The Distribution of Investor Beliefs, Stock Ownership and Stock Returns

GIKAS HARDOUVELIS
University of Piraeus and CEPR

GEORGIOS KARALAS
University Carlos III of Madrid

DIMITRI VAYANOS
London School of Economics, CEPR and NBER

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Abstract

We study the relationship between the distribution of investor beliefs, the breadth of ownership and expected returns in a model where stocks differ in the intensity of disagreement and in the extent to which beliefs are polarized, as measured by the number of optimists and pessimists relative to moderates. Polarization explains the size-dependent relationship between breadth and expected returns that we find empirically: positive for large stocks and negative for small stocks. We also find empirical support for the underlying mechanism: polarized stocks earn lower expected returns and are held more broadly if small and less broadly if large.

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*`Ghardouv@unipi.gr`, `gkaralas@emp.uc3m.es`, `d.vayanos@lse.ac.uk`. We thank Harrison Hong, George Leledakis, Dong Lou, Anna Pavlova, Luca Pezzo, Christopher Polk, Spyros Skouras, David Sraer, Michela Verardo, and two anonymous referees, as well as seminar participants at Amherst, Athens, Bristol, Carlos III, Cyprus and Piraeus, and participants at the Miami Behavioral Finance conference for helpful comments. Financial support from Comunidad de Madrid (Programa Excelencia para el Profesorado Universitario, convenio con Universidad Carlos III de Madrid, V Plan Regional de Investigacion Cientifica e Innovacion Tecnologica) and the LSE Paul Woolley Centre is gratefully acknowledged.

1 Introduction

Why do some assets have a narrow and some a broad investor base? Do assets with a broad base have different expected returns than assets with a narrow base? Two influential theories characterize the breadth of ownership and its relationship to expected returns, and yield opposite predictions. The first theory, formulated by Merton (1987) as the investor-recognition hypothesis, emphasizes costs of entering asset markets. According to that theory, an asset for which entry costs are high attracts few investors and trades at a deep discount because of imperfect risk-sharing. Hence, a narrow investor base indicates an undervalued asset with high expected return. The second theory, formulated by Miller (1977) and Harrison and Kreps (1978), and further developed by Chen, Hong, and Stein (2002, CHS), Scheinkman and Xiong (2003), Hong and Stein (2007) and Hong and Sraer (2016), emphasizes differences in beliefs across investors, combined with short-sale constraints. According to that theory, large disagreements across investors about an asset's payoff result in the asset being held only by the most optimistic investors. Since optimists push the asset price up, a narrow investor base indicates an overvalued asset with low expected return.

Some empirical findings are consistent with the entry-cost theory. For example, Hong and Kacperczyk (2009) find that social norms prevent some institutional investors from holding stocks in "sin industries" (alcohol, gaming and tobacco), and this raises the stocks' expected returns. Lou (2014) finds that increased advertising by firms brings in more investors, raises their stock prices and lowers their expected returns. Other findings support the differences-in-beliefs theory. CHS find that stocks with a narrow investor base earn low expected returns in the cross-section. Diether, Malloy, and Scherbina (2002, DMS) find that stocks for which financial analysts disagree the most earn low expected returns.

In this paper we show that neither the entry-cost theory nor existing formulations of the differences-in-beliefs theory can explain the empirical relationship between breadth of ownership and expected returns. We propose a richer formulation of the differences-in-beliefs theory that provides an explanation, and argue that existing formulations may overlook the most empirically relevant dimension of differences in beliefs. We provide direct evidence for our explanation by testing not only for the relationship between breadth and expected returns but also for that between each of the two variables and investor beliefs.

We show that the empirical relationship between breadth and expected returns changes sign depending on stock size. For large stocks, a narrow investor base predicts low future returns, consistent with the differences-in-beliefs theory and the findings of CHS. For small stocks, by contrast, a narrow base predicts high future returns, consistent with the entry-cost theory. Moreover, this relationship is stronger in absolute value than for large stocks. Our findings for small stocks are puzzling for the differences-in-beliefs theory especially because that theory should be more relevant for small stocks. Indeed, since optimists are better able to absorb a smaller supply, the overvaluation that they generate should be more severe for small stocks (Hong, Scheinkman, and Xiong (2006)).

One could argue for a hybrid explanation of the puzzle: the breadth-return relationship is explained by the differences-in-beliefs theory for large stocks and by the entry-cost theory for small stocks. We propose instead an explanation that is based only on differences in beliefs. In CHS and most other papers in the literature, the distribution of investor beliefs is described only by the intensity of disagreement, as measured by the difference between the beliefs of optimists and pessimists. We describe the distribution of beliefs by an additional dimension, which we show is more relevant empirically. This is the extent to which beliefs are polarized, as measured by the number of optimists and pessimists relative to investors with moderate beliefs. We show that polarization moves breadth in opposite directions for small and large stocks, and that this causes the sign of the breadth-return relationship to change with size. Our explanation is more parsimonious than the hybrid explanation. It is also supported by the direct evidence that we present on the relationship that investor beliefs have with breadth and expected returns.

In our model, presented in Section 2, there are multiple stocks and a continuum of investors with different beliefs and short-sale constraints. Beliefs for each stock are described by a general distribution with finite support. The intensity of disagreement maps to the *range* of the distribution. The extent of polarization maps to the distribution's *kurtosis*. Holding range constant, higher kurtosis indicates fewer optimists and pessimists relative to moderates. Stocks differ in their range and kurtosis, as well as in their size, idiosyncratic variance, and exposure to systematic risk. Only differences in kurtosis and size across stocks suffice, however, to generate our main empirical hypotheses.

Stocks' expected returns in equilibrium are negatively related to range and positively to kurtosis. The effect of range follows the same logic as in CHS and DMS. Stocks for which the range is high exhibit more disagreement: more extreme positive beliefs by optimists and negative beliefs by pessimists. Therefore, when short-sale constraints keep pessimists out of the market, optimists render the prices of these stocks higher than of otherwise identical stocks with less extreme beliefs. The effect of kurtosis follows a different logic. Stocks for which the kurtosis is low exhibit more polarization: more optimists and pessimists, and fewer moderates. Therefore, when pessimists drop out of the market, optimists push the prices of these stocks higher than of otherwise identical stocks with fewer optimists.

Following CHS, we measure breadth of ownership by the fraction of investors holding a stock. Breadth is negatively related to range. Indeed, stocks are held only by optimists when optimistic beliefs are extreme, and by both optimists and moderates when beliefs are less extreme. The effect of kurtosis changes sign with size. For small stocks, breadth is negatively related to kurtosis because these stocks are held only by optimists, and high kurtosis indicates few optimists. For large stocks instead, breadth is positively related to kurtosis, because these stocks are held by both optimists and moderates, and high kurtosis indicates few pessimists, who are the only investors not holding the stocks.

Our theoretical results yield three main empirical hypotheses. Hypothesis 1 concerns the relationship between investor beliefs and expected returns. Expected returns should be negatively related to range holding kurtosis constant, and positively related to kurtosis holding range constant. Hypothesis 2 concerns the relationship between investor beliefs and breadth. Holding kurtosis constant, breadth should be negatively related to range. Holding range constant, breadth should be negatively related to kurtosis for small stocks and positively for large stocks. Hypothesis 3 follows from Hypotheses 1 and 2, and concerns the relationship between breadth and expected returns. For large stocks, the effects of range and kurtosis reinforce each other, generating a positive relationship. For small stocks instead, the effects work in opposite directions, and the relationship is negative if the effect of kurtosis dominates.

We test the three hypotheses using CRSP data on US stock prices and returns, Thompson Reuters data on holdings by 13-F institutional investors, and I/B/E/S data on analyst forecasts. We proxy ownership patterns by those within 13-F investors, computing breadth as the number of 13-F investors holding a stock, divided by the total number of 13-F investors. We proxy investor beliefs by analyst forecasts, and compute the range and kurtosis of the distribution of forecasts across analysts. We describe our dataset and empirical measures in Section 3.

Section 4 presents our empirical findings on Hypothesis 3. We perform two related tests, in which, following CHS, we use the first difference ΔB of breadth rather than the level to account for the high autocorrelation of breadth and its correlation with size. One test is to regress future returns on ΔB, its interaction with size, and various controls. The other test is to double-sort stocks into portfolios based on size and ΔB, and compare the returns and alphas of the high- relative to the low-ΔB portfolio across size groups. Both tests paint a consistent picture: ΔB predicts negatively the returns of small stocks and positively those of large stocks, with the effect of ΔB for small stocks being larger in absolute value than the effect for large stocks.

Our findings on Hypothesis 3, shown for annual returns, weaken in statistical significance for quarterly returns but strengthen for return horizons ranging from two to five years, with the regression coefficients increasing approximately linearly with horizon. Our findings weaken somewhat in the first half of the sample. They hold when measuring breadth by the Herfindahl index rather than by the fraction of investors holding a stock.

Section 5 presents our empirical findings on Hypotheses 1 and 2. When regressing breadth on range, kurtosis and the interaction of kurtosis with size, we find that range is negatively related to breadth and that kurtosis is negatively related to breadth for small stocks and positively for large stocks. When regressing future returns on range and kurtosis, we find that range predicts returns negatively and kurtosis predicts them positively. Moreover, the coefficient of kurtosis is significant for return horizons ranging from one quarter to five years, and increases approximately linearly with horizon, while the coefficient of range becomes insignificant at horizons of one year and longer. The effect of kurtosis on expected returns thus seems to dominate that of range at horizons of one year and longer, a finding that is consistent with the breadth-return relationship turning negative for small stocks.

Our findings on Hypotheses 1 and 2 remain almost identical when replacing range by standard deviation, as a measure of the intensity of disagreement. They also remain similar when replacing kurtosis by a ratio of the number of moderate to extreme forecasts. Range and kurtosis thus appear to be robust measures of the intensity of disagreement and the extent of polarization, respectively. Our findings also hold, somewhat more weakly, when truncating the distribution of forecasts by removing one forecast from each of the two extremes. They also hold when adding the skewness of the distribution alongside range and kurtosis in the regressions.

A final robustness test, in Section 6, is to perform the analysis at the level of investment styles rather than individual investors. Our model can be applied to styles by assuming that each style is held by a disjoint group of investors. The findings in Section 4 become stronger at the level of styles. This rules out alternative explanations for the relationship between breadth of ownership and expected returns that apply to the level of investors but not to styles, such as monitoring or rent extraction by large shareholders, and asymmetric information by corporate insiders.

Our paper is most closely related to CHS. CHS describe the distribution of investor beliefs by the intensity of disagreement, and find that breadth is positively related to expected returns. We find instead a negative relationship for small stocks and a weaker positive one for large stocks, and explain this finding by describing investor beliefs by the intensity of disagreement and the extent of polarization. Our finding that breadth is negatively related to expected return for small stocks is consistent with CHS. This is because CHS bunch small stocks together with mid-cap stocks, which makes the negative relationship for small stocks hard to detect.

DMS examine empirically how the intensity of disagreement relates to expected returns, thus adopting the same one-dimensional description of beliefs as CHS. Proxying investor beliefs by analyst forecasts, they find that stocks for which disagreement is large earn low returns. We show that the effects of the intensity of disagreement on expected returns weaken rapidly as horizon increases while the effects of the extent of polarization—the new dimension that we introduce—

remain strong.

