ell)exp(-\overline{\beta}h) \right\}, \tag{B1}$$

for the demand

$$\overline{x}_\ell(p_\ell) = B\left(\frac{h}{1-e^{-rh}}d_\ell - p_\ell\right) - \overline{A}_{\overline{e}}\overline{e}_{\ell-1} - \overline{A}_s s_{\ell-1} - \Delta\overline{e}_{\ell-1} \tag{B2}$$

and the optimal consumption. We then show that the demand in equation (B2) and the optimal consumption satisfy the transversality condition (A6). These results will imply that the demand in equation (B2) solves ($\overline{P}$) and the function (18) is the value function. The demand in equation (B2) is equal to the demand in equation (5) minus $\Delta\overline{e}_{\ell-1}$, and produces the trade $x_\ell + u_\ell - \Delta\overline{e}_{\ell-1}$. The two demands are equal along the optimal path. Indeed, equation (9) implies that if $\overline{x}_\ell(p_\ell) = x_\ell + u_\ell - \Delta\overline{e}_{\ell-1}$ then $\Delta\overline{e}_\ell = 0$.

Bellman Equation

We proceed in 3 steps. First, we show that the optimality conditions are sufficient for the demand in equation (B2) to maximize the RHS of the Bellman equation (B1). (This is why we refer to these conditions as "optimality conditions".) Second, we compute the expectation of the RHS conditional on period $\ell - 1$ information. Finally, we show that the Bellman conditions are sufficient for the function (18) to satisfy the Bellman equation. (This is why we refer to these conditions as "Bellman conditions".)

Step 1: Optimal Demand

We define the vector $\overline{v}_{\ell-1}$ by

$$\overline{v}_{\ell-1} = \begin{pmatrix} \Delta\overline{e}_{\ell-1} \\ \overline{e}_{\ell-1} \\ s_{\ell-1} \end{pmatrix}.$$

A market maker chooses his demand $\overline{x}_\ell(p_\ell)$ to maximize the expectation of the period ℓ value function w.r.t. ζ_ℓ. Using the budget constraint (A3) and equation (A5), we can write this expectation as

$$-E_{\zeta_\ell} exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}+d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})h-\overline{c}_{\ell-1}h)-\frac{1-e^{-rh}}{h}p_\ell\overline{x}_\ell(p_\ell)$$

$$+d_\ell(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1}+\overline{x}_\ell(p_\ell))+F(\overline{Q},\overline{N}(\overline{v}_{\ell-1}+\begin{pmatrix}\overline{x}_\ell(p_\ell)-(\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1})\\0\\0\end{pmatrix})+\overline{n}\zeta_\ell)+\overline{q})), \tag{B3}$$

where p_ℓ is given by equation (A4). The market maker can condition his trade $\overline{x}_\ell$ on ζ_ℓ, since he can infer ζ_ℓ from the price. Therefore, his problem is the same as choosing a trade $\overline{x}_\ell$ to maximize equation (B3) without the expectation sign. The first-order condition is

$$-\frac{1-e^{-rh}}{h}p_\ell+d_\ell+(1,0,0)\overline{Q}(\overline{N}(\overline{v}_{\ell-1}+\begin{pmatrix}\overline{x}_\ell-(\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1})\\0\\0\end{pmatrix})+\overline{n}\zeta_\ell)=0. \tag{B4}$$

The first-order condition determines a maximum since $\overline{Q}_{1,1}<0$. Denoting by $\overline{G}$ the row vector formed by the LHS of equations (A11), (A12), and (A13), we can write the first-order condition as

$$\overline{G}\begin{pmatrix}\overline{e}_{\ell-1}\\s_{\ell-1}\\\zeta_\ell\end{pmatrix}+\overline{Q}_{1,1}(\overline{x}_\ell-(\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1}-\Delta\overline{e}_{\ell-1}))=0.$$

Therefore, the optimal trade is

$$\overline{x}_\ell=\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1}-\Delta\overline{e}_{\ell-1}. \tag{B5}$$

The demand in equation (B2) is optimal since it produces the optimal trade. Substituting p_ℓ from equation (A4), and using the definition of $\overline{Q}'$, we can write equation (B3), evaluated for the optimal trade, as

$$-E_{\zeta_\ell} exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}+d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})h-\overline{c}_{\ell-1}h)$$

$$+d_\ell(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})+F(\overline{Q}',\begin{pmatrix}\overline{v}_{\ell-1}\\\zeta_\ell\end{pmatrix})+\overline{q})). \tag{B6}$$

Step 2: Computing the Expectation

We have to compute the expectation of equation (B6) conditional on period $\ell-1$ information, i.e. w.r.t. δ_ℓ and ζ_ℓ. Computing the expectation w.r.t. δ_ℓ is straightforward. We get

$$-E_{\zeta_\ell} exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})$$

$$-\frac{1}{2}\overline{\alpha}\sigma^2 h(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})^2+F(\overline{Q}',\begin{pmatrix}\overline{v}_{\ell-1}\\ \zeta_\ell\end{pmatrix})+\overline{q})). \tag{B7}$$

To compute the expectation w.r.t. ζ_ℓ, we use the formula

$$E(exp(-\overline{\alpha}(ax+\frac{1}{2}bx^2)))=exp(-\overline{\alpha}(-\frac{1}{2}\overline{\alpha}\Sigma^2(1+\overline{\alpha}\Sigma^2 b)^{-1}a^2+\frac{1}{2\overline{\alpha}}log(1+\overline{\alpha}\Sigma^2 b))), \tag{B8}$$

where x is normal with mean 0 and variance Σ^2, and a and b are constants. (Equation (B8) gives simply the moment generating function of the normal distribution for $b=0$. We can always assume $b=0$ by also assuming that x is normal with mean 0 and variance $\Sigma^2(1+\overline{\alpha}\Sigma^2 b)^{-1}$.)

We set $x=\zeta_\ell$, $\Sigma^2=a_e^2(\Sigma_e^2+\sigma_e^2 h)+\sigma_u^2 h$, $a=\overline{Q}'_{\{4\},\{1,2,3\}}\overline{v}_{\ell-1}$, and $b=\overline{Q}'_{4,4}$. Using the definitions of $\overline{R}$ and $\overline{R}'$, we can write equation (B7) as

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})$$

$$-\frac{1}{2}\overline{\alpha}\sigma^2 h(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})^2+F(\overline{Q}'_{\{1,2,3\}}-\overline{R}',\overline{v}_{\ell-1})+\frac{1}{2\overline{\alpha}}log(\overline{R})+\overline{q})).$$

Finally, using the definition of $\overline{P}$, we can rewrite this equation as

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})+e^{rh}F(\overline{P},\overline{v}_{\ell-1})+\frac{1}{2\overline{\alpha}}log(\overline{R})+\overline{q})).$$

Step 3: Bellman Equation

To compute the RHS of the Bellman equation, we have to maximize w.r.t. $\overline{c}_{\ell-1}$

$$-exp(-\overline{\alpha}\overline{c}_{\ell-1})h-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})$$

$$+e^{rh}F(\overline{P},\overline{v}_{\ell-1})+\frac{1}{2\overline{\alpha}}log(\overline{R})+\overline{q})-\overline{\beta}h). \tag{B9}$$

Simple calculations show that the maximum is

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}\overline{M}_{\ell-1}+d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})+F(\overline{P},\overline{v}_{\ell-1})$$

$$+\frac{1}{2\overline{\alpha}}log(\overline{R})e^{-rh}+\overline{q}e^{-rh}+\frac{(\overline{\beta}e^{-rh}-r)h}{\overline{\alpha}}-\frac{1}{\overline{\alpha}}(1-e^{-rh})log(\frac{h}{e^{rh}-1}))). \tag{B10}$$

This is equal to the period $\ell-1$ value function from equations (A14) and (A15).

Transversality Condition

It is easy to check that, by substituting the optimal $\overline{c}_{\ell-1}$ in the second term of equation (B9), we find equation (B10) times e^{-rh}. Therefore, the expectation of the period ℓ value function in period $\ell-1$, is the period $\ell-1$ value function times e^{-rh}. Recursive use of this equation implies equation (A6). Q.E.D.

B. Proof of Proposition 2

We show that the function (20) solves the Bellman equation for the market order in equation (4) and the optimal consumption. The proof that the market order in equation (4) and the optimal consumption satisfy the transversality condition (A22) is as in Section A.

We proceed in 4 steps. First, we compute the expectation of the RHS of the Bellman equation w.r.t. u_ℓ. Second, we use the optimality conditions to show that the market order in equation (4) maximizes the RHS. Third, we compute the expectation of the RHS w.r.t. the remaining information revealed in period ℓ , i.e. δ_ℓ and ϵ_ℓ. Finally, we use the Bellman conditions to show that the function (20) satisfies the Bellman equation. Notice that we take expectations w.r.t. u_ℓ before determining the optimal market order. This is because the large trader does not know u_ℓ and, unlike the market makers, cannot condition his order on price.

Step 1: Expectation w.r.t. u_ℓ

We have to compute the expectation of the period ℓ value function w.r.t. u_ℓ. Using the budget constraint (19), equations (12) and (A21), and the vector $v_{\ell-1}$ defined by equation (A34), we have to compute

$$-E_{u_\ell} exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}+d_{\ell-1}e_{\ell-1}h-c_{\ell-1}h)+\frac{1-e^{-rh}}{h}p_\ell x_\ell$$

$$+d_\ell(e_{\ell-1}-x_\ell)+F(Q,Nv_{\ell-1}+nu_\ell+\hat{n}\Delta x_\ell)+q)),$$

where p_ℓ and x_ℓ are given by equations (A20) and (A19), respectively. The term inside the exponential is a quadratic function of u_ℓ. The coefficient of $u_\ell^2/2$ is n^tQn, and equations (A19) and (A20) imply that the coefficient of u_ℓ is

$$Q_u\begin{pmatrix} v_{\ell-1} \\ \Delta x_\ell \end{pmatrix}.$$

To compute the expectation, we use equation (B8) and set $x = u_\ell$, $\Sigma^2 = \sigma_u^2 h$, a the coefficient of u_ℓ, and b the coefficient of $u_\ell^2/2$. The expectation is

$$-exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}+d_{\ell-1}e_{\ell-1}h-c_{\ell-1}h)+\frac{1-e^{-rh}}{h}p_\ell x_\ell+d_\ell(e_{\ell-1}-x_\ell)$$

$$+F(Q,Nv_{\ell-1}+\hat{n}\Delta x_\ell)-F(R'_u,\begin{pmatrix} v_{\ell-1} \\ \Delta x_\ell \end{pmatrix})+\frac{1}{2\alpha}log(R_u)+q)), \tag{B11}$$

where p_ℓ is evaluated for $u_\ell = 0$.

Step 2: Optimal Market Order

The large trader chooses Δx_ℓ to maximize equation (B11). Since $dp_\ell/d\Delta x_\ell = -1/B$ and $dx_\ell/d\Delta x_\ell = 1$, the first-order condition is

$$\frac{1-e^{-rh}}{h}\left(p_\ell-\frac{1}{B}x_\ell\right)-d_\ell+\hat{n}^t Q\left(Nv_{\ell-1}+\hat{n}\Delta x_\ell\right)-(R'_u)_{\{4\},\{1,2,3,4\}}\begin{pmatrix} v_{\ell-1} \\ \Delta x_\ell \end{pmatrix}=0. \tag{B12}$$

The first-order condition determines a maximum because of equation (A31). Denoting by G the LHS of equation (A30) and by $\mathcal{G}$ the LHS of equation (A31), we can write the first-order condition as

$$Gv_{\ell-1}+\mathcal{G}\Delta x_\ell = 0.$$

Therefore, $\Delta x_\ell = 0$, i.e. the market order in equation (4) is optimal. Substituting p_ℓ and x_ℓ from equations (A20) and (A19), and using the definition of Q', we can write equation (B11), evaluated for $\Delta x_\ell = 0$, as

$$-exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}+d_{\ell-1}e_{\ell-1}h-c_{\ell-1}h)+d_\ell e_{\ell-1}+F(Q',v_{\ell-1})+\frac{1}{2\alpha}log(R_u)+q)). \tag{B13}$$

Step 3: Expectation w.r.t. δ_ℓ and ϵ_ℓ

We have to compute the expectation of equation (B13) w.r.t. δ_ℓ and ϵ_ℓ. Computing the expectation w.r.t. δ_ℓ is straightforward. To compute the expectation w.r.t. ϵ_ℓ, we use equation (B8) and set $x = \epsilon_\ell$, $\Sigma^2 = \sigma_e^2 h$,

$$a = Q'_{\{1\},\{1,2,3\}}\begin{pmatrix} e_{\ell-1}-s_{\ell-1} \\ s_{\ell-1} \\ \overline{e}_{\ell-1} \end{pmatrix},$$

and $b = Q'_{1,1}$. Proceeding as in Section A, and using the definitions of R, R', and P, we get

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}-c_{\ell-1}h)+e^{rh}d_{\ell-1}e_{\ell-1}$$

$$+e^{rh}F(P, \begin{pmatrix} e_{\ell-1}-s_{\ell-1} \\ s_{\ell-1} \\ \overline{e}_{\ell-1} \end{pmatrix}) + \frac{1}{2\alpha} log(R_u R)+q)).$$

Step 4: Bellman Equation

We proceed as in Section A. Q.E.D.

C. Proof of Theorem 1

We prove Theorem 1 in Sections C.1, C.2, and C.3. In Section C.1 we replace (S) by an equivalent system, (S'), which is easier to solve. In Section C.2 we show that for $\sigma_e^2 = 0$, (S') collapses to (s), and that (s) has a solution. In Section C.3 we extend the solution of (S') for small σ_e^2.

C.1. The Equivalent System

To form the system (S'), we replace the Bellman conditions (A32) of the large trader's optimization problem, by a new set of conditions, the "envelope conditions". Under both the Bellman and the envelope conditions, the matrix Q can be interpreted as a matrix of marginal benefits. The coefficient $Q_{1,2}$, for instance, is the marginal benefit of increasing $e_\ell - s_\ell$, the "first" state variable, when s_ℓ, the "second" state variable is 1 and the other state variables are 0. The Bellman conditions compute this marginal benefit under the assumption that the large trader changes his strategy in response to the change in $e_\ell - s_\ell$, while the envelope conditions assume that the large trader does not change his strategy. The Bellman and the envelope conditions are of course equivalent, when the large trader's strategy is optimal, i.e. when the optimality conditions hold. We use the envelope conditions instead of the Bellman conditions because they are much easier to solve.

To state the envelope conditions, we define the matrix N_e by

$$N_e = \begin{pmatrix} 1 & a_s(1+g) & -a_{\overline{e}}(1+g) \\ 0 & 1-a_s(1+g) & a_{\overline{e}}(1+g) \\ 0 & 0 & 1 \end{pmatrix}$$

and the matrix $\hat{Q}'$ by

$$\hat{Q}' = -\frac{1-e^{-rh}}{h} \begin{pmatrix} 0 \\ \frac{A_s - a_s}{B} \\ \frac{A_{\overline{e}} + a_{\overline{e}}}{B} \end{pmatrix} (a_e, a_s, -a_{\overline{e}}) + N_e^t Q N - (R'_u)_{\{1,2,3\}} + \begin{pmatrix} a_e \\ a_s \\ -a_{\overline{e}} \end{pmatrix} (R'_u)_{\{4\},\{1,2,3\}}. \tag{B14}$$

We also define the scalar $\hat{R}$ and the matrices $\hat{R}'$ and $\hat{P}$ by proceeding as in Section B and using $\hat{Q}'$ instead of Q'. The envelope conditions are $Q = \hat{P}$. Notice that the matrices $\hat{Q}'$ and $\hat{P}$ are not symmetric a priori. Therefore, the system $(S)'$ consists of 23 equations (since there are nine envelope conditions) and 23 unknowns (since the matrix Q is not symmetric a priori). We first show that the solution of (S') produces a symmetric matrix Q. We then show that the solution of (S') satisfies the Bellman conditions, and is thus the solution of (S).

The Matrix Q is Symmetric

We use the vector $v_{\ell-1}$ defined by equation (A34). We define the vector a by

$$a = \begin{pmatrix} a_e \\ a_s \\ -a_{\overline{e}} \end{pmatrix}. \tag{B15}$$

Finally, we denote by p_ℓ and x_ℓ those given by equations (A20) and (A19) for $u_\ell = \Delta x_\ell = 0$.