Cen, Lu, and Yang (2013, CLY) show theoretically and empirically that the relationship between breadth and expected returns turns negative during periods when investor sentiment is volatile. Sentiment in their model is driven by irrational investors who trade with rational arbitrageurs. During periods when irrational investors become optimistic, breadth increases and expected return decreases. The variation in the number of optimists is central to our model as well, in the cross-section rather than the time-series.[1]

While our model is based on differences in beliefs, it is related to the alternative theory based on entry costs (Merton (1987)). Small stocks in our model are held only by the optimists, and variation in the number of optimists could arise from greater awareness about a stock due to lower entry costs. We take the distribution of investor beliefs as exogenous and do not examine how it could be affected by entry costs.

Barberis and Shleifer (2003) show that style investing affects asset prices and returns through the flows of funds across styles. Flows in their model generate return predictability in the form of momentum, reversal and lead-lag effects. Similar mechanisms are at play with rational investors in Vayanos and Woolley (2013). Our style-level findings indicate that variables associated to styles predict stock returns over horizons longer than those of momentum and lead-lag effects. Moreover, the direction of the predictability switches sign as stock size increases.

2 Theory

We derive our empirical hypotheses from a model in which investors disagree about stocks' payoffs and there are short-sale constraints. Stocks can differ in size, riskiness and the distribution of investor beliefs. Our model's main results characterize how the distribution of beliefs relates to expected stock returns and the breadth of ownership, and how these relationships change with stock size and riskiness.

2.1 Model

There are two periods 0 and 1. There are $I+1$ assets, indexed by $i = 0, 1, .., I$, which pay off in period 1. Asset 0 is riskless. We take it as the numeraire and set its price in period 0 and its payoff in period 1 to one. Assets $1, .., I$ are risky and we refer to them as stocks. Stock $i = 1, .., I$ is in

[1] Additional papers that find a non-monotone relationship between breadth and expected returns include Choi, Jin, and Yan (2012) and Cao and Wu (2022). The former paper finds that breadth is positively related to expected returns when measured based on the holdings of institutional investors, but is negatively related for retail investors. The latter paper finds an inverted U-shaped relationship between changes in breadth and expected returns.

supply of $\theta_i > 0$ shares, trades at price S_i per share in period 0, and pays dividend

$$b_i D + D_i \tag{2.1}$$

per share in period 1. The term $b_i D$ is a systematic component, equal to the product of a factor D that is common to all stocks, times stock i's sensitivity b_i to the factor. The term D_i is an idiosyncratic component specific to stock i. The factor D and the idiosyncratic components $\{D_i\}_{i=1,..,I}$ are given by

$$D = \bar{D} + \eta, \tag{2.2}$$

$$D_i = \bar{D}_i + \eta_i, \tag{2.3}$$

where $(\bar{D}, \{\bar{D}_i\}_{i=1,..,I})$ are positive constants and $(\eta, \{\eta_i\}_{i=1,..,I})$ are independent, normally distributed random variables with mean zero and variances $(\sigma^2, \{\sigma_i^2\}_{i=1,..,I})$. Setting the means of $(\eta, \{\eta_i\}_{i=1,..,I})$ to zero is without loss of generality because we can redefine $(\bar{D}, \{\bar{D}_i\}_{i=1,..,I})$. We assume that $(\bar{D}, \{\bar{D}_i\}_{i=1,..,I})$ are large enough so that equilibrium prices are always positive. We define the return of stock i as $\frac{b_i D + D_i - S_i}{S_i}$ and the stock's expected return as $\frac{b_i \bar{D} + \bar{D}_i - S_i}{S_i}$.

There is a mass one continuum of competitive investors indexed by $n \in [0, 1]$. All investors have CARA utility with risk-aversion coefficient a. Different investors hold different opinions about the dividends of each stock. We assume that investors agree on the systematic component but disagree on the idiosyncratic component. Investor n believes that the mean of the idiosyncratic component for stock i is $\bar{D}_i + \epsilon_i(n)$ instead of the true value $\bar{D}_i$. We assume that the function $n \to \epsilon_i(n)$ is measurable and refer to $\epsilon_i(n)$ as the belief of investor n for stock i. We refer to investors with $\epsilon_i(n) > 0$ as optimists for stock i and to investors with $\epsilon_i(n) < 0$ as pessimists. We denote by $x_i(n)$ the number of shares of stock i held by investor n. Investors are subject to short-sale constraints: $x_i(n)$ must be non-negative.

2.2 Equilibrium

Using CARA utility, the dividends' one-factor structure (2.1)-(2.3), and the dividends' normality, we can write the maximization problem of investor n in the mean-variance form

$$\max_{\substack{\{x_i(n)\}_{i=1,..,I} \\ x_i(n) \geq 0}} \sum_{i=1}^{I} \left(b_i \bar{D} + \bar{D}_i + \epsilon_i(n) - S_i \right) x_i(n) - \frac{a}{2} \left[\sigma^2 \left(\sum_{i=1}^{I} b_i x_i(n) \right)^2 + \sum_{i=1}^{I} \sigma_i^2 x_i(n)^2 \right].$$

The first-order condition for stock i is

$$b_i\bar{D} + \bar{D}_i + \epsilon_i(n) - S_i - b_i\lambda(n) - a\sigma_i^2 x_i(n) = 0 \quad \text{if} \quad x_i(n) > 0, \tag{2.4}$$

$$b_i\bar{D} + \bar{D}_i + \epsilon_i(n) - S_i - b_i\lambda(n) - a\sigma_i^2 x_i(n) \leq 0 \quad \text{if} \quad x_i(n) = 0, \tag{2.5}$$

where $\lambda(n) \equiv a\sigma^2 \sum_{j=1}^{I} b_j x_j(n)$. Using (2.4) and (2.5), we can write the investor's demand for stock i as

$$x_i(n) = \max\left\{\frac{b_i\bar{D} + \bar{D}_i + \epsilon_i(n) - S_i - b_i\lambda(n)}{a\sigma_i^2}, 0\right\}. \tag{2.6}$$

Demand is positive if the investor's expectation $b_i\bar{D} + \bar{D}_i + \epsilon_i(n)$ of the stock's dividend, minus a premium $b_i\lambda(n)$ for systematic risk, exceeds the stock's price S_i. The premium for systematic risk is the product of the stock's sensitivity b_i to the factor, times a factor premium $\lambda(n)$. The factor premium $\lambda(n)$ is investor-specific and equal to the product of the investor's risk aversion a, times the variance σ^2 of the factor, times the investor's portfolio sensitivity $\sum_{j=1}^{I} b_j x_j(n)$ to the factor.

Aggregating across investors and using market clearing

$$\int_0^1 x_i(n)dn = \theta_i, \tag{2.7}$$

we find that the equilibrium price S_i of stock i solves

$$\int_0^1 \max\left\{\frac{b_i\bar{D} + \bar{D}_i + \epsilon_i(n) - S_i - b_i\lambda(n)}{a\theta_i\sigma_i^2}, 0\right\} dn = 1. \tag{2.8}$$

Solving for equilibrium amounts to solving for stock prices $\{S_i\}_{i=1,..,I}$ and factor premia $\{\lambda(n)\}_{n\in[0,1]}$.

We solve for the equilibrium assuming that the factor premia $\{\lambda(n)\}_{n\in[0,1]}$ are equal across investors. Equality of the factor premia simplifies the equilibrium considerably because the holdings of a stock i by an investor n depend only on the investor's belief $\epsilon_i(n)$ for that stock and not on the beliefs for the other stocks. This rules out situations such as a stock being held mostly by its pessimists because its optimists hold larger average positions across the other stocks.

The factor premia $\{\lambda(n)\}_{n\in[0,1]}$ differ across investors if investors differ in their average optimism across stocks. This is because investors with higher average optimism hold larger average positions in stocks relative to less optimistic investors. Even if investors have the same average optimism, factor premia differ across investors if investors differ in the correlation, in the cross-section of stocks, between their optimism and that of other investors. This is because investors who are optimists for stocks for which most other investors are pessimists hold larger average positions relative to investors whose optimism correlates positively with that of other investors. In both cases (differences in average optimism or in optimism correlation) investors holding larger average positions are more exposed to the factor and have larger factor premia $\lambda(n)$. Therefore, they hold smaller positions in a given stock i relative to investors with smaller average positions but same belief for stock i.

Proposition 2.1 characterizes the equilibrium when the factor premia $\{\lambda(n)\}_{n=1,..,N}$ are equal across investors. In Appendix A, which contains the proof of Proposition 2.1 and of all subsequent theoretical results, we show that factor premia are equal across investors under either one of two sufficient conditions. The first condition is that investors can trade an additional asset $I+1$ whose payoff in period 1 is the factor D. Asset $I+1$ is in zero supply, can be traded without short-sale constraints, and can be interpreted as a futures contract on an aggregate market index. Because investors hold the same belief on the factor, trading the factor equalizes factor premia across investors. The second condition is that investors are symmetric in their beliefs across stocks in the sense that (i) for each stock i the function $n \to \epsilon_i(n)$ takes values in a finite set and (ii) for each permutation within that set there exists one stock i' that shares the same other characteristics $(\theta_i, b_i, \sigma_i)$ as stock i and is such that the function $n \to \epsilon_{i'}(n)$ is obtained from $n \to \epsilon_i(n)$ through that permutation. Symmetry ensures that investors are identical in their average optimism across stocks and in the correlation between their optimism and that of other investors.

Proposition 2.1. *Suppose that in equilibrium the factor premia $\{\lambda(n)\}_{n=1,..,N}$ are equal across investors. The price S_i of stock i is*

$$S_i = b_i\bar{D} + \bar{D}_i - a\sigma^2 b_i \sum_{j=1}^{I} b_j\theta_j + a\theta_i\sigma_i^2\phi_i, \tag{2.9}$$

where ϕ_i is the unique solution of

$$\int_0^1 \max\left\{\frac{\epsilon_i(n)}{a\theta_i\sigma_i^2} - \phi_i, 0\right\} dn = 1. \tag{2.10}$$

Holdings for stock i depend only on investor beliefs for that stock, and are given by

$$x_i(n) = \max\left\{\frac{\epsilon_i(n)}{a\sigma_i^2} - \theta_i\phi_i, 0\right\} \tag{2.11}$$

for investor n.

The equilibrium price S_i of stock i is equal to the stock's expected dividend, minus a premium $a\sigma^2 b_i \sum_{j=1}^{I} b_j\theta_j$ for systematic risk, plus a term $a\theta_i\sigma_i^2\phi_i$ that captures the joint effects of investor disagreement and idiosyncratic risk. The premium for systematic risk is the product of the stock's sensitivity b_i to the factor, times the common value of the investor-specific factor premia $\{\lambda(n)\}_{n\in[0,1]}$. The common value of $\{\lambda(n)\}_{n\in[0,1]}$ is derived by multiplying investor risk aversion a, times factor variance σ^2, times portfolio sensitivity to factor risk. Portfolio sensitivity is $\sum_{j=1}^{I} b_j\theta_j$, the same as if each investor in the mass one continuum were holding the supply of each stock.

The term $a\theta_i\sigma_i^2\phi_i$, which captures how investor disagreement and idiosyncratic risk impact the price of stock i, is central to our analysis. We determine how it depends on the distribution of investor beliefs in Section 2.3. Using $a\theta_i\sigma_i^2\phi_i$, we also determine how the distribution of beliefs impacts the breadth of ownership of stock i.

2.3 Expected Returns and Breadth of Ownership

We perform two comparative statics on the distribution of investor beliefs. Comparative static (I) concerns the Intensity of investor disagreement, which we measure by the range of the distribution of analysts' forecasts in our empirical analysis. Comparative static (P) concerns the extent to which beliefs are Polarized, which we measure by the kurtosis of the distribution of analysts' forecasts.