We will show that $Q = Q^t$. Using the envelope conditions and the fact that $\hat{R}'$ is symmetric, we get

$$Q - Q^t = (\hat{Q}' - (\hat{Q}')^t)e^{-rh}.$$

Using the definition of $\hat{Q}'$ and noting that $a^t v_{\ell-1} = x_\ell$, we get

$$\hat{Q}' v_{\ell-1} = -\frac{1-e^{-rh}}{h} \begin{pmatrix} 0 \\ \frac{A_s - a_s}{B} \\ \frac{A_{\overline{e}} + a_{\overline{e}}}{B} \end{pmatrix} x_\ell + N_e^t Q N v_{\ell-1} - (R'_u)_{\{1,2,3\}} v_{\ell-1} + a (R'_u)_{\{4\},\{1,2,3\}} v_{\ell-1}. \tag{B16}$$

Equations (A20) and (A19) imply that

$$-\frac{A_s - a_s}{B} s_{\ell-1} - \frac{A_{\overline{e}} + a_{\overline{e}}}{B} \overline{e}_{\ell-1} = p_\ell + \frac{x_\ell}{B} - \frac{h}{1-e^{-rh}} d_\ell.$$

Using this fact we get

$$(\hat{Q}')^t v_{\ell-1} = a\left(\frac{1-e^{-rh}}{h}\left(p_\ell + \frac{1}{B}x_\ell\right) - d_\ell\right) + N^t Q^t N_e v_{\ell-1} - (R'_u)^t_{\{1,2,3\}} v_{\ell-1} + (R'_u)_{\{1,2,3\},\{4\}} x_\ell. \tag{B17}$$

Since the first-order condition (B12) holds for $\Delta x_\ell = 0$, we have

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell\right) - d_\ell + \hat{n}^t Q N v_{\ell-1} - (R'_u)_{\{4\},\{1,2,3\}} v_{\ell-1} = 0. \tag{B18}$$

We subtract equation (B17) from equation (B16), add the transpose of equation (A30) times x_ℓ, and subtract equation (B18) times a. Noting that the matrix R'_u is symmetric, and that

$$N_e + \hat{n}a^t = N, \tag{B19}$$

we get

$$(\hat{Q}' - (\hat{Q}')^t)v_{\ell-1} = N^t(Q - Q^t)Nv_{\ell-1}.$$

Since this holds for all $v_{\ell-1}$, we get

$$Q - Q^t = (\hat{Q}' - (\hat{Q}')^t)e^{-rh} = N^t(Q - Q^t)Ne^{-rh}.$$

It is easy to check that this equation produces a system of three linear equations in $Q_{1,2} - Q_{2,1}$, $Q_{1,3} - Q_{3,1}$, and $Q_{2,3} - Q_{3,2}$. Moreover, the solution of this system is zero provided that $1 - a_e(1+g)$ and $1 - a_s - a_{\overline{e}} \in [0,1)$. In Section C.2 we will show that the solution of (S') indeed satisfies $1 - a_e(1+g)$ and $1 - a_s - a_{\overline{e}} \in [0,1)$.

The Bellman Conditions Hold

We only need to show that $Q' = \hat{Q}'$. We subtract equation (B14) from equation (A26), and add the vector a times equation (A30). Using equation (B19), we get $Q' = \hat{Q}'$.

C.2. The Solution for $\sigma_e^2 = 0$

We first assume that (s) has a solution a_s, a_e, g, and $\overline{\Sigma}_e^2$, such that $1-a_e(1+g)$, $1-a_s(1+g)$, and $1 - a_s - a_{\overline{e}} \in (0,1)$. (We define $a_{\overline{e}}$ by equation (25).) Starting from this solution, we construct a solution of (S'). We then show that (s) has a solution with the above properties.

The Solution of (S')

We proceed in three steps. First, we use the equations of the market makers' optimization problem to solve for $A_{\overline{e}}$, A_s, B, and $\overline{Q}$. Second, we use the envelope conditions of the large trader's problem to solve for Q. Finally, we show that the optimality conditions of the large trader's problem are satisfied.

Step 1: The Market Makers' Problem

We first use the Bellman conditions to solve for $\overline{Q}$, as a function of $A_{\overline{e}}$, A_s, and B. We then plug $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ into the optimality conditions, and solve for $A_{\overline{e}}$, A_s, and B.

For $\sigma_e^2 = 0$, $\overline{R}' = 0$. Therefore, the Bellman conditions (A14) become

$$\overline{Q} = (\overline{Q}'_{\{1,2,3\}} - \overline{\alpha}\sigma^2 h\Gamma)e^{-rh}.$$

The equation for $\overline{Q}_{1,1}$ is $\overline{Q}_{1,1} = -\overline{\alpha}\sigma^2 he^{-rh} < 0$. The equations for $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ are

$$\overline{Q}_{1,2} = -\frac{1-e^{-rh}}{h}\frac{A_{\overline{e}}}{B}e^{-rh} - \overline{\alpha}\sigma^2 he^{-rh} \tag{B20}$$

and

$$\overline{Q}_{1,3} = -\frac{1-e^{-rh}}{h}\frac{A_s}{B}e^{-rh}, \tag{B21}$$

respectively. The equations for $\overline{Q}_{2,2}$, $\overline{Q}_{2,3}$, and $\overline{Q}_{3,3}$ form a system of three linear equations. We omit the solution of this system, since we do not use it in what follows.

Plugging $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ into the optimality conditions (A11) and (A12), we get a system of two linear equations in $A_{\overline{e}}/B$ and A_s/B. Solving this system, we get

$$\frac{A_s}{B} = \frac{a_s\overline{\alpha}\sigma^2 h^2 e^{-rh}}{(1-e^{-rh})^2 D_1}, \tag{B22}$$

where

$$D_1 = 1 - (1 - a_s - a_{\overline{e}})e^{-rh}.$$

We omit $A_{\overline{e}}/B$ since we do not use it in what follows. To determine $1/B$, we multiply the optimality condition (A13) by a_s, and subtract it from the optimality condition (A12). Plugging $\overline{Q}_{1,3}$ in the resulting equation, we get

$$\frac{A_s - a_s}{B} = (1 - a_s(1+g))\frac{A_s}{B}e^{-rh}. \tag{B23}$$

Combining equations (B22) and (B23), we get

$$\frac{1}{B} = \frac{D_2\overline{\alpha}\sigma^2 h^2 e^{-rh}}{(1-e^{-rh})^2 D_1}, \tag{B24}$$

where

$$D_2 = 1 - (1 - a_s(1+g))e^{-rh}.$$

Step 2: The Envelope Conditions

For $\sigma_e^2 = 0$, R'_u and R' are equal to 0. Therefore, the envelope conditions become

$$Q = (\hat{Q} - \alpha\sigma^2 h\Gamma)e^{-rh} = (-\frac{1-e^{-rh}}{h}\begin{pmatrix} 0 \\ \frac{A_s - a_s}{B} \\ \frac{A_{\overline{e}} + a_{\overline{e}}}{B} \end{pmatrix}(a_e, a_s, -a_{\overline{e}}) + N_e^t QN - \alpha\sigma^2 h\Gamma)e^{-rh}. \tag{B25}$$

Equation (B25) produces a system of nine linear equations in the elements of the matrix Q. We will "break" this system into three subsystems of three equations each. To obtain the first subsystem, we multiply equation (B25) from the left by the vector $(-1, 1, 0)$. We get

$$(-1,1,0)Q = (-\frac{1-e^{-rh}}{h}\frac{A_s - a_s}{B}(a_e, a_s, -a_{\overline{e}}) + (1 - a_s(1+g))(-1,1,0)QN)e^{-rh}, \quad \text{(B26)}$$

i.e. a system in $Q_{2,1} - Q_{1,1}$, $Q_{2,2} - Q_{1,2}$, and $Q_{2,3} - Q_{1,3}$. The solution of this system is

$$Q_{2,1} - Q_{1,1} = -\frac{1-e^{-rh}}{h}\frac{A_s - a_s}{B}\frac{a_e(1-(1-a_s(1+g))^2 e^{-rh})e^{-rh}}{D_3 D_4}, \quad \text{(B27)}$$

$$Q_{2,2} - Q_{1,2} = -\frac{1-e^{-rh}}{h}\frac{A_s - a_s}{B}\frac{a_s e^{-rh}}{D_3}, \quad \text{(B28)}$$

and

$$Q_{2,3} - Q_{1,3} = \frac{1-e^{-rh}}{h}\frac{A_s - a_s}{B}\frac{a_{\overline{e}} e^{-rh}}{D_3}, \quad \text{(B29)}$$

where

$$D_3 = 1 - (1 - a_s(1+g))(1 - a_s - a_{\overline{e}})e^{-rh}$$

and

$$D_4 = 1 - (1 - a_s(1+g))(1 - a_e(1+g))e^{-rh}.$$

To obtain the second and third subsystems, we multiply equation (B25) from the left by the vectors $(1, 0, 0)$ and $(0, 0, 1)$, respectively. The second subsystem is in $Q_{1,1}$, $Q_{1,2}$, and $Q_{1,3}$. The third subsystem is in $Q_{3,1}$, $Q_{3,2}$, and $Q_{3,3}$, and in $Q_{2,1} - Q_{1,1}$, $Q_{2,2} - Q_{1,2}$, and $Q_{2,3} - Q_{1,3}$ that we already have determined. We omit the solutions of the second and third subsystems, since we do not use them in what follows.

Step 3: The Optimality Conditions

We proceed in three steps. First, we show that the equations of (S') imply the market maker and large trader equations (26) and (27). Second, we show that the three optimality conditions (A30) are satisfied. Finally, we show that equation (A31) is satisfied.

Step 3.1: The Market Maker and Large Trader Equations

We first derive the market maker equation (26). Plugging the optimal trade of equation (B5) into the first-order condition (B4), and using equation (A5), we get

$$-\frac{1-e^{-rh}}{h}p_\ell + d_\ell + (1,0,0)\overline{Q}\begin{pmatrix} 0 \\ \overline{e}_\ell \\ s_\ell \end{pmatrix} = -\frac{1-e^{-rh}}{h}p_\ell + d_\ell + \left(\overline{Q}_{1,2}\overline{e}_\ell + \overline{Q}_{1,3}s_\ell\right) = 0. \quad \text{(B30)}$$

Substituting $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ from equations (B20) and (B21) into equation (B30), we get

$$-\frac{1-e^{-rh}}{h}p_\ell + d_\ell - \overline{\alpha}\sigma^2 h e^{-rh}\overline{e}_\ell - \frac{1-e^{-rh}}{h}\left(\frac{A_{\overline{e}}}{B}\overline{e}_\ell + \frac{A_s}{B}s_\ell\right)e^{-rh} = 0. \tag{B31}$$

Equation (A4) implies that

$$\overline{E}_\ell p_{\ell+1} = \overline{E}_\ell\left(\frac{h}{1-e^{-rh}}d_{\ell+1} - \frac{A_{\overline{e}}}{B}\overline{e}_\ell - \frac{A_s}{B}s_\ell - \frac{1}{B}\zeta_{\ell+1}\right) = \frac{h}{1-e^{-rh}}d_\ell - \frac{A_{\overline{e}}}{B}\overline{e}_\ell - \frac{A_s}{B}s_\ell. \tag{B32}$$

Combining equations (B31) and (B32), we get equation (26).

We next derive the large trader equation (27). We use the vectors $v_{\ell-1}$ and a defined by equations (A34) and (B15), respectively. We denote by p_ℓ, x_ℓ, and $(e_\ell, s_\ell, \overline{e}_\ell)$ those given by equations (A20), (A19), and (A21) for $u_\ell = \Delta x_\ell = 0$. Finally. we denote by $p_{\ell+1}$ and $x_{\ell+1}$ those given by equations (A20) and (A19) for $u_{\ell+1} = \Delta x_{\ell+1} = 0$.

For $\sigma_e^2 = 0$, $R_u' = 0$. Since the first-order condition (B12) holds for $\Delta x_\ell = 0$, we have

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell\right) - d_\ell + \hat{n}^t Q N v_{\ell-1} = 0. \tag{B33}$$

Equation (B25) implies that

$$\hat{n}^t Q N v_{\ell-1} = \hat{n}^t(-\frac{1-e^{-rh}}{h}\begin{pmatrix} 0 \\ \frac{A_s - a_s}{B} \\ \frac{A_{\overline{e}} + a_{\overline{e}}}{B}\end{pmatrix} a^t + N_e^t Q N - \alpha\sigma^2 h\Gamma)e^{-rh} N v_{\ell-1}. \tag{B34}$$

Combining equation (B33) with equation (B34), and noting that

$$a^t N v_{\ell-1} = a^t E_\ell v_\ell = E_\ell a^t v_\ell = E_\ell x_{\ell+1},$$

$$\hat{n}^t N_e^t = \hat{n}^t - (g a_s - a_{\overline{e}})(1+g)(-1,1,0),$$

and

$$\hat{n}^t \Gamma N v_{\ell-1} = -(1,1,0) N v_{\ell-1} = -e_\ell,$$

we get

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell\right) - d_\ell - \frac{1-e^{-rh}}{h}\left(g\frac{A_s - a_s}{B} + \frac{A_{\overline{e}} + a_{\overline{e}}}{B}\right)E_\ell x_{\ell+1} e^{-rh}$$
$$+\left(\hat{n}^t - (g a_s - a_{\overline{e}})(1+g)(-1,1,0)\right) Q N^2 v_{\ell-1} e^{-rh} + \alpha\sigma^2 h e_\ell e^{-rh} = 0. \tag{B35}$$

Noting that $E_\ell v_\ell = N v_{\ell-1}$, we can write the expectation in period ℓ, of equation (B33) in period $\ell+1$, as

$$\frac{1-e^{-rh}}{h}\left(E_\ell p_{\ell+1} - \frac{1}{B}E_\ell x_{\ell+1}\right) - d_\ell + \hat{n}^t Q N^2 v_{\ell-1} = 0. \tag{B36}$$

We multiply equation (B36) by e^{-rh} and subtract it from equation (B35). To simplify the resulting equation, we use two facts. The first fact is

$$\frac{1}{B} - g\frac{A_s - a_s}{B} - \frac{A_{\overline{e}} + a_{\overline{e}}}{B} = (ga_s - a_{\overline{e}})(1+g)\frac{A_s - a_s}{B}\frac{1}{1 - a_s(1+g)}.$$

To derive this fact, we multiply the optimality conditions (A11), (A12), and (A13) by -1, $-g$ and $1 + ga_s - a_{\overline{e}}$, respectively, and add them up. We then use equations (B21) and (B23). The second fact is

$$(-1,1,0)QNv_{\ell-1} = (-\frac{1-e^{-rh}}{h}\frac{A_s - a_s}{B}E_\ell x_{\ell+1} + (1 - a_s(1+g))(-1,1,0)QN^2 v_{\ell-1})e^{-rh}. \tag{B37}$$

To derive this fact, we multiply equation (B26) by $Nv_{\ell-1}$. Using these two facts, we get

$$\begin{aligned}\frac{1-e^{-rh}}{h}&\left(p_\ell - \frac{1}{B}x_\ell - d_\ell h - E_\ell p_{\ell+1}e^{-rh}\right) + \alpha\sigma^2 h e_\ell e^{-rh}\\ &- \frac{(ga_s - a_{\overline{e}})(1+g)}{1 - a_s(1+g)}(-1,1,0)QNv_{\ell-1} = 0.\end{aligned} \tag{B38}$$

Combining equations (B37) and (B38), and noting that $E_\ell v_\ell = Nv_{\ell-1}$, we get equation (27).

Step 3.2: The Three Optimality Conditions (A30)

We will show that the three optimality conditions (A30) are equivalent to equations (21), (22), and (25), which are satisfied. To show the equivalence, we first show that

$$\begin{aligned}G(1 - Ne^{-rh})v_{\ell-1} = &(\alpha e_\ell - \overline{\alpha}\overline{e}_\ell)\sigma^2 h e^{-rh} + \frac{1-e^{-rh}}{h}\left(\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1}\right)e^{-rh}\\ &- \frac{1-e^{-rh}}{h}\frac{1}{B}x_\ell - \frac{(ga_s - a_{\overline{e}})(1+g)}{1 - a_s(1+g)}(-1,1,0)QNv_{\ell-1},\end{aligned} \tag{B39}$$

where G is the row vector formed by the LHS of the three optimality conditions (A30), $\overline{E}_\ell$ is the expectation w.r.t. the market makers' information, and E_ℓ is the expectation w.r.t. the large trader's information. In Section B we wrote the LHS of the first-order condition (B12) as $Gv_{\ell-1} + \mathcal{G}\Delta x_\ell$. Therefore, the LHS of equation (B33) is equal to $Gv_{\ell-1}$. Moreover, since $E_\ell v_\ell = Nv_{\ell-1}$, the LHS of equation (B36) is equal to $GNv_{\ell-1}$, and the LHS of equation (B38) is equal to the LHS of equation (B39). The LHS of equation (B38) is also equal to the RHS of equation (B39). This follows by substituting the price p_ℓ from equation (26) into the LHS of equation (B38).