We describe the distribution of investor beliefs for stock i by its cumulative distribution function

$$F_i(\epsilon) \equiv \mathcal{L}\{n : \epsilon_i(n) \leq \epsilon\},$$

where $\mathcal{L}$ is the Lebesgue measure on the set $\mathbb{R}$ of real numbers. We denote the mean of that distribution by μ_i. For both comparative statics, we compare two stocks i and i' that differ in the distribution of investor beliefs but share the same mean μ_i of that distribution and the same other characteristics $(\theta_i, b_i, \bar{D}_i, \sigma_i)$. For comparative static (I), we assume that the distribution for stock i' is derived from the distribution for stock i by spreading the latter uniformly around its mean by a factor $\chi > 1$. This amounts to assuming

$$F_{i'}(\epsilon + \Delta\epsilon) - F_{i'}(\epsilon) = F_i\left(\mu_i + \frac{1}{\chi}(\epsilon + \Delta\epsilon - \mu_i)\right) - F_i\left(\mu_i + \frac{1}{\chi}(\epsilon - \mu_i)\right) \tag{2.12}$$

for all ϵ and $\Delta\epsilon > 0$. Intensity of disagreement is larger for stock i' than for stock i. For comparative static (P), we assume that the distribution for stock i' is derived from the distribution for stock i by reducing probability mass uniformly for all values by a factor $\psi \in (0, 1)$ and adding mass ψ to the mean μ_i as an atom. This amounts to assuming

$$F_{i'}(\epsilon + \Delta\epsilon) - F_{i'}(\epsilon) = \psi\left[F_i(\epsilon + \Delta\epsilon) - F_i(\epsilon)\right] \tag{2.13}$$

for all ϵ and $\Delta\epsilon > 0$ such that $\epsilon > \mu_i$ or $\epsilon + \Delta\epsilon < \mu_i$. Polarization of beliefs is smaller for stock i' than for stock i.

Lemma 2.1 determines how the range, standard deviation and kurtosis of the distribution of investor beliefs for stock i' compare to their counterparts for stock i. We assume that the distribution for stock i has finite support, so that the range is finite. This assumption also ensures finite standard deviation and kurtosis.

Lemma 2.1. *Suppose that the distribution of investor beliefs for stock i has finite support.*

- *Under comparative static (I), the range and standard deviation of the distribution of investor beliefs for stock i' are equal to χ times their counterparts for stock i. The kurtosis for stock i' is equal to that for stock i.*
- *Under comparative static (P), the range of the distribution of investor beliefs for stock i' is equal to that for stock i. The standard deviation for stock i' is equal to $\sqrt{\psi}$ times its counterpart for stock i, and the kurtosis for stock i' is equal to $\frac{1}{\psi}$ times its counterpart for stock i.*

Under comparative static (I), the distribution of investor beliefs for stock i' has higher range and standard deviation than the distribution for stock i because it is derived by spreading out the latter around its mean. Since the spreading is uniform, the kurtosis is equal across the two distributions. Under comparative static (P), the distribution for stock i' has lower standard deviation than the distribution for stock i because probability mass shifts to the mean. Since the probability of all values other than the mean decreases by the same percentage, the variance and the fourth central moment decrease by that percentage. Since kurtosis is the ratio of the fourth central moment to the square of the variance, it is higher for the stock i' distribution than for the stock i one. The range is equal across the two distributions because they have the same support.

Because comparative static (I) changes the range but not the kurtosis, while comparative static (P) changes the kurtosis but not the range, we identify the former with the range and the latter with the kurtosis in our empirical analysis. Thus, an increase in the intensity of disagreement, as when switching from stock i to stock i' under comparative static (I), corresponds to an increase in the range holding the kurtosis constant. Conversely, a decrease in the extent to which beliefs are polarized, as when switching from stock i to stock i' under comparative static (P), corresponds to an increase in the kurtosis holding the range constant. We do not use the standard deviation in our empirical analysis because it changes under both comparative statics and could thus reflect the effect of both.

Proposition 2.2 determines how the price and expected return of stock i' compare to their counterparts for stock i. We denote the supremum and the infimum of the support of the distribution of investor beliefs for stock i by $\overline{\epsilon}_i$ and $\underline{\epsilon}_i$, respectively.

Proposition 2.2. *Suppose that in equilibrium the premia $\{\lambda(n)\}_{n=1,..,N}$ are equal across investors, and that the distribution of investor beliefs for stock i has finite support.*

- *Under comparative static (I), stock i' trades at a higher price and earns lower expected return than stock i when*

$$\frac{\mu_i - \underline{\epsilon}_{i'}}{a\theta_i\sigma_i^2} > 1. \tag{2.14}$$

  When instead (2.14) is violated, price and expected return are equal across the two stocks.

- *Under comparative static (P), stock i' trades at a lower price and earns higher expected return than stock i when (2.14) holds. When instead (2.14) is violated, price and expected return are equal across the two stocks.*

Under comparative static (I), stock i' trades at a higher price and earns lower expected return than stock i, except when the short-sale constraint for the two stocks does not bind for any investor, in which case price and expected return are equal across the two stocks. Thus, when the constraint binds, an increase in the range of the distribution of investor beliefs, holding the kurtosis constant, raises stock prices and lowers stock expected returns. As in Miller (1977), Harrison and Kreps (1978), CHS, Scheinkman and Xiong (2003), Hong and Stein (2007) and Hong and Sraer (2016), when investor beliefs about a stock become more spread out, some pessimists drop out, and the stock's price rises because it is driven by the optimists to a larger extent.

Under comparative static (P), stock i' trades at a lower price and earns higher expected return than stock i, except when the short-sale constraint for the two stocks does not bind for any investor, in which case price and expected return are equal across the two stocks. Thus, when the constraint binds, an increase in the kurtosis of the distribution of investor beliefs, holding the range constant, lowers stock prices and raises stock expected returns. The intuition is that when investor beliefs about a stock become less polarized, the stock's price drops because it is driven by the optimists to a smaller extent.

Condition (2.14) ensures that the short-sale constraint for stock i' binds for some investors. To derive that condition, we note that when the short-sale constraint for stock i' does not bind for any investor, (2.10) implies

$$\int_0^1 \left(\frac{\epsilon_i}{a\theta_i\sigma_i^2} - \phi_{i'} \right) dn = 1 \Rightarrow \frac{\mu_i}{a\theta_i\sigma_i^2} - \phi_{i'} = 1 \Rightarrow \phi_{i'} = \frac{\mu_i}{a\theta_i\sigma_i^2} - 1. \tag{2.15}$$

The short-sale constraint for stock i' does not bind for any investor if it does not bind for the stock's most pessimistic investors. Substituting (2.15) into (2.11) for $\epsilon_i(n) = \underline{\epsilon}_i$, we find that the latter condition becomes

$$\frac{\underline{\epsilon}_{i'}}{a\theta_i\sigma_i^2} - \phi_i \geq 0 \Leftrightarrow \frac{\underline{\epsilon}_{i'}}{a\theta_i\sigma_i^2} - \left(\frac{\mu_i}{a\theta_i\sigma_i^2} - 1\right) \geq 0 \Leftrightarrow \frac{\mu_i - \underline{\epsilon}_{i'}}{a\theta_i\sigma_i^2} \leq 1, \tag{2.16}$$

which is the opposite to (2.14). If (2.16) holds, then the short-sale constraint does not bind for both stocks i and i' (i.e., not only for stock i') because the range of the distribution of investor beliefs for stock i' is higher than or equal to that for stock i.

The intuition for (2.14) is that the most pessimistic investors for stock i' drop out of that stock if the difference between their belief $\underline{\epsilon}_{i'}$ and the average belief μ_i is sufficiently large relative to the stock's supply θ_i and idiosyncratic variance σ_i^2. The pessimists' reward from holding the stock is the premium from bearing the stock's idiosyncratic risk. When (2.14) holds, that reward is small relative to their pessimistic belief.

Proposition 2.3 determines how the breadth of ownership of stock i' compares to its counterpart for stock i. We define breadth as the fraction of investors holding a stock. The breadth for stock i is

$$B_i \equiv \mathcal{L}\{n : x_i(n) > 0\}. \tag{2.17}$$

Proposition 2.3. *Suppose that in equilibrium the premia $\{\lambda(n)\}_{n=1,..,N}$ are equal across investors, and that the distribution of investor beliefs for stock i has finite support.*

- *Under comparative static (I), the breadth of ownership of stock i' is not larger than the breadth of stock i when (2.14) holds, and is strictly smaller when in addition $F_i(\epsilon)$ increases in $\epsilon \in [\underline{\epsilon}_i, \overline{\epsilon}_i]$. When instead (2.14) is violated, breadth is equal across the two stocks.*
- *Under comparative static (P), stock i' has smaller breadth of ownership than stock i when*

$$\phi_{i'} \geq \frac{\mu_i}{a\theta_i\sigma_i^2}, \tag{2.18}$$

(2.14) holds, and $F_i(\epsilon)$ has a density $f_i(\epsilon)$ in $[\mu_i, \overline{\epsilon}_i]$ that is positive and non-decreasing in ϵ. When instead

$$\phi_i < \frac{\mu_i}{a\theta_i\sigma_i^2}, \tag{2.19}$$

and (2.14) holds, stock i' has larger breadth than stock i. When finally (2.14) is violated, *breadth is equal across the two stocks. Moreover, there exist thresholds $\Theta_\psi^* < \Theta_1^*$ such that (2.18) holds for $\theta_i\sigma_i^2 \leq \Theta_\psi^*$ and (2.19) holds for $\theta_i\sigma_i^2 > \Theta_1^*$.*

Under comparative static (I), stock i' has smaller breadth of ownership than stock i, except when (i) the short-sale constraint for the two stocks is not binding for any investor, in which case breadth is equal across the two stocks, or (ii) the distribution of investor beliefs has gaps (zero-probability intervals) in $[\underline{\epsilon}_i, \overline{\epsilon}_i]$, in which case equality can also hold. Thus, barring the equality cases, an increase in the range of the distribution of investor beliefs, holding the kurtosis constant, lowers breadth of ownership. As in Miller (1977), Harrison and Kreps (1978), CHS, Scheinkman and Xiong (2003), Hong and Stein (2007) and Hong and Sraer (2016), when the distribution of investor beliefs about a stock becomes more spread out, some pessimists drop out and breadth decreases.

Under comparative static (P), the comparison between the breadth of ownership of stocks i and i' depends on their size. If the stocks are large (large number of shares θ_i), then the breadth of stock i' exceeds that of stock i. If instead the stocks are small (small θ_i), then the comparison reverses under the sufficient condition that the distribution of investor beliefs has a density in $[\mu_i, \overline{\epsilon}_i]$ that is positive and non-decreasing. Both comparisons require that the short-sale constraint binds for some investors, otherwise breadth is equal across the two stocks. Thus, when the constraint binds, an increase in the kurtosis of the distribution of investor beliefs, holding the range constant, has a non-monotone effect on breadth: breadth decreases for small stocks and increases for large stocks. The non-monotonicity result is central to our theory and empirical analysis.

The intuition for the non-monotonicity is as follows. An increase in the kurtosis, holding the range constant, corresponds to an increase in the fraction of investors with moderate beliefs and a corresponding decrease in the fraction of optimists and pessimists. Small stocks are held only by optimists because the pessimists' reward from bearing the stocks' idiosyncratic risk is small relative to their pessimistic belief. Since the fraction of optimists is larger for a small stock i than for a stock i' with less polarized beliefs and same other characteristics, stock i has higher breadth. By contrast, large stocks are held by both optimists and moderates. Since the fraction of pessimists is larger for a large stock i than for a stock i' with less polarized beliefs and same other characteristics, stock i has a smaller combined fraction of optimists and moderates, and thus has lower breadth.

Condition (2.19) ensures that "perfect" moderates, with beliefs right at the mean μ_i of the distribution, hold stock i. The condition follows by setting $\epsilon_i(n) = \mu_i$ in (2.11). It ensures that perfect moderates hold stock i' (as well as stock i). This is because Proposition 2.2 implies that stock i' is cheaper than stock i due to its less polarized distribution of beliefs. Condition (2.19) holds when stocks i and i' are large (large θ_i) or have high idiosyncratic variance (large σ_i^2). In both cases, the reward from bearing the stocks' idiosyncratic risk is sufficiently large to induce the moderates to hold the stocks.

Condition (2.18) conversely ensures that perfect moderates do not hold stock i'. Since Proposition 2.2 implies that stock i' is cheaper than stock i, Condition (2.18) also ensures that perfect moderates do not hold stock i. Condition (2.18) holds when stocks i and i' are small (small θ_i) or have low idiosyncratic variance (small σ_i^2). For Condition (2.18) to imply larger breadth for a small stock with more polarized beliefs, an additional condition on the distribution of investor beliefs is required. Intuitively, a stock with more polarized beliefs has a larger fraction of optimists, but because it trades at a higher price, some of the non-extreme optimists drop out. The first effect dominates, causing breadth to increase, when the distribution of investor beliefs has a density in $\epsilon \in [\mu_i, \overline{\epsilon}_i]$ that is non-decreasing in ϵ, ensuring that there are sufficient extreme optimists relative to non-extreme ones. The condition on the density can be dispensed with when stocks i and i' are so small so that they are held only by extreme optimists.

Proposition 2.4 examines how breadth depends on size. The breadth of a stock i is compared to that of a stock i' that is in larger number of shares $\theta_{i'} > \theta_i$ and has the same other characteristics $(b_i, \bar{D}_i, \sigma_i, F_i(\epsilon))$.

Proposition 2.4. *Suppose that in equilibrium the premia $\{\lambda(n)\}_{n=1,..,N}$ are equal across investors, and that the distribution of investor beliefs for stock i has finite support. The breadth of ownership of stock i' that is in larger number of shares $\theta_{i'} > \theta_i$ than stock i and has the same other characteristics $(b_i, \bar{D}_i, \sigma_i, F_i(\epsilon))$ is not smaller than the breadth of stock i when*

$$\frac{\mu_i - \underline{\epsilon}_i}{a\theta_i\sigma_i^2} > 1 \tag{2.20}$$

holds, and is strictly larger when in addition $F_i(\epsilon)$ increases in $\epsilon \in [\underline{\epsilon}_i, \overline{\epsilon}_i]$. When instead (2.20) is violated, breadth is equal across the two stocks.

Stock i' has larger breadth than stock i, except when (i) the short-sale constraint for the two stocks is not binding for any investor, in which case breadth is equal across the two stocks, or (ii) the distribution of investor beliefs has gaps in $[\underline{\epsilon}_i, \overline{\epsilon}_i]$, in which case equality can also hold. Thus, barring the equality cases, breadth is larger for larger stocks. This is because the reward from bearing idiosyncratic risk is larger for larger stocks, and thus investors with a wider set of beliefs are drawn to hold them.

Using Propositions 2.2-2.4, we derive our empirical hypotheses. Our first hypothesis follows from Proposition 2.2 and concerns the relationship that range and kurtosis have with expected returns.

Hypothesis 1 (**Investor beliefs and expected returns**). *Expected returns are predictable from the range and the kurtosis of the distribution of investor beliefs as follows:*

- *Expected returns are negatively related to range.*
- *Expected returns are positively related to kurtosis.*

Our second hypothesis follows from Proposition 2.3 and concerns the relationship that range and kurtosis have with breadth.

Hypothesis 2 (**Investor beliefs and breadth of ownership**). *Breadth of ownership is related to the range and the kurtosis of the distribution of investor beliefs as follows:*

- *Breadth is negatively related to range.*
- *Breadth is negatively related to kurtosis for small stocks. The relationship turns positive for large stocks.*

Our third hypothesis, which we term Hypothesis 2a because it relates to Hypothesis 2, concerns the relationship between stock size and breadth. The monotonicity of this relationship follows from Proposition 2.4. The cross-effect with kurtosis follows from Hypothesis 2.

Hypothesis 2a (**Stock size and breadth of ownership**). *Breadth of ownership is increasing in stock size. The effect of size on breadth is stronger for stocks with a high kurtosis of the distribution of investor beliefs.*

Our final hypothesis concerns the relationship between breadth and expected returns. Variation in the range of the distribution of investor beliefs generates a positive relationship between breadth and expected returns, as in CHS. This is because range is negatively related to breadth (Hypothesis 2) and to expected returns (Hypothesis 1). Variation in the kurtosis of the distribution of investor beliefs generates a negative relationship between breadth and expected returns for small stocks and a positive relationship for large stocks. This is because kurtosis is negatively related to breadth for small stocks and positively related to it for large stocks (Hypothesis 2) and is positively related to expected returns (Hypothesis 1). The effects of range and kurtosis reinforce each other for large stocks, generating a positive relationship between breadth and expected returns. The effects work in opposite directions for small stocks. Assuming that the effect of kurtosis dominates that of range, the relationship between breadth and expected returns is negative for small stocks.

Hypothesis 3 (Breadth of ownership and expected returns). *Expected returns are predictable from the breadth of ownership as follows:*

- *Expected returns are negatively related to breadth for small stocks.*
- *Expected returns are positively related to breadth for large stocks.*

3 Data Sources and Variables

Our sample consists of common stocks (codes 10 and 11 of CRSP) trading on NYSE, NASDAQ and AMEX between the first quarter of 1980 and the fourth quarter of 2018. The frequency of the sample is quarterly. The length and frequency of the sample are driven by the availability of the ownership data. Ownership data pertaining to investment styles are available only between the first quarter of 1997 and the fourth quarter of 2015, so our analysis of style-level ownership is limited accordingly.

3.1 Stock Returns

We source data on stock prices, stock returns including dividends, trading volume, and number of outstanding shares from CRSP. We calculate a stock's return over any given horizon by compounding the stock's monthly returns during that horizon. We measure a stock's size in any given quarter by market capitalization, which we calculate by multiplying the stock's share price at the end of the quarter times the number of outstanding shares on the same day. We define small stocks as those with size below the 30th percentile of our sample, mid-cap stocks as those with size between the 30th and the 70th percentile, and large stocks as those with size above the 70th percentile.

We construct a number of stock-level variables that we use as controls. These include idiosyncratic volatility, market beta, book-to-market ratio, momentum and turnover. We calculate market beta and idiosyncratic volatility in any given quarter using a within-quarter time-series regression of daily excess stock returns over the riskless rate on the daily market excess return. Idiosyncratic volatility is the standard deviation of the regression residuals. If more than ten observations are missing within the quarter, then we treat the market beta and idiosyncratic volatility observations as missing. We source the riskless rate and the market return from Kenneth French's website.

We source the ratio of book value of equity to market value from the Financial Ratios Suite of WRDS. WRDS calculates the book-to-market ratio on a quarterly basis and lags all observations by two months to ensure no look-ahead biases. We construct our momentum variable in any given quarter by compounding monthly returns during the nine-month period ending at the end of the quarter. We construct turnover in any given month by dividing the number of shares traded in that month by the total outstanding shares in the same month. Because reported share volume is estimated differently by NYSE/AMEX and NASDAQ (Atkins and Dyl (1997)), with the latter roughly double-counting, we divide NASDAQ share volume by two (Nagel (2005)).

3.2 Institutional Ownership

We source data on institutional ownership from Thomson Reuters (TR). That data are derived from institutional investors' 13-F filings. Institutional investors with more than \$100 million in assets are required to report their stock-level holdings to the SEC on a quarterly basis, within 45 days from the end of the quarter.

We use two different databases of TR. From the first database, TR Stock Ownership, available in WRDS, we source the number of 13-F institutional investors who hold any given stock, the total number of 13-F investors, the fraction of the stock held by all 13-F investors, and the stock's Herfindahl index H. The fraction of the stock held by all 13-F investors, which we term institutional ownership (IO), is calculated by dividing the number of shares held by all 13-F investors by the total number of outstanding shares of the stock. The Herfindahl index H is calculated by dividing the number of shares held by a given 13-F investor by the number of shares held by all 13-F investors, squaring that fraction, and summing across investors. We use H as an alternative measure of breadth of ownership in robustness tests.

The second database, Thomson Eikon, groups 13-F institutional investors into investment styles based on their portfolio characteristics and/or their business type. From that database, we source the number of investment styles that hold any given stock and the fraction of the stock held by each style. In our sample, stocks are held by 32 different styles. The 32 styles include seventeen general styles (e.g. aggressive growth, core growth, core value, deep value, index, etc) and fifteen hedge fund styles. Appendix B provides more details on the styles and the TR classification procedure.

We construct Breadth B and Herfindahl index H at the investor and at the style level. The investor-level variables are calculated as follows. Breadth for stock i and quarter t is the number of 13-F investors who hold the stock in that quarter, divided by the total number of 13-F investors in the same quarter. Herfindahl Index for stock i in quarter t is calculated by TR as described above. The style-level variables are calculated as follows. Breadth for stock i and quarter t is the number of different styles that hold the stock in that quarter. (We do not divide by the total number of styles as it is constant over time in our sample.) The Herfindahl Index for stock i and quarter t is calculated by dividing the number of shares of the stock held by any given style by the number of shares held by all styles, squaring that fraction, and summing across styles.

In some of our tests we use changes in Breadth B and Herfindahl index H, rather than levels. We calculate the change ΔB in Breadth for stock i and quarter t as the percentage change in the number of 13-F investors who hold the stock in quarter $t-1$ and in quarter t, considering only investors present in the TR database in both quarters. This calculation follows CHS. We calculate analogously the change ΔH in Herfindahl index.

3.3 Analyst Forecasts

We source data on analyst forecasts from the Detail History file of the I/B/E/S database, which is provided by TR. The data cover the period between the second quarter of 1982 and the fourth quarter of 2018. We use analyst forecasts for earnings per share (EPS) one fiscal year ahead (FY1). Following DMS, we consider a forecast as active during its publication month and all subsequent months until a revision is published by the same analyst or the earnings data are released. We use the forecast in the last month of any given quarter as the quarterly value of the forecast.

Following DMS, we standardize forecasts for any given stock and quarter by dividing them by the absolute value of the mean forecast at the end of the quarter. This allows us to express the dispersion in forecasts in relative terms: a given dispersion in dollar terms is more significant economically when EPS is low. It also helps us avoid omitted variable problems: not standardizing can confound the effects of range and kurtosis with the effect of the level of earnings, while standardizing by lagged price can confound the same effects with the effect of the earnings-to-price ratio.

We calculate the range, standard deviation, skewness and kurtosis of the standardized forecasts. We calculate the range as the maximum minus the minimum. We calculate the standard deviation, skewness and kurtosis with finite sample corrections to avoid a mechanical relationship between them and the number of analysts.[2] The finite sample corrections require a sample size of at least four (two for standard deviation, three for skewness and four for kurtosis). We thus include in our

[2]We calculate standard deviation as $\sqrt{\frac{n}{n-1}}\sqrt{m_2}$, skewness as

$$\frac{\sqrt{n(n-1)}}{n-2}\frac{m_3}{m_2^{\frac{3}{2}}}$$

(`https://www.mathworks.com/help/stats/skewness.html`) and kurtosis as

$$\frac{n-1}{(n-2)(n-3)}\left[(n+1)\frac{m_4}{m_2^2}-3(n-1)\right]+3$$

(`https://www.mathworks.com/help/stats/kurtosis.html`), where n is the sample size (number of analysts), m_2 is the sample variance, m_3 is the sample third central moment and m_4 is the sample fourth central moment. Without the finite sample corrections, standard deviation is $\sqrt{m_2}$, skewness is $\frac{m_3}{m_2^{\frac{3}{2}}}$ and kurtosis is $\frac{m_4}{m_2^2}$.

analysis only stock/quarter observations with at least four analysts. Not applying the finite sample corrections strengthens our results.