We next evaluate the RHS of equation (B39) for three values of $v_{\ell-1}$, the column vectors of the matrix

$$\hat{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{a_{\overline{e}}}{1-e^{-rh}} \\ 0 & 0 & \frac{a_s}{1-e^{-rh}} \end{pmatrix}.$$

We divide the result by $\sigma^2 h e^{-rh}/(1-e^{-rh})$, and denote it by $\hat{G}_e$ for the first column vector, G_s for the second, and $G_{\overline{e}}$ for the third. We have

$$G(1-Ne^{-rh})\hat{N} = \frac{\sigma^2 h e^{-rh}}{1-e^{-rh}}(\hat{G}_e, G_s, G_{\overline{e}}).$$

Since $1-a_e(1+g)$ and $1-a_s-a_{\overline{e}} \in (0,1)$, the matrices $1-Ne^{-rh}$ and $\hat{N}$ are invertible. Therefore, the three optimality conditions (A30), i.e. $G=0$, are equivalent to $\hat{G}_e = G_s = G_{\overline{e}} = 0$. We will show that $G_s = 0$ and $G_{\overline{e}} = 0$ are equations (22) and (25), respectively. Moreover, we will show that $\hat{G}_e = k_e G_e + k_s G_s + k_{\overline{e}} G_{\overline{e}}$, where G_e is the LHS of equation (21), and $k_e \neq 0$. Therefore, the three optimality conditions (A30) will be equivalent to equations (21), (22), and (25).

We first compute $G_{\overline{e}}$. Equation (A19) implies that $x_\ell = 0$. Equation (A21) implies that

$$\begin{pmatrix} e_\ell - s_\ell \\ s_\ell \\ \overline{e}_\ell \end{pmatrix} = N v_{\ell-1} = \begin{pmatrix} 0 \\ \frac{a_{\overline{e}}}{1-e^{-rh}} \\ \frac{a_s}{1-e^{-rh}} \end{pmatrix}.$$

Equations (A20) and (B32) imply that

$$\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1} = \frac{1}{B} a_e (e_\ell - s_\ell) = 0. \tag{B40}$$

Plugging into equation (B38), we find that $G_{\overline{e}}$ is the LHS of equation (25).

We next compute G_s. Equation (A19) implies that $x_\ell = a_s$. Equation (A21) implies that $(e_\ell - s_\ell, s_\ell, \overline{e}_\ell) = (0, 1-a_s, a_s)$. Equation (B40) implies that $\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1} = 0$. Equations (B22), (B23), (B28), and (B29), imply that

$$(1-a_s)(Q_{2,2}-Q_{1,2}) + a_s(Q_{2,3}-Q_{1,3}) = -(1-a_s(1+g))\frac{a_s^2(1-a_s-a_{\overline{e}})\overline{\alpha}\sigma^2 h e^{-3rh}}{(1-e^{-rh})D_1 D_3}.$$

Plugging into equation (B38), we get

$$G_s = (\alpha(1-a_s) - \overline{\alpha} a_s)(1-e^{-rh}) - \frac{a_s D_2 \overline{\alpha}}{D_1} + (g a_s - a_{\overline{e}})(1+g)\frac{a_s^2(1-a_s-a_{\overline{e}})\overline{\alpha} e^{-2rh}}{D_1 D_3}.$$

It is easy to check that G_s is in fact the LHS of equation (22).

We finally compute $\hat{G}_e$. Equation (A19) implies that $x_\ell = a_e$. Equation (A21) implies that $(e_\ell - s_\ell, s_\ell, \overline{e}_\ell) = (1-a_e(1+g), ga_e, a_e)$. Equation (B40) implies that $\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1} = a_e(1-a_e(1+g))/B$. Plugging into equation (B38), we get

$$\hat{G}_e = (\alpha(1-a_e) - \overline{\alpha}a_e)(1-e^{-rh}) + \frac{a_e(1-a_e(1+g))D_2\overline{\alpha}e^{-rh}}{D_1} - \frac{a_e D_2 \overline{\alpha}}{D_1}$$

$$+(ga_s - a_{\overline{e}})(1+g)a_s a_e \frac{(1-a_e(1+g))D_3 + ga_s - a_{\overline{e}}}{D_1 D_3 D_4}\overline{\alpha}e^{-2rh}.$$

Setting

$$k_e = -\frac{D_2 D_6}{D_1 D_4(1-a_s-a_{\overline{e}})},$$

$$k_s = -\frac{1}{D_1 D_4}\left[(ga_s - a_{\overline{e}})(1+g)a_e e^{-rh} + \frac{\left(\frac{\overline{\alpha}}{\alpha} - g\right) D_3 a_e (1-(1-a_e(1+g))e^{-rh})}{1-a_s-a_{\overline{e}}}\right],$$

$$k_{\overline{e}} = \frac{1}{D_1 D_4}\left[a_e(1+g)(1-e^{-rh})D_2 + \frac{a_e(1-(1-a_e(1+g))e^{-rh})}{1-a_s-a_{\overline{e}}}\hat{k}_{\overline{e}}\right],$$

$$\hat{k}_{\overline{e}} = \frac{\overline{\alpha}}{\alpha}D_5 + (1-e^{-rh})\left(\frac{\overline{\alpha}}{\alpha} - g\right) D_3 - (1-a_s-a_{\overline{e}})(1-e^{-rh})(1+g),$$

$$D_5 = 1 - (1-a_s(1+g)(2-a_s-a_{\overline{e}}))e^{-rh},$$

and

$$D_6 = 1-(1-a_e(1+g))(1-a_s-a_{\overline{e}})e^{-rh},$$

we have the identity $\hat{G}_e = k_e G_e + k_s G_s + k_{\overline{e}} G_{\overline{e}}$. The proof of this identity is omitted, and is available from the author upon request. For the solution of (s), we have $1-a_e(1+g)$, $1-a_s(1+g)$, and $1-a_s-a_{\overline{e}} \in (0,1)$. Therefore, $k_e = -D_2 D_6 / D_1 D_4 (1-a_s-a_{\overline{e}}) < 0$.

Step 3.3: Equation (A31)

We use the vector b defined by equation (A34). We also define the vector $\hat{b}$ by

$$\hat{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{B41}$$

We can write the first of the three optimality conditions (A30) as

$$-\frac{1-e^{-rh}}{h}\frac{2a_e}{B} + \hat{n}^t Q(a_e \hat{n} + b) = 0.$$

Therefore, equation (A31) is equivalent to $\hat{n}^t Q b > 0$, or, since Q is symmetric, $b^t Q \hat{n} > 0$. Using the envelope conditions (B25), we get

$$b^t Q \hat{n} = b^t(\hat{Q}' - \alpha\sigma^2 h\Gamma)e^{-rh}\hat{n} = (b^t Q N \hat{n} + \alpha\sigma^2 h)e^{-rh}$$

$$= (b^t Q((1 - a_e(1+g))\hat{n} + (a_{\overline{e}} - g a_s)\hat{b}) + \alpha\sigma^2 h)e^{-rh}.$$

Therefore,

$$b^t Q\hat{n} = \frac{\left((a_{\overline{e}} - g a_s) b^t Q\hat{b} + \alpha\sigma^2 h\right) e^{-rh}}{1 - (1 - a_e(1+g))e^{-rh}}.$$

Using the envelope conditions again, we get

$$b^t Q\hat{b} = ((1 - a_s - a_{\overline{e}}) b^t Q\hat{b} - \alpha\sigma^2 h)e^{-rh}.$$

Therefore, $b^t Q\hat{b} = -\alpha\sigma^2 h/D_1$, and

$$b^t Q\hat{n} = \frac{D_2 \alpha\sigma^2 h}{D_1(1 - (1 - a_e(1+g))e^{-rh})} > 0.$$

The Solution of (s)

We proceed in three steps. First, we use equations (23) and (24) to solve for g and $\overline{\Sigma}_e^2$, as functions of $a_e \in (0,1)$. Second, we use equation (22) to solve for a_s as a function of g. Finally, we plug a_s and g into equation (21), obtain an equation only in a_e, and show that this equation has a solution $a_e \in (0,1)$.

Step 1: Determination of g and $\overline{\Sigma}_e^2$

We define the function $f(a_e)$ by

$$f(a_e) = \frac{2\overline{\sigma}_u^2 - a_e \overline{\sigma}_u^2 + a_e}{2(1 - a_e)},$$

and the function $F(x, a_e)$ by

$$F(x, a_e) = x^2 + 2f(a_e)x - \overline{\sigma}_u^2.$$

Since $a_e \in (0,1)$, we have $f(a_e) > 0$ and $f'(a_e) > 0$. Moreover, equation $F(x, a_e) = 0$ has a unique positive solution, that we denote by $x(a_e)$.