Dividing forecasts by the absolute value of the mean forecast can generate inflated standardized forecasts, and thus an inflated value of the range and standard deviation, when the mean forecast is close to zero. We mitigate the effects of inflated values by mapping observations into deciles. Following Nagel (2005), we transform range, standard deviation, skewness and kurtosis into deciles across the population of stocks in any given quarter, and normalize the units so that the smallest decile corresponds to zero and the largest to one. We perform the same transformation for most other variables in our regressions. The decile transformation removes any time trends from our regression results.

We update the values of range, standard deviation, skewness and kurtosis each quarter. We do the same for the ownership variables B, H and IO, and for the changes ΔB, ΔH and ΔIO, as well as for momentum. For size, share price, idiosyncratic volatility, market beta, book-to-market ratio and turnover, we update the values at the end of the second quarter of each year. Our results remain similar when updating the values of these variables quarterly rather than annually.

3.4 Sample Size

The full sample includes stock/quarters for which the following criteria are met. There should be data on the return, size and IO of the stock during the quarter. There should additionally be data on the breadth B of the stock during the quarter and the previous quarter, so that we can compute the change ΔB. Finally, the IO of the stock during any of the five quarters surrounding quarter t (quarters $t-2$, $t-1$, t, $t+1$ and $t+2$) should not exceed 100%. These criteria leave us with a total of 19311 stocks, 621688 stock/quarters, and an average of 4117 stocks per quarter. When using additional controls, the number of stock/quarters drops to a minimum of 520719 and the average number of stocks per quarter drops to a minimum of 3448.

We additionally report results for a more restricted sample in which we include a stock/quarter if the IO of the stock during the quarter, the two previous quarters and the two subsequent quarters is larger or equal than 30%. We impose this criterion because we measure B as the fraction of 13-F investors holding a stock, and that fraction can become an imprecise measure of how widely the stock is held across all investors (including non-13-F ones) for low values of IO. Imposing the $IO \geq 30\%$ criterion reduces our sample to 265296 stock/quarters, and an average of 1757 stocks per quarter. When using additional controls, the number of stock/quarters drops to a minimum of 245779 and the average number of stocks per quarter drops to a minimum of 1628. The excluded stock/quarters with the $IO \geq 30\%$ criterion constitute 91% of the total stock/quarters corresponding to small stocks, 59% of those corresponding to mid-cap stocks, and 24% of those corresponding to large stocks. The results for the $IO \geq 30\%$ sample are generally similar to those for the main sample.

In Section 5, where we use data on analyst forecasts, the sample is reduced to 183215 stock/quarters (147034 for $IO \geq 30\%$). This is because we exclude stock/quarters with fewer than four analysts. The excluded stock/quarters constitute 99% of the total stock/quarters corresponding to small stocks, 81% of those corresponding to mid-cap stocks, and 30% of those corresponding to large stocks. When using additional controls, the number of stock/quarters drops to a minimum of 176574 (143191 for $IO \geq 30\%$).

3.5 Descriptive Statistics

Table I presents descriptive statistics of breadth B, the change ΔB in breadth, the range of the distribution of analyst forecasts, the kurtosis of that distribution, and the number of analysts. We report statistics over the full sample and over the subsamples of stock/quarters involving small, mid-cap and large stocks. The mean of B across the full sample is 4.55%, meaning that the average stock is held by only 4.55% of 13-F investors. Consistent with Hypothesis 2a, there is a strong positive relationship between B and size. The mean of B for small stocks is 0.50%, for mid-cap stocks is 2.06% and for large stocks is 11.55%. In the subsample of stock/quarters where institutional ownership IO exceeds 30%, the mean of B rises to 8.55%, reflecting the higher fraction of large stocks in that subsample.

4 Breadth of Ownership and Expected Returns

In this section we test Hypothesis 3, which concerns the relationship between breadth B and expected returns. The tests in this section provide only indirect evidence on how investor beliefs and disagreement affect breadth and expected returns. They do not require, however, proxies of beliefs, which can be noisy.

4.1 Main Results

Tables II and III present our main tests of Hypothesis 3. In these tables we use the change ΔB in B, rather than the level. This is because B is highly autocorrelated (first-order autocorrelation is 0.997) and highly correlated with size (correlation of breadth and size deciles is 0.904). Hence, using levels may confound the effects of B on returns with the effects of size. CHS use ΔB rather than B for similar reasons. In robustness tests in Table V we report results using B rather than

Table I: **Descriptive statistics of breadth, range and kurtosis**

	Panel A: Full Sample				Panel B: $IO \geq 30\%$			
	Full	Small	Mid-cap	Large	Full	Small	Mid-cap	Large
				Breadth (B)				
mean	4.55%	0.50%	2.06%	11.55%	8.55%	1.06%	3.02%	13.24%
stdev	7.54%	0.41%	1.36%	10.43%	9.60%	0.55%	1.28%	10.77%
				ΔB				
mean	0.06%	0.01%	0.05%	0.12%	0.10%	0.02%	0.06%	0.13%
stdev	0.59%	0.20%	0.36%	0.96%	0.67%	0.19%	0.37%	0.85%
				Range				
mean	45.16%	82.32%	63.44%	38.40%	41.15%	83.27%	59.10%	35.74%
stdev	123.28%	164.95%	152.34%	110.05%	116.04%	165.87%	146.06%	104.83%
				Kurtosis				
mean	4.16	3.54	3.73	4.32	4.23	3.54	3.75	4.38
stdev	3.07	2.68	2.70	3.17	3.11	2.57	2.67	3.21
				Number of analysts				
mean	9.53	4.67	5.64	10.93	10.04	4.78	5.78	11.30
stdev	5.91	1.11	2.13	6.19	6.09	1.15	2.18	6.29
				Number of observations				
$B/\Delta B$	621688	178971	250215	192502	265296	16272	102398	146626
Range/Kurt.	183215	1694	46496	135025	147034	754	32547	113733

Note: Mean and standard deviation of breadth B, the change ΔB in B, the range of the distribution of analyst forecasts, the kurtosis of that distribution, and the number of analysts. Panel A reports statistics for the full sample, broken down by size. Panel B reports statistics for the subsample where institutional ownership IO exceeds 30%.

ΔB.

Table II presents results from pooled OLS regressions of stock returns in the year formed by quarters $t+1$ to $t+4$ on ΔB in quarter t and on its interaction with size in the same quarter. As controls we include size, IO, and quarterly dummies to control for time fixed effects. In the specifications termed "Extra controls" we additionally include share price, idiosyncratic volatility, market beta, book-to-market ratio, momentum and turnover. In the specifications termed "Contemp. ΔIO" we additionally include ΔIO in quarters $t+1$ to $t+4$ (four variables). The variable ΔIO is contemporaneous with returns. Controlling for it, as is done also in CHS, removes a relationship between ΔB and expected returns that arises if ΔB predicts changes in IO, and these changes affect returns.

As described in Section 3.3, we transform ΔB, size, IO, share price, idiosyncratic volatility, market beta, book-to-market ratio, momentum and turnover into deciles across the population of

Table II: **Returns on breadth**

	Panel A: Full Sample				Panel B: $IO \geq 30\%$			
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
ΔB	-0.078***	-0.079***	-0.195***	-0.175***	-0.077**	-0.063**	-0.206***	-0.171***
	(-3.56)	(-4.30)	(-8.24)	(-8.93)	(-1.96)	(-2.36)	(-5.29)	(-6.54)
$\Delta B \times$ size	0.124***	0.125***	0.260***	0.260***	0.101***	0.093***	0.254***	0.233***
	(5.66)	(5.31)	(10.71)	(9.99)	(2.59)	(2.74)	(6.61)	(7.05)
size	-0.205***	-0.083***	-0.343***	-0.215***	-0.123***	-0.069***	-0.240***	-0.169***
	(-7.66)	(-4.70)	(-11.00)	(-11.29)	(-3.59)	(-3.02)	(-6.62)	(-7.05)
IO	0.080***	0.076***	0.135***	0.147***	-0.051***	-0.054***	0.064***	0.073***
	(6.23)	(4.72)	(10.65)	(9.39)	(-3.41)	(-2.94)	(4.60)	(4.59)
Extra controls	No	Yes	No	Yes	No	Yes	No	Yes
Contemp. ΔIO	No	No	Yes	Yes	No	No	Yes	Yes

Note: Pooled OLS regressions of stock returns in the year formed by quarters $t+1$ to $t+4$ on the change ΔB in breadth in quarter t and on its interaction with size in the same quarter. The regressions include as controls size, IO, and quarterly dummies. The regressions under "Extra controls" additionally include share price, idiosyncratic volatility, market beta, book-to-market ratio, momentum and turnover. The regressions under "Contemp. ΔIO" additionally include ΔIO in quarters $t+1$ to $t+4$. Size, share price, idiosyncratic volatility, market beta, book-to-market ratio and turnover are measured at the end of the last June. ΔB, IO and momentum are measured at the end of quarter t. Both sets of variables are transformed into deciles across the population of stocks at the time when each variable is measured, and the units are normalized so that the smallest decile corresponds to zero and the largest to one. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter. In this and all subsequent tables, three asterisks (***) denote statistical significance at the 1% level, two asterisks (**) at the 5% level, and one asterisk (*) at 10% level.

stocks, and normalize the units so that the smallest decile corresponds to zero and the largest to one. Thus, the coefficient of ΔB measures the relationship between ΔB and return for stocks in the bottom size decile. Moreover, the sum of that coefficient and of the coefficient of the interaction with size measures the relationship between ΔB and return for stocks in the top decile. The t-statistics are calculated using robust standard errors double-clustered by stock to address autocorrelation at the stock level (Petersen (2009)) and by quarter to address contemporaneous correlation across stocks. We double-cluster by stock and quarter in all subsequent tables as well, except Table III.

Fama and MacBeth (1973) regressions for this and subsequent tables yield broadly similar conclusions regarding statistical significance, even after adjusting for autocorrelation using Newey-West standard errors. Conclusions are similar also when using non-overlapping returns to address the autocorrelation.

The findings in Table II are consistent with Hypothesis 3. Consider small stocks first. The coefficient of ΔB, which measures the effect of ΔB on expected returns for stocks in the bottom size decile, is negative. Thus, ΔB predicts the returns of small stocks negatively.

Consider large stocks next. The coefficient of the interaction with size is positive, meaning that the negative effect of ΔB on expected returns weakens as size increases. Moreover, the sum of the interaction coefficient and of the coefficient of ΔB is positive, meaning that ΔB and expected returns are positively related for stocks in the top size decile. Thus, ΔB predicts the returns of large stocks positively.

The negative effect of ΔB on expected returns is significant at the 1% level for size deciles 1-4 in the full sample with extra controls (second column in Panel A), which we take as our baseline specification. The positive effect of ΔB is significant at the 1% level for size deciles 8-10 in the baseline specification.

Turning to economic significance, the coefficient -0.079 on ΔB in the baseline specification means that within the smallest size decile, a stock in the top ΔB decile ($\Delta B = 1$) earns 7.9% lower annual expected return than a stock in the bottom ΔB decile ($\Delta B = 0$). The sum of that coefficient and that of the interaction term is 0.046, meaning that within the largest size decile, a stock in the top ΔB decile earns 4.6% higher annual expected return than a stock in the bottom ΔB decile. The negative predictive effect of ΔB for small stocks is larger than the positive effect for large stocks.

The effects remain comparable in magnitude in the subsample $IO \geq 30\%$. They become about twice as large when controlling for ΔIO. Thus, the indirect relationship between ΔB and expected returns that arises because ΔB predicts changes in IO, and these changes affect returns, is quantitatively important. Moreover, that relationship works in the opposite direction than the direct relationship arising in our model.

Table III complements Table II by presenting results from portfolio sorts. We construct nine portfolios based on a 3×3 double sort, first on size and then on ΔB. As described in Section 3.1, we define small stocks as those with size below the 30th percentile of our sample, mid-cap stocks as those with size between the 30th and the 70th percentile, and large stocks as those with size above the 70th percentile. Within each size group, we define low ΔB stocks as those with ΔB below the 20th percentile, mid ΔB stocks as those between the 20th and the 80th percentile, and high ΔB stocks as those above the 80th percentile.[3]

Panel A reports the returns of the nine portfolios. The portfolios are formed at the end of quarter t. Annual returns are measured in the subsequent year, in the year beginning one month after the end of quarter t, and in the year beginning two months after the end of quarter t. The average of the entire series of annual returns is calculated. Portfolio returns are the equally weighted

[3] Our classification procedure leaves some ΔB portfolios empty for small stocks in 22 quarters in the early part of our sample. In those quarters we use a different rule to allocate small stocks into ΔB portfolios: we define low ΔB stocks as those with $\Delta B < 0$, mid ΔB stocks as those with $\Delta B = 0$, and high ΔB stocks as those with $\Delta B > 0$.

Table III: **Portfolio double-sorts on size and changes in breadth**

Panel A: Average returns of nine size - ΔB portfolios						
	Panel A1: Full Sample			Panel A2: $IO \geq 30\%$		
	Small	Mid-cap	Large	Small	Mid-cap	Large
Low ΔB	22.03%	13.61%	12.31%	28.51%	15.63%	13.33%
	(6.71)	(5.55)	(7.20)	(8.47)	(6.53)	(7.84)
Mid ΔB	18.91%	13.57%	13.26%	19.39%	15.70%	14.11%
	(8.10)	(7.66)	(9.47)	(8.49)	(8.69)	(10.02)
High ΔB	15.98%	13.87%	13.85%	23.25%	15.39%	13.98%
	(6.53)	(7.21)	(9.19)	(7.62)	(8.32)	(9.58)

Panel B: High ΔB minus Low ΔB								
	Panel B1: Full Sample				Panel B2: $IO > 30\%$			
	Small	Mid-cap	Large	Large minus Small	Small	Mid-cap	Large	Large minus Small
Average returns	-6.05%***	0.25%	1.55%	7.60%***	-5.26%*	-0.24%	0.65%	5.90%**
	(-3.86)	(0.18)	(1.48)	(6.41)	(-1.89)	(-0.16)	(0.64)	(2.22)
CAPM alpha	-3.74%***	1.36%	1.83%*	5.57%***	-4.85%**	0.98%	1.01%	5.87%**
	(-3.23)	(1.32)	(1.65)	(5.52)	(-2.02)	(0.77)	(0.95)	(2.24)
Carhart-4 alpha	-7.17%***	-1.80%	-0.69%	6.48%***	-10.64%***	-2.25%	-1.31%	9.33%***
	(-4.63)	(-1.20)	(-0.70)	(4.58)	(-3.25)	(-1.44)	(-1.39)	(3.10)
FF-5 & UMD alpha	-7.21%***	-1.75%	0.34%	7.55%***	-11.65%***	-0.07%	-0.14%	11.51%***
	(-4.35)	(-1.06)	(0.31)	(4.58)	(-3.04)	(-0.04)	(-0.13)	(3.11)

Note: Average returns of nine portfolios formed by a 3×3 double sort, first on size and then on the change ΔB in breadth in quarter t. We define small stocks as those with size below the 30th percentile of our sample, mid-cap stocks as those with size between the 30th and the 70th percentile, and large stocks as those with size above the 70th percentile. Within each size group, we define low ΔB stocks as those with ΔB below the 20th percentile, mid ΔB stocks as those between the 20th and the 80th percentile, and high ΔB stocks as those above the 80th percentile. Panel A reports the returns of the nine portfolios. The portfolios are formed at the end of quarter t. Annual returns are measured in the subsequent year, in the year beginning one month after the end of quarter t, and in the year beginning two months after the end of quarter t. The average of the entire series of annual returns is calculated. Portfolio returns are the equally weighted averages of the returns of the stocks in the portfolio. Panel B reports average annual returns and alphas for long-short strategies that go long in the high ΔB portfolio and short in the low ΔB portfolio. Alphas are computed using the CAPM, the Carhart (1997) four-factor model, and the Fama and French (2015) five-factor model augmented by the Carhart (1997) momentum factor. The t-statistics, in parentheses, are computed using Newey-West standard errors with two lags.

averages of the returns of the stocks in the portfolio. Results for value-weighted averages are similar. Panel B reports average annual returns and alphas for long-short strategies that go long in the high ΔB portfolio and short in the low ΔB portfolio. Alphas are computed using the CAPM, the Carhart (1997) four-factor model, and the Fama and French (2015) five-factor model augmented by the Carhart (1997) momentum factor. The t-statistics are computed using Newey-West standard errors with two lags, to address the serial correlation induced by the overlap of the annual returns.

Panel A shows that for small stocks there is a clear decreasing pattern in average return when moving from the low ΔB to the high ΔB portfolio. The average annual return drops from 22.03% for low ΔB to 15.98% for high ΔB. Going long in the high ΔB portfolio and short in the low ΔB portfolio yields an average annual return of -6.05%, with t-statistic -3.86. The strategy's CAPM, four-factor and five-factor alphas are similar to its average return.

For mid-cap stocks, there is no clear pattern in average return across the ΔB portfolios. For large stocks, a pattern reappears in the full sample and is the opposite to that for small stocks, but is insignificant. The change in the long-short strategies' returns when moving from small to large stocks is significant, however. Going long in the large-stock long-short ΔB portfolio and short in the small-stock long-short ΔB portfolio (Panel B) yields an average annual return of 7.60%, with t-statistic 6.41.

Tables II and III paint a consistent picture. ΔB predicts negatively the returns of small stocks and positively those of large stocks. The effect of ΔB for small stocks is significant. It is also larger in absolute value than the effect for large stocks, which is insignificant in some specifications. The change in the effect of ΔB from small to large stocks is significant.

The results in Tables II and III differ from CHS. CHS find a relationship between breadth and expected returns that is nonlinear in size. The relationship on annual returns is negative and insignificant for size quintile 1, becomes positive and significant for size quintiles 2 and 3, and becomes weaker positive and significant for size quintiles 4 and 5. Our results differ from CHS mainly because CHS use the NYSE size breakpoints, while we use size breakpoints based on all NYSE, NASDAQ, and AMEX stocks. Because NYSE stocks are larger on average than NASDAQ and AMEX stocks, NYSE size breakpoints bunch small stocks together with mid-cap stocks and spread out large stocks more thinly. For example, CHS size quintile 1 consists of all stocks in our size deciles 1-4, and of 99.45%, 84.27% and 17.26% of the stocks in our size deciles 5, 6 and 7, respectively. Likewise, CHS size quintile 3 consists of 45.71% and 50% of the stocks in our size deciles 8 and 9, respectively. Because CHS bunch small stocks together with mid-cap stocks, a negative relationship between breadth and expected returns for small stocks is hard to detect. CHS's nonlinear effect of size on the relationship between breadth and expected returns can also be reconciled with our results. Indeed, while our regression specification in Table II can only capture a linear effect, the portfolio results in Table III are consistent with a non-linear effect. This is because the returns of high ΔB relative to low ΔB portfolios increase sharply when moving from small to mid-cap stocks, but not when moving from mid-cap to large stocks.

Table IV presents results for return horizons other than one year. The regressions are the same as in the second and fourth columns of Panel A of Table II. We evaluate returns from quarter $t+1$ to $t+k$ and consider horizons of one quarter ($k=1$), two years ($k=8$), three years ($k=12$), four years ($k=16$) and five years ($k=20$). We do not express stock returns in annualized terms, but leave them as cumulative returns. As in Table II, we compute t-statistics using robust standard errors double-clustered by stock and by quarter. Using Newey-West standard errors with $k-1$ lags, corresponding to the overlap between the periods over which returns are evaluated, yields similar standard errors.

The findings in Table IV are consistent with Hypothesis 3. As in Table II, ΔB predicts returns of small stocks negatively. Moreover, the effect weakens when moving from small to large stocks, and becomes positive for large stocks. The coefficients of ΔB and of the interaction term scale up approximately linearly with horizon, indicating a long-lived effect of breadth on expected returns. The coefficients are significant at the 1% level at all horizons, except at the one-quarter horizon when not controlling for ΔIO. Their sum is significant at the 1% level at all horizons.

4.2 Robustness

Tables V and VI present robustness tests for the results in Section 4.1. In column groups (1) and (2) of Table V the sample period is split into two sub-periods. In column groups (3) and (4) the sample is split into two sub-samples depending on whether the absolute value of changes in the Baker and Wurgler (2006) market-wide sentiment index is below or above its 90$^{\text{th}}$ percentile. This allows us to relate our analysis to CLY, who use the same variable and cutoff, and find that the relationship between breadth and expected returns is positive when the variable is below the cutoff and negative when the variable is above. We source the sentiment index from Jeffrey Wurgler's website. In column group (5) the level of B is used rather than the change ΔB. In column group (6) the one-quarter lagged level of B is used as an additional control. The regressions are the same as in the second and fourth columns of Panel A of Table II.

The results in the two sub-periods are consistent with Hypothesis 3: the coefficient of ΔB is negative, the coefficient of the interaction term is positive, and the sum of the two coefficients is positive. The coefficients of ΔB and of the interaction term, as well as their sum, are significant at the 1% level, except for the coefficient of ΔB in the first sub-period when not controlling for ΔIO, which is insignificant.

The results in the low sentiment-variation sub-sample are consistent with Hypothesis 3. The coefficients of ΔB and of the interaction term, as well as their sum, are significant at the 1% level. The positive relationship between breadth and expected return for stocks in the larger size deciles is consistent with CLY. This is because CLY limit their sample to stocks in NYSE size quintiles

Table IV: **Long-horizon return regressions**

	1Q		8Q		12Q		16Q		20Q	
ΔB	-0.009	-0.028***	-0.188***	-0.352***	-0.262***	-0.481***	-0.357***	-0.652***	-0.410***	-0.760***
	(-1.10)	(-3.33)	(-6.21)	(-10.85)	(-6.46)	(-11.55)	(-6.43)	(-10.96)	(-4.26)	(-7.63)
ΔB*size	0.021**	0.044***	0.286***	0.523***	0.377***	0.691***	0.492***	0.898***	0.573***	1.053***
	(2.14)	(4.42)	(7.30)	(12.03)	(7.29)	(12.60)	(7.15)	(11.93)	(5.02)	(8.82)
size	-0.019**	-0.040***	-0.155***	-0.423***	-0.244***	-0.693***	-0.331***	-0.875***	-0.396***	-1.107***
	(-2.34)	(-4.97)	(-4.96)	(-11.35)	(-5.00)	(-10.83)	(-4.84)	(-10.55)	(-4.17)	(-9.52)
IO	0.024***	0.038***	0.134***	0.283***	0.205***	0.428***	0.314***	0.614***	0.484***	0.880***
	(3.55)	(5.79)	(5.66)	(11.43)	(5.93)	(12.10)	(7.06)	(13.31)	(7.07)	(11.89)
Extra controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Contemp. ΔIO	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes

Note: Pooled OLS regressions of stock returns from end of quarter t to end of quarter $t+k$ with $k = 1, 8, 12, 16, 20$, on the change ΔB in Breadth in quarter t and the interaction of ΔB with size in the same quarter. The regressions additionally include size, IO, quarterly dummies and the extra controls in Table II. The regressions are run with or without ΔIO, as indicated. Stock returns are not expressed in annualized terms, but are left as cumulative returns. Variables are transformed into deciles and normalized, as described in Table II. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter.