We can write equation (24) as

$$F\left(\frac{a_e \overline{\Sigma}_e^2}{(1 - a_e)h}, a_e\right) = 0 \Rightarrow \overline{\Sigma}_e^2 = \frac{(1 - a_e)h x(a_e)}{a_e}.$$

Dividing equation (23) by equation (24), we get

$$g = \frac{a_e \overline{\Sigma}_e^2}{(1 - a_e)h\overline{\sigma}_u^2} \Rightarrow g = \frac{x(a_e)}{\overline{\sigma}_u^2}.$$

We will show that $0 > dg/da_e > -(1+g)/a_e$, a fact that we will use in step 3. Differentiating equation $F(x(a_e), a_e) = 0$, we get

$$\frac{dg}{da_e} = \frac{1}{\overline{\sigma}_u^2}\frac{dx(a_e)}{da_e} = -\frac{f'(a_e)x(a_e)}{\overline{\sigma}_u^2(x(a_e) + f(a_e))} = -\frac{f'(a_e)g}{x(a_e) + f(a_e)} < 0.$$

To show that $dg/da_e > -(1+g)/a_e$ we will show that

$$\frac{1+g}{a_e} + \frac{dg}{da_e} = \frac{1}{a_e} + g\left(\frac{1}{a_e} - \frac{f'(a_e)}{x(a_e)+f(a_e)}\right) > 0.$$

Since $F(x(a_e), a_e) = 0$, we have $x(a_e) < \overline{\sigma}_u^2/2f(a_e)$, i.e. $g < 1/2f(a_e)$ Therefore, it suffices to show that

$$\frac{1}{a_e} + \frac{1}{2f(a_e)}\left(\frac{1}{a_e} - \frac{f'(a_e)}{f(a_e)}\right) > 0.$$

Noting that $f'(a_e)/f(a_e) < 1/a_e(1-a_e)$, it is easy to show this result.

Step 2: Determination of a_s

We can write equation (22) as

$$G_s(a_s, g) \equiv -a_s \frac{D_5}{D_3}\overline{\alpha} + (\alpha(1-a_s) - \overline{\alpha}a_s)(1-e^{-rh}) = 0, \tag{B42}$$

where D_3 and D_5 were defined in Section C.2, and $a_{\overline{e}} = a_s\overline{\alpha}/\alpha$. For $a_s \in (0, \alpha/(\alpha+\overline{\alpha}))$, $1 - a_s - a_{\overline{e}} \in (0,1)$ and $D_3 > 0$. The function $G_s(a_s, g)D_3$ is a third-order polynomial in a_s, which is strictly positive for $a_s = 0$, strictly negative for $a_s = \alpha/(\alpha+\overline{\alpha})$, and goes to ∞ when a_s goes to ∞. Therefore, equation (B42) has a solution $a_s \in (0, \alpha/(\alpha+\overline{\alpha}))$, which is unique in $(0, \alpha/(\alpha+\overline{\alpha}))$. Moreover, at the solution we have $\partial G_s(a_s, g)/\partial a_s < 0$.

We will show that $da_s/dg < 0$, a fact that we will use in step 3. Noting that

$$G_s(a_s, g) = -a_s\left(1 + \frac{a_s\left(g - \frac{\overline{\alpha}}{\alpha}\right)e^{-rh}}{D_3}\right)\overline{\alpha} + (\alpha(1-a_s) - \overline{\alpha}a_s)(1-e^{-rh}), \tag{B43}$$

we get

$$\frac{\partial G_s(a_s, g)}{\partial g} = -a_s^2 \frac{1 - (1 - a_s\left(1 + \frac{\overline{\alpha}}{\alpha}\right))^2 e^{-rh}}{(D_3)^2}\overline{\alpha}e^{-rh} < 0. \tag{B44}$$

Differentiating equation $G_s(a_s, g) = 0$, we get $da_s/dg < 0$.

We will also show that for $g > \overline{\alpha}/\alpha$ and $1 - a_s(1+g) \in [0,1)$, we have $da_s/dg > -a_s/(1+g)$. Using equation (B43) and noting that

$$\frac{\partial}{\partial a_s}\frac{a_s^2}{D_3} = \frac{a_s}{(D_3)^2}(2 - (2 - a_s\left(2 + g + \frac{\overline{\alpha}}{\alpha}\right))e^{-rh}) > 0,$$

we get $\partial G_s(a_s, g)/\partial a_s < -\overline{\alpha}$. Using equation (B44) and noting that

$$D_3 > 1 - (1 - a_s\left(1 + \frac{\overline{\alpha}}{\alpha}\right))^2 e^{-rh} > 0,$$

and $D_3 > a_s(1+g)e^{-rh}$, we get $\partial G_s(a_s, g)/\partial g > -a_s\overline{\alpha}/(1+g)$. Differentiating equation $G_s(a_s, g) = 0$, we get $da_s/dg > -a_s/(1+g)$.

Step 3: Determination of a_e

We denote by $G_e(a_e, a_s, g)$ the LHS of equation (21), and by $G_e(a_e)$ the LHS when we plug g and a_s as functions of a_e. The function $G_e(a_e)$ is continuous in $a_e \in (0, 1)$. When a_e goes to 0, g goes to a limit $g(0) > 0$, and a_s goes to a limit $a_s(0) \in (0, \alpha/(\alpha + \overline{\alpha}))$. The function $G_e(a_e)$ goes to

$$-(1 - a_s(0)\left(1 + \frac{\overline{\alpha}}{\alpha}\right))(1 - e^{-rh})\alpha < 0.$$

When a_e goes to 1, g goes to 0, and a_s goes to a limit $a_s(1) \in (0, \alpha/(\alpha + \overline{\alpha}))$. The function $G_e(a_e)$ goes to

$$(1 - a_s(1))\overline{\alpha} > 0.$$

Therefore, equation (21) has a solution $a_e \in (0, 1)$. Since $a_s \in (0, \alpha/(\alpha + \overline{\alpha}))$, we have $1 - a_s - a_{\overline{e}} \in (0, 1)$. Equation (23) implies that $g < (1 - a_e)/a_e$. Therefore, $1 - a_e(1 + g) \in (0, 1)$. Finally, equation (21) implies that $1 - a_s(1 + g) > 0$. Therefore, $1 - a_s(1 + g) \in (0, 1)$.

We will show that at the solution a_e, we have $dG_e(a_e)/da_e > 0$. This fact implies that the solution is unique and, as we show in Section C.3, allows us to use the implicit function theorem. We have

$$\frac{dG_e(a_e)}{da_e} = \frac{\partial G_e(a_e, a_s, g)}{\partial a_e} + \left(\frac{\partial G_e(a_e, a_s, g)}{\partial g} + \frac{\partial G_e(a_e, a_s, g)}{\partial a_s}\frac{da_s}{dg}\right)\frac{dg}{da_e}.$$

Since $\partial G_e(a_e, a_s, g)/\partial a_e > 0$ and $0 > dg/da_e > -(1 + g)/a_e$, we have $dG(a_e)/da_e > 0$ if

$$\frac{\partial G_e(a_e, a_s, g)}{\partial a_e} - \left(\frac{\partial G_e(a_e, a_s, g)}{\partial g} + \frac{\partial G_e(a_e, a_s, g)}{\partial a_s}\frac{da_s}{dg}\right)\frac{1 + g}{a_e} > 0.$$

We have

$$\frac{\partial G_e(a_e, a_s, g)}{\partial a_e} - \frac{\partial G_e(a_e, a_s, g)}{\partial g}\frac{1 + g}{a_e} = (1 - (1 - a_e(1 + g))e^{-rh})\overline{\alpha},$$

and

$$\frac{\partial G_e(a_e, a_s, g)}{\partial a_s} = -a_e(1 - (1 - a_e(1 + g))e^{-rh})(1 + g)\overline{\alpha} + (1 - a_e(1 + g))\left(1 + \frac{\overline{\alpha}}{\alpha}\right)(1 - e^{-rh})\alpha.$$

Using equation (21), we can write $\partial G_e(a_e, a_s, g)/\partial a_s$ as

$$-\frac{a_e(1 - (1 - a_e(1 + g))e^{-rh})\overline{\alpha}}{1 - a_s\left(1 + \frac{\overline{\alpha}}{\alpha}\right)}\left(g - \frac{\overline{\alpha}}{\alpha}\right).$$

Therefore, we have $dG_e(a_e)/da_e > 0$ if

$$1 + \frac{(1 + g)\left(g - \frac{\overline{\alpha}}{\alpha}\right)}{1 - a_s\left(1 + \frac{\overline{\alpha}}{\alpha}\right)}\frac{da_s}{dg} > 0.$$

If $g \leq \overline{\alpha}/\alpha$, this condition is satisfied, since $da_s/dg < 0$. If $g > \overline{\alpha}/\alpha$, this condition is satisfied since $da_s/dg > -a_s/(1 + g)$.