Table V: **Robustness tests: Sub-periods, sentiment, level of breadth, and lagged breadth**

	(1) First part of sample (2Q-1980 to 4Q-1997)		(2) Second part of sample (1Q-1998 to 4Q-2017)		(3) $\|\Delta$ SENT$\|$ below 90^{th} percentile		(4) $\|\Delta$ SENT$\|$ above 90^{th} percentile		(5) Level of B		(6) Lagged B as control	
ΔB	-0.016	-0.088***	-0.128***	-0.246***	-0.087**	-0.183***	0.026	-0.065**			-0.055***	-0.156***
	(-1.02)	(-5.45)	(-4.03)	(-7.35)	(-4.42)	(-8.70)	(0.69)	(-2.17)			(-3.13)	(-8.39)
ΔB*size	0.055***	0.157***	0.184***	0.349***	0.140***	0.277***	-0.043	0.078**			0.101***	0.242***
	(2.61)	(7.02)	(4.38)	(7.63)	(5.54)	(9.88)	(-0.96)	(2.46)			(4.48)	(9.76)
B									0.116***	-0.077*		
									(2.63)	(-1.91)		
B*size									0.154***	0.360***		
									(4.52)	(10.11)		
Extra controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Contemp. ΔIO	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes

Note: In column groups (1) and (2) the sample period is split into two sub-periods. In column groups (3) and (4) the sample is split into two sub-subsamples depending on whether the absolute value of changes in sentiment is above or below its 90^{th} percentile. In column group (3) the level of B is used rather than the change ΔB. In column group (6) the one-quarter lagged level of B is used as an additional control. All columns present results from pooled OLS regressions of stock returns from end of quarter t to end of quarter $t+4$ on B or ΔB in quarter t and the interaction between B or ΔB with size in the same quarter. The regressions additionally include size, IO, quarterly dummies and the extra controls in Table II. The regressions are run with or without ΔIO, as indicated. Variables are transformed into deciles and normalized, as described in Table II. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter.

2-5. In the high sentiment-variation sub-sample, the coefficients of ΔB and of the interaction term change sign but are insignificant. The sum of the two coefficients changes sign as well, implying a negative relationship between breadth and expected return for the larger size deciles, consistent with CLY. It is insignificant as well, however. Controlling for ΔIO reverses all signs, rendering them consistent with Hypothesis 3. Moreover, the coefficients of ΔB and of the interaction term become significant at the 5% level. Splitting the sample based on whether the absolute value of changes in sentiment is below or above its median (rather than the 90^{th} percentile) yields results consistent with Hypothesis 3 in each sub-sample.

The results using the level of B rather than the change ΔB are consistent with Hypothesis 3 when controlling for ΔIO. When not controlling for ΔIO, the coefficient of the interaction term is positive, consistent with Hypothesis 3, but the coefficient of B is positive, at odds with Hypothesis 3. The results using the one-quarter lagged level of B as an additional control are consistent with Hypothesis 3. Unreported results using book equity rather than market capitalization to measure size are also consistent with Hypothesis 3.

Table VI presents results using the Herfindahl index H instead of B as a measure of breadth of ownership. Column group (1) presents results from the main regressions with annual returns. In column groups (2) and (3), the sample period is split into two sub-periods. In column group (4), the level of H is used rather than the change ΔH. The regressions in each column group are analogous to those in the second and fourth columns of Panel A of Table II.

Since B and H are negatively related, Hypothesis 3 implies that the coefficient of ΔH or H should be positive, the coefficient of the interaction with size should be negative, and the sum of the two coefficients should be negative. These predictions are borne out in the main regressions with annual returns and in the second sub-period. In the first sub-period and in the regressions with the level of H rather than the change ΔH, the signs are consistent with Hypothesis 3, but some of the coefficients are insignificant.

5 Analyst Forecasts

In this section we test Hypotheses 1 and 2, which concern the relationship between expected return and breadth on one hand and the distribution of investor beliefs on the other. We proxy investor beliefs by the forecasts made by financial analysts.

Table VI: **Robustness tests: Herfindahl index**

	(1): ΔH		(2): ΔH - First part of sample (2Q-1980 to 4Q-1997)		(3): ΔH - Second part of sample (1Q-1998 to 4Q-2017)		(4): Level of H	
ΔH	0.030***	0.047***	0.001	0.010	0.040***	0.062***		
	(3.13)	(4.66)	(0.09)	(1.00)	(3.09)	(4.53)		
ΔH*size	-0.054***	-0.070***	-0.022	-0.029*	-0.055**	-0.075***		
	(-3.51)	(-4.38)	(-1.27)	(-1.71)	(-2.45)	(-3.13)		
H							0.037	0.192***
							(1.48)	(7.24)
H*size							-0.175***	-0.354***
							(-5.18)	(-9.63)
Extra controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Contemp. ΔIO	No	Yes	No	Yes	No	Yes	No	Yes

Note: Column group (1) presents results from the main regressions with annual returns, with the change ΔH in the Herfindahl index rather than the change ΔB in breadth. In column groups (2) and (3), the sample period is split into two sub-periods. In column group (4), the level of H is used rather than the change ΔH. All columns present results from pooled OLS regressions of stock returns from end of quarter t to end of quarter $t+1$ or $t+4$ on H or ΔH in quarter t and the interaction between H or ΔH and size in the same quarter. The regressions additionally include size, IO, quarterly dummies and the extra controls in Table II. The regressions are run with or without ΔIO, as indicated. Variables are transformed into deciles and normalized, as described in Table II. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter.

5.1 Analyst Forecasts and Expected Returns

Table VII presents results from pooled OLS regressions of stock returns in quarter $t+1$ and in the $k = 1, .., 5$ years formed by quarters t to $t+4k$ on the range and kurtosis of the distribution of analyst forecasts in quarter t. We include as additional independent variables size, quarterly dummies, IO, share price, idiosyncratic volatility, market beta, book-to-market ratio, momentum and turnover. We transform range, kurtosis, size, IO, share price, idiosyncratic volatility, market beta, book-to-market ratio, momentum and turnover into deciles across the population of stocks, and normalize the units so that the smallest decile corresponds to zero and the largest to one. We report results for the full sample of stocks and for two size sub-samples constructed using the median of the NYSE size distribution as cut-off.

The findings in Table VII are consistent with Hypothesis 1, except for the significance of the coefficient of range at horizons of one year and longer. According to Hypothesis 1, the coefficient of range should be negative and the coefficient of kurtosis should be positive. The coefficient of range is indeed negative, is significant at the 5% level at the one-quarter horizon, but becomes insignificant at horizons of one year and longer. The coefficient of kurtosis is positive, scales approximately linearly with horizon, and is significant at the 5% level at all horizons.

Turning to economic significance, the coefficient -0.014 on range at the one-quarter horizon

Table VII: **Returns on range and kurtosis of analyst forecasts**

	1Q	4Q	8Q	12Q	16Q	20Q
range	-0.014**	-0.012	-0.022	-0.036	-0.050	-0.047
	(-1.98)	(-0.61)	(-0.70)	(-0.88)	(-0.98)	(-0.70)
kurtosis	0.005**	0.010**	0.018**	0.026**	0.034*	0.069**
	(2.24)	(2.20)	(1.99)	(2.00)	(1.92)	(2.41)
Extra controls	Yes	Yes	Yes	Yes	Yes	Yes

Note: Pooled OLS regressions of stock returns in quarter $t+1$ and in the $k = 1, .., 5$ years formed by quarters t to $t + 4k$ on the range and the kurtosis of the distribution of analyst forecasts in quarter t. The regressions include as additional independent variables size, quarterly dummies, IO, share price, idiosyncratic volatility, market beta, book-to-market ratio, momentum and turnover. Size, share price, idiosyncratic volatility, market beta, book-to-market ratio and turnover are measured at the end of the last June. Range, kurtosis, IO and momentum are measured at the end of quarter t. Variables are transformed into deciles and normalized, as described in Table II. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter.

means that the expected return difference between lowest and highest range decile is 1.4% (0.014) per quarter. Likewise, the coefficient 0.005 on kurtosis means that the expected return difference between highest and lowest kurtosis decile is 0.5% (0.005) per quarter.

Since the coefficient of range loses its significance at horizons of one year and longer, while the coefficient of kurtosis remains significant, the effect of kurtosis on expected returns appears longer-lived than that of range. This finding is consistent with the breadth-return relationship turning negative for small stocks, especially at horizons of one year and longer (Tables II and IV). Indeed, a negative relationship between breadth and expected returns arises when the effect of kurtosis dominates that of range.

The effect of range on expected returns in Table VII is weaker than in DMS. DMS find that expected returns of stocks in the lowest disagreement quintile exceed those in the highest quintile by 0.79% per month. We find a difference between lowest and highest range decile of 1.4% (0.014) per quarter. The implied difference between lowest and highest range quintile is 1.26% (=1.4% $\times$ 0.9) per quarter, or 0.42% (=1.26%/3) per month, which is about one-half of the effect in DMS.

Our results differ from DMS because they forecast monthly returns while we forecast returns at horizons from one quarter to five years, and because the effect of range on expected returns appears to be short lived. Unreported regressions as in Table VII for returns at the one-month horizon yield similar results as in DMS. The difference in expected returns between lowest and highest range decile is 1% per month, and is significant at the 1% level. The implied difference between lowest and highest range quintile is 0.9% (=1% $\times$ 0.9), which is almost the same as the effect in DMS. An additional reason why our results differ from DMS is that we restrict our sample to stock/quarters with four or more analysts, so that we can calculate the kurtosis, while DMS

Table VIII: **Breadth on range and kurtosis of analyst forecasts**

	(1): Full sample	(2): $IO \geq 30\%$
range	-0.024***	-0.022***
	(-6.28)	(-7.95)
kurtosis	-0.007	-0.006*
	(-1.64)	(-1.74)
kurtosis*size	0.016***	0.016***
	(2.99)	(3.79)
size	0.727***	0.763***
	(82.17)	(114.38)

Note: Contemporaneous pooled OLS regressions of breadth B on the range of the distribution of analyst forecasts, the kurtosis of that distribution, size, IO, the interaction of kurtosis with size, and quarterly dummies. Size is measured at the end of the last June. Range, kurtosis and IO are measured at the end of quarter t. Variables are transformed into deciles and normalized, as described in Table II. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter.

allow for two or more analysts. This excludes primarily stock/quarters involving small stocks, for which DMS find stronger effects of range.