C.3. The Solution for Small σ_e^2

We first prove Lemma 1. To state the lemma, we use the following notation. Consider a function $F(x,y)$, where x is a $1\times N$ vector, y a $1\times M$ vector, and F a $K\times 1$ vector. We denote by $J_xF(x,y)$ the $K\times N$ matrix of partial derivatives of $F(x,y)$ w.r.t. x, i.e. the Jacobian matrix of $F(x,y)$ w.r.t. x.

Lemma 1 *Consider a function*

$$F(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix},$$

where x is a $1\times N$ vector, y a $1\times M$ vector, $F_1(x,y)$ a $N\times 1$ vector, and $F_2(x,y)$ a $M\times 1$ vector. Suppose that (i) there exists a function $y(x)$ such that $F_2(x,y(x))=0$, (ii) $J_yF_2(x,y)$ is invertible, and (iii) $J_xF_1(x,y(x))$ is invertible. Then $J_{x,y}F(x,y)$ is invertible for $y=y(x)$.

In words, Lemma 1 says that the Jacobian matrix of $F(x,y)$ w.r.t. (x,y) is invertible if (i) we can solve equation $F_2(x,y)=0$ for y, and (ii) the Jacobian matrix of the function $F_1(x,y(x))$, that we obtain by plugging $y(x)$ in the function $F_1(x,y)$, is invertible. Lemma 1 allows us to "eliminate" the function $F_2(x,y)$ and consider a smaller Jacobian matrix.

Proof: We have

$$J_{x,y}F(x,y) = \begin{pmatrix} J_xF_1(x,y) & J_yF_1(x,y) \\ J_xF_2(x,y) & J_yF_2(x,y) \end{pmatrix}.$$

The matrix $J_{x,y}F(x,y)$ is invertible if the matrix obtained by multiplying the last M columns by $J_yF_2(x,y)^{-1}J_xF_2(x,y)$ and subtracting them from the first N, is invertible. This matrix is

$$\begin{pmatrix} J_xF_1(x,y) - J_yF_1(x,y)J_yF_2(x,y)^{-1}J_xF_2(x,y) & J_yF_1(x,y) \\ 0 & J_yF_2(x,y) \end{pmatrix},$$

and is invertible if the matrix

$$J_xF_1(x,y) - J_yF_1(x,y)J_yF_2(x,y)^{-1}J_xF_2(x,y)$$

is invertible. We will show that for $y=y(x)$, this matrix is $J_xF_1(x,y(x))$. Differentiating $F_2(x,y(x))=0$, we get

$$J_xF_2(x,y(x)) + J_yF_2(x,y(x))J_xy(x) = 0 \Rightarrow J_xy(x) = -J_yF_2(x,y(x))^{-1}J_xF_2(x,y(x)).$$

Therefore,

$$J_x F_1(x, y(x)) = J_x F_1(x, y(x)) + J_y F_1(x, y(x)) J_x y(x)$$

$$= J_x F_1(x, y(x)) - J_y F_1(x, y(x)) J_y F_2(x, y(x))^{-1} J_x F_2(x, y(x)).$$

Q.E.D.

To extend the solution of (S') for small σ_e^2, we use the implicit function theorem. We denote by z the 1×23 vector of unknowns a_e, a_s, $a_{\overline{e}}$, g, $\overline{\Sigma}_e^2$, $A_{\overline{e}}/B$, A_s/B, $1/B$, $\overline{Q}$, and Q. We denote by $K(z, \sigma_e^2) = 0$ the 23×1 vector of optimality conditions of the large trader's problem, equations (23) and (24) of the recursive filtering problem, equations of the market makers' problem, and envelope conditions of the large trader's problem. Finally, we denote by z_0 the solution of (S') for $\sigma_e^2 = 0$. The function $K(z, \sigma_e^2)$ is C^1 at $(z_0, 0)$, and $K(z_0, 0) = 0$. The implicit function theorem applies if the matrix $J_z K(z, 0)$ is invertible for $z = z_0$.

To show that $J_z K(z, 0)$ is invertible, we use Lemma 1. We set $(x, y) = z$ and $F(x, y) = K(z, 0)$, denote by y the 1×15 vector of unknowns $A_{\overline{e}}/B$, A_s/B, $1/B$, $\overline{Q}$, and Q, and by $F_2(x, y) = 0$ the 15×1 vector of equations of the market makers' problem and envelope conditions of the large trader's problem. In Section C.2 we solved equation $F_2(x, y) = 0$ for y. Since $F_2(x, y)$ is linear in y, and since we could solve for y, the matrix $J_y F_2(x, y)$ is invertible. Lemma 1 implies that $J_z K(z, 0)$ is invertible if $J_x F_1(x, y(x))$ is invertible.

In Section C.2 we showed that the optimality conditions of the large trader's problem are connected to equations (21), (22), and (25), though an invertible linear transformation. Therefore, we can assume that $F_1(x, y(x)) = 0$ consists of equations (21), (22), (25), (23), and (24).

To show that $J_x F_1(x, y(x))$ is invertible, we use Lemma 1 for the function $F_1(x, y(x))$. The "new" (x, y) is the "old" x, the new $F(x, y)$ is $F_1(x, y(x))$, the new y is the 1×4 vector of unknowns a_s, $a_{\overline{e}}$, g, and $\overline{\Sigma}_e^2$, and the new $F_2(x, y) = 0$ is the 4×1 vector of equations (22), (25), (23), and (24). In Section C.2 we solved $F_2(x, y) = 0$ for y. Using the results of this section, it is easy to check that the matrix $J_y F_2(x, y)$ is invertible. Lemma 1 implies that $J_{x,y} F(x, y)$ is invertible if $J_x F_1(x, y(x))$ is invertible. Using the notation of Section (25), $J_x F_1(x, y(x)) = dG_e(a_e)/da_e > 0$. Q.E.D.

D. Proofs of Propositions 3, 5, and 6

Proof of Proposition 3: We denote by a_e^* and a_s^*, the a_e and a_s that solve (S). We will show that $a_e^* > a_s^*$ for $\sigma_e^2 = 0$, and conclude by continuity. We proceed by contradiction and assume that $a_e^* \leq a_s^*$. As in Section C.2, we fix $a_e \in (0,1)$ and define g and $\overline{\Sigma}_e^2$ from equations (23) and (24), and a_s from equation (22). Since $a_s < \alpha/(\alpha+\overline{\alpha}) < 1$, there exists $a_e \geq a_e^*$ such that $a_e = a_s$. For this a_e we have $G_e(a_e) \geq 0$, since equation $G_e(a_e) = 0$ has a unique solution $a_e^* \in (0,1)$, and the function $G_e(a_e)$ goes to a strictly positive limit when a_e goes to 1.

Noting that $a_s = a_e$ and $a_{\overline{e}} = a_s\overline{\alpha}/\alpha$, we have

$$G_e(a_e) = (1 - a_e(1+g))\left(a_e(1-(1-a_e(1+g))e^{-rh})\overline{\alpha} - (\alpha(1-a_e) - \overline{\alpha}a_e)(1-e^{-rh})\right).$$

Using equation (22), we get

$$G_e(a_e) = (1 - a_e(1+g))a_e\overline{\alpha}F,$$

where

$$F = 1 - (1 - a_e(1+g))e^{-rh} - \frac{1-(1-a_e(1+g)(2-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right)))e^{-rh}}{1-(1-a_e(1+g))(1-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right))e^{-rh}}.$$

$$= -(1-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right))e^{-rh}\frac{1-(1-a_e(1+g))^2e^{-rh}}{1-(1-a_e(1+g))(1-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right))e^{-rh}}.$$

Since $1-a_e(1+\overline{\alpha}/\alpha) = 1-a_s(1+\overline{\alpha}/\alpha) \in (0,1)$ and $1-a_e(1+g) \in (0,1)$, we have $G_e(a_e) < 0$, a contradiction. Q.E.D.

Proof of Proposition 5: We first show that if $ga_s - a_{\overline{e}} \geq 0$, the large trader's stock holdings decrease over time. Since $a_e > a_s$, we have

$$a_e(1+g) - a_s - a_{\overline{e}} > ga_s - a_{\overline{e}} \geq 0.$$

The coefficients of $(1-a_s-a_{\overline{e}})^{\ell'-\ell}$ and $(1-a_e(1+g))^{\ell'-\ell}$ in equation (30) are thus positive, and stock holdings decrease over time.

We next show that if $ga_s - a_{\overline{e}} < 0$, stock holdings decrease and then increase over time. Stock holdings are equal to 1 for $\ell' = \ell$, and go to $a_{\overline{e}}/(a_s + a_{\overline{e}}) < 1$ when ℓ' goes to ∞. Their derivative w.r.t. ℓ' changes sign at most once. Therefore, stock holdings decrease and then increase over time if they increase for large ℓ'. We distinguish two cases. If $a_e(1+g) - a_s - a_{\overline{e}} > 0$, then $1 - a_s - a_{\overline{e}} > 1 - a_e(1+g)$. Stock holdings are approximately

$$\frac{a_{\overline{e}}}{a_s + a_{\overline{e}}} + \frac{a_e(ga_s - a_{\overline{e}})}{(a_e(1+g) - a_s - a_{\overline{e}})(a_s + a_{\overline{e}})}(1-a_s-a_{\overline{e}})^{\ell'-\ell}$$

for large ℓ', and they increase. If $a_e(1+g) - a_s - a_{\overline{e}} < 0$, stock holdings are approximately

$$\frac{a_{\overline{e}}}{a_s + a_{\overline{e}}} + \frac{a_e - a_s}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_e(1+g))^{\ell'-\ell},$$

and they also increase.

We next show that if $ga_s - a_{\overline{e}} \geq 0$, the trading rate decreases over time. The trading rate is

$$\frac{x_{\ell'}}{\sum_{\ell'' \geq \ell'} x_{\ell''}} = \frac{x_{\ell'}}{e_{\ell'-1} - e_\infty}.$$

Equation (A37) implies that

$$x_{\ell'} = a_e(e_{\ell'-1} - s_{\ell'-1}) + a_s s_{\ell'-1} - a_{\overline{e}}\overline{e}_{\ell'-1}$$

$$= \frac{a_e(ga_s - a_{\overline{e}})}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_s - a_{\overline{e}})^{\ell'-\ell} + \frac{a_e(1+g)(a_e - a_s)}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_e(1+g))^{\ell'-\ell}. \quad \text{(B45)}$$

Using equations (30) and (B45), we can write the trading rate as

$$\frac{ga_s - a_{\overline{e}} + (1+g)(a_e - a_s)f(\ell')}{\frac{ga_s - a_{\overline{e}}}{a_s + a_{\overline{e}}} + \frac{a_e - a_s}{a_e} f(\ell')},$$

where

$$f(\ell') = \frac{(1 - a_e(1+g))^{\ell'-\ell}}{(1 - a_s - a_{\overline{e}})^{\ell'-\ell}}.$$

Using the inequalities $a_e > a_s$ and $a_e(1+g) - a_s - a_{\overline{e}} > 0$, it is easy to check that the trading rate increases in $f(\ell')$, and that $f(\ell')$ decreases in ℓ'. Therefore, the trading rate decreases over time.

We finally show that if $ga_s - a_{\overline{e}} \geq 0$, the price impact decreases over time. We can write the expected price change, $p_{\ell'-1} - p_{\ell'}$, successively as

$$\left(\frac{h}{1 - e^{-rh}}\overline{Q}_{1,2} + \frac{A_{\overline{e}}}{B}\right)\overline{e}_{\ell'-1} + \left(\frac{h}{1 - e^{-rh}}\overline{Q}_{1,3} + \frac{A_s}{B}\right)s_{\ell'-1} + \frac{a_e}{B}(e_{\ell'-1} - s_{\ell'-1})$$

$$= \frac{h}{1 - e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})(a_s s_{\ell'-1} - a_{\overline{e}}\overline{e}_{\ell'-1}) + \frac{a_e}{B}(e_{\ell'-1} - s_{\ell'-1})$$

$$= \frac{h}{1 - e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})\frac{a_e(ga_s - a_{\overline{e}})}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_s - a_{\overline{e}})^{\ell'-\ell}$$

$$+ \left(-\frac{h}{1 - e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})\frac{a_e(ga_s - a_{\overline{e}})}{a_e(1+g) - a_s - a_{\overline{e}}} + \frac{a_e}{B}\right)(1 - a_e(1+g))^{\ell'-\ell}. \quad \text{(B46)}$$

For the first equality we use equation (8) for $p_{\ell'}$ and equation (B30) for $p_{\ell'-1}$, for the second equality we use the optimality conditions (A11) and (A12), and for the third equality we use equation (A37). Using equations (B45) and (B46), we can write the price impact as

$$\frac{\frac{h}{1 - e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})(ga_s - a_{\overline{e}}) + \left(-\frac{h}{1 - e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})(ga_s - a_{\overline{e}}) + \frac{a_e(1+g) - a_s - a_{\overline{e}}}{B}\right)f(\ell')}{ga_s - a_{\overline{e}} + (1+g)(a_e - a_s)f(\ell')}.$$

The function $f(\ell')$ decreases in ℓ'. Therefore, the price impact decreases over time if it increases in $f(\ell')$. It is easy to check that the price impact increases in $f(\ell')$ if and only if

$$\frac{1}{B} - \frac{h}{1-e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2}) > 0. \tag{B47}$$

Substituting $1/B$ from the optimality condition (A13), we can write inequality (B47) as

$$\frac{h}{1-e^{-rh}}(1+g)\overline{Q}_{1,3} < 0.$$

This inequality holds as long as $\overline{Q}_{1,3} < 0$. Equations (B21) and (B22) imply that for $\sigma_e^2 = 0$, $\overline{Q}_{1,3} < 0$. Therefore, for small σ_e^2, $\overline{Q}_{1,3} < 0$. Our numerical solutions confirm that $\overline{Q}_{1,3} < 0$ for large σ_e^2. Q.E.D.

Proof of Proposition 6: We replace (S') by an equivalent system (S'_c), that we obtain as follows. We set $a_e = \phi_e\sqrt{h}$, $a_s = \phi_s h$, $a_{\overline{e}} = \phi_{\overline{e}} h$, and $\overline{\Sigma}_e^2 = \phi_\Sigma\sqrt{h}$, and replace the unknowns a_e, a_s, $a_{\overline{e}}$, and $\overline{\Sigma}_e^2$, by ϕ_e, ϕ_s, $\phi_{\overline{e}}$, and ϕ_Σ. We divide the Bellman conditions of the market makers' problem by h, the envelope conditions of the large trader's problem by $\sqrt{h}$ if they correspond to $Q_{1,1}$, $Q_{2,1}$, and $Q_{3,1}$, and by h otherwise, and equation (24) of the recursive filtering problem by h. Finally, we multiply the optimality conditions of the large trader's problem by the invertible matrix

$$(1-Ne^{-rh})\hat{N}\frac{1-e^{-rh}}{\sigma^2 h^2 e^{-rh}}\begin{pmatrix} k_e & 0 & 0 \\ k_s & 1 & 0 \\ k_{\overline{e}} & 0 & 1 \end{pmatrix}^{-1}.$$

For $\sigma_e^2 = 0$, we get equations $G_e/h = G_s/h = G_{\overline{e}}/h = 0$. We denote by z the vector of unknowns of (S'_c) and by $K(z, \sigma_e^2, h) = 0$ the vector of equations.

We will use the implicit function theorem for $\sigma_e^2 = h = 0$. It is easy to check the following. First, the function $K(z, \sigma_e^2, h)$ can be extended by continuity for $h = 0$, and is C^1. (We use the fact that the matrices $\overline{R}'$, R'_u, and R', are "of order" h.) Second, for $\sigma_e^2 = h = 0$, the optimality conditions of the market makers' problem, the equations obtained from the Bellman conditions of the market makers' problem, and the equations obtained from the envelope conditions of the large trader's problem, are linear in $A_{\overline{e}}/B$, A_s/B, $1/B$, $\overline{Q}$, and Q, and can be solved in these unknowns. Third, for $\sigma_e^2 = h = 0$, the equations obtained from the optimality conditions of the large trader's problem, equation (23), and the equation obtained from equation (24), become

$$(\phi_e)^2(1+g)\overline{\alpha} - r\alpha = 0,$$

$$-\phi_s \frac{r + 2\phi_s(1+g)}{r + \phi_s(2+g) + \phi_{\overline{e}}} \overline{\alpha} + r\alpha = 0,$$

$$\phi_{\overline{e}}\alpha - \phi_s\overline{\alpha} = 0,$$

$$g - \frac{\phi_e \phi_\Sigma}{\overline{\sigma}_u^2} = 0,$$

and

$$(\phi_e\phi_\Sigma)^2 + 2\overline{\sigma}_u^2\phi_e\phi_\Sigma - \overline{\sigma}_u^2 = 0,$$

respectively. Fourth, the ϕ_e, ϕ_s, $\phi_{\overline{e}}$, and g_0 of the Proposition solve these equations. Therefore, for $\sigma_e^2 = h = 0$, we have a solution, z_0, to $K(z, \sigma_e^2, h) = 0$. To show that the matrix $J_z K(z, 0, 0)$ is invertible for $z = z_0$, we proceed as in Section C.3. Q.E.D.