5.2 Analyst Forecasts and Breadth of Ownership

Table VIII presents results from contemporaneous pooled OLS regressions of breadth B on the range of the distribution of analyst forecasts, the kurtosis of that distribution, size, IO, the interaction of kurtosis with size, and quarterly dummies. We transform range, kurtosis, size, and IO into deciles across the population of stocks, and normalize the units so that the smallest decile corresponds to zero and the largest to one.

The findings in Table VIII are consistent with Hypothesis 2. According to Hypothesis 2, the coefficient of range should be negative, the coefficient of kurtosis should be negative because it measures the effect of kurtosis on B for stocks in the bottom size decile, and the sum of the interaction coefficient and the coefficient of kurtosis should be positive because it measures the effect of kurtosis on B for stocks in the top size decile. The coefficient of range is indeed negative, and is significant at the 1% level. The coefficient of kurtosis is negative, is significant at the 10% level for the subsample $IO \geq 30\%$, and is at the threhold of 10% significance for the full sample. The sum of that coefficient and of the coefficient of the interaction term is positive, and is significant at the 1% level.

The findings in Table VIII are consistent with Hypothesis 2a as well. The positive relationship between B and size is stronger for high kurtosis stocks because the coefficient of the interaction term is positive. That coefficient is significant at the 1% level.

When augmenting Table VIII with an interaction term between range and size, we find that the coefficient of range is positive and significant and the coefficient of the interaction term is negative and significant (with their sum being negative and significant, consistent with the negative sign of range in Table VIII). The positive coefficient of range is at odds with our model because range in our model is negatively related to B for any size group. A positive coefficient could arise because stocks for which disagreement is high attract more attention in the press, triggering greater awareness by investors. The entry costs in Merton (1987) could thus be lower for those stocks.

5.3 Robustness

Table IX presents robustness tests for the results in Sections 5.1 and 5.2. Panel A concerns expected returns. The regressions are the same as in Table VII for quarterly returns. Panel B concerns breadth B. The regressions are the same as in Table VIII. In column (1) of each panel, range is replaced by standard deviation. In column (2), range and kurtosis are computed for a truncated distribution of forecasts derived by dropping one forecast from each of the two extremes of the distribution. This requires restricting the sample to stock/quarters with at least six analysts. In column (3), kurtosis is replaced by a ratio of the number of moderate to extreme forecasts. The numerator of that ratio is the number of forecasts that lie in an interval centered at the average of the minimum and the maximum forecast, with length on each side equal to 25% of the range. The denominator is the number of remaining forecasts.[4] We term the ratio polar25 because it measures the extent to which beliefs are Polarized with a 25% cutoff. A decrease in polarization corresponds to an increase in both kurtosis and polar25. In column (4), the skewness of the distribution of forecasts is included as an additional variable.

Replacing range by standard deviation yields almost identical results to those in Sections 5.1 and 5.2. Thus, range and standard deviation seem to be almost equivalent in measuring the intensity of disagreement. Truncating the distribution of forecasts renders the effect of range on expected returns insignificant, but preserves the significance of the effect of range on B and of the effects of kurtosis on expected returns and B. Truncation can reduce significance because the sample becomes restricted to stock/quarters with at least six rather than four analysts.

Replacing kurtosis by polar25 yields almost identical results to those in Section 5.1 for expected returns and stronger results than those in Section 5.2 for B. Thus, kurtosis and polar25 seem to be similar measures of (the lack of) belief polarization. Unreported results using cutoffs for polar

[4]The numerator of polar is the number of forecasts that lie within the interval $(\underline{\epsilon} + 0.25(\overline{\epsilon} - \underline{\epsilon}), \overline{\epsilon} - 0.25(\overline{\epsilon} - \underline{\epsilon}))$, where $\underline{\epsilon}$ is the minimum forecast and $\overline{\epsilon}$ is the maximum forecast. The denominator of polar is the number of forecasts that lie within the union of the intervals $[\underline{\epsilon}, \underline{\epsilon} + 0.25(\overline{\epsilon} - \underline{\epsilon})]$ and $[\overline{\epsilon} - 0.25(\overline{\epsilon} - \underline{\epsilon}), \overline{\epsilon}]$.

Table IX: **Robustness tests: Standard deviation, truncation, polar, skewness**

	Panel A: Returns - 1Q			
	(1)	(2)	(3)	(4)
range		-0.007	-0.014*	-0.014**
		(-1.16)	(-1.90)	(-2.00)
st.dev.	-0.015**			
	(-2.07)			
kurtosis	0.004**	0.005**		0.005**
	(1.97)	(1.97)		(2.28)
polar25			0.005*	
			(1.89)	
skewness				0.006*
				(1.75)
Extra controls	Yes	Yes	Yes	Yes

	Panel B: Breadth			
	(1)	(2)	(3)	(4)
range		-0.021***	-0.024***	-0.024***
		(-6.58)	(-6.46)	(-6.27)
st.dev.	-0.031***			
	(-8.01)			
kurtosis	-0.006	-0.033***		-0.007
	(-1.38)	(-3.77)		(-1.64)
kurtosis*size	0.014**	0.053***		0.016***
	(2.56)	(5.26)		(2.98)
polar25			-0.038***	
			(-6.23)	
polar25*size			0.068***	
			(8.76)	
skewness				-0.000
				(-0.44)

Note: In column (1) of each panel, range is replaced by standard deviation. In column (2), range and kurtosis are computed for a truncated distribution of forecasts derived by dropping one forecast from each of the two extremes of the distribution. In column (3), kurtosis is replaced a ratio of moderate to extreme forecasts. The numerator of that ratio is the number of forecasts that lie in an interval centered at the average of the minimum and the maximum forecast, with length on each side equal to 25% of the range. The denominator is the number of remaining forecasts. In column (4), the skewness of the distribution of forecasts is included as an additional variable. All columns in Panel A present results from pooled OLS regressions of stock returns in quarter $t+1$ on the range and the kurtosis of the distribution of analyst forecasts in quarter t, or on the alternative variables to range and kurtosis described above. The regressions additionally include size, quarterly dummies, and the extra controls in Table VII. All columns in Panel B present results from pooled OLS regressions of breadth B on the range of the distribution of analyst forecasts, the kurtosis of that distribution, and the interaction of kurtosis with size, or on the alternative variables to range and kurtosis described above. The regressions additionally include size, IO, and quarterly dummies. Variables are transformed into deciles and normalized, as described in Table II. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter.

ranging from 15% to 35% render the effects of polar on expected returns insignificant for some cutoffs. The effects are significant for all cutoffs, however, when annual instead of quarterly returns are used. The effects of polar on B remain significant for all cutoffs.

Adding skewness to the regression yields almost identical results to those in Sections 5.1 and 5.2 for range and kurtosis. Thus, skewness seems to capture an economically different aspect of the distribution of investor beliefs. Skewness has an insignificant effect of breadth but its effect on expected returns is significant at the 10% level.

In unreported regressions we do not use the decile transformation for range and kurtosis, but instead express them in logs to reduce the importance of extreme values. The results are similar to those in Sections 5.1 and 5.2.

6 Ownership at the Style Level

In this section we extend the analysis in Section 4 to the level of investment styles. Investors can adopt different styles, such as value and growth, because of different preferences or beliefs. Assuming that different styles are adopted by disjoint groups of investors, we can map each style to an investor group and interpret the investors in our model as styles. With that interpretation, we can test the model using measures of ownership computed at the style rather than the investor level. The style-level analysis can be viewed as an additional robustness test. It can also help rule out alternative explanations of our findings on the relationship between breadth of ownership and expected returns that apply to the level of individual investors but not to aggregate styles. Examples are explanations based on monitoring or rent extraction by large shareholders (e.g., Admati, Pfleiderer, and Zechner (1994), Burkart, Gromb, and Panunzi (1997), Bolton and Von Thadden (1998)) and on asymmetric information by corporate insiders (e.g., Kyle (1985)).

Breadth is correlated at the investor and at the style level, with a correlation of 0.50. This correlation is driven partly by size, but remains important even within size groups. The correlation within the groups of small, mid-cap and large stocks is 0.64, 0.60 and 0.28, respectively. Given the positive correlation between breadth at the investor and at the style level, we expect our findings to extend to styles.

Table X presents descriptive statistics of B and ΔB at the style level. We denote B at the style level by B_{style}. Consistent with Hypothesis 2a and the findings in Table I, there is a positive relationship between B_{style} and size. The mean of B_{style} for small stocks is 5.73, meaning that the average small stock is held by 5.73 out of the 29 styles in our data. The mean of B_{style} rises to 9.28 for mid-cap stocks and to 11.90 for large stocks.

Table X: **Descriptive statistics of style-level breadth**

	Panel A: Full Sample				Panel B: $IO \geq 30\%$			
	Full	Small	Mid-cap	Large	Full	Small	Mid-cap	Large
	Breadth of Styles (B-styles)							
mean	9.18	5.73	9.28	11.90	11.17	8.35	10.48	12.18
stdev	3.69	2.82	2.96	2.64	2.62	2.33	2.25	2.44
	ΔB-style							
mean	0.08	0.09	0.10	0.06	0.06	0.08	0.07	0.05
stdev	0.97	0.92	1.00	0.97	0.95	0.84	0.98	0.95
	Number of Observations							
Obs.	261596	70444	105434	85718	150210	11215	63809	75186

Note: Mean and standard deviation of style-level breadth B_{style} and of its first difference ΔB_{style}. Panel A reports statistics for the full sample, broken down by stock size. Panel B reports statistics for the subsample where institutional ownership IO exceeds 30%. The statistics are based on pooled cross-sectional and time-series samples.

Table XI presents results using style-level measures of ownership. Column group (1) presents results from the main regressions with annual returns. In column groups (2) and (3), returns are quarterly and bi-annual, respectively. In column group (4), the level of B_{style} is used rather than the first difference ΔB_{style}. In column group (5), the first difference ΔH_{style} of the style-level Herfindahl index is used instead of ΔB_{style}. The regressions in each column group are analogous to those in the second and fourth columns of Panel A of Table II. We do not consider sub-periods because ownership data pertaining to investment styles are available only from 1997 to 2015.

Results using style-level measures of ownership provide strong support for Hypothesis 3. In all regressions, the coefficients of B_{style}, ΔB_{style} or ΔH_{style}, and of the interaction term with size, have signs consistent with Hypothesis 3. These coefficients are significant except for two cases, and their sum has sign consistent with Hypothesis 3 except for one case.

7 Conclusion

We study the relationship between the distribution of investor beliefs, the breadth of ownership and expected returns. The distribution of beliefs in our model is described by two dimensions: the intensity of disagreement, as measured by the difference between the beliefs of optimists and pessimists, and the extent to which beliefs are polarized, as measured by the number of optimists and pessimists relative to moderates. We map these dimensions to the range and the kurtosis of the distribution of beliefs. When the effect of kurtosis dominates that of range, our two-dimensional description of beliefs generates the empirical relationship between breadth and expected returns:

Table XI: **Returns on style-level measures of ownership**

	(1): ΔB-style - 4Q		(2): ΔB-style - 1Q		(3): ΔB-style - 8Q		(4): Level of B-style		(5): ΔH-style	
ΔB-style	-0.060***	-0.090***	-0.010	-0.017**	-0.123***	-0.174***				
	(-3.73)	(-5.14)	(-1.35)	(-2.39)	(-5.63)	(-7.45)				
ΔB-style*size	0.081***	0.113***	0.021*	0.026**	0.164***	0.224***				
	(3.08)	(3.98)	(1.83)	(2.30)	(5.88)	(7.28)				
B-style							-0.040	-0.226***		
							(-0.86)	(-5.18)		
B-style*size							0.137**	0.339***		
							(2.44)	(5.95)		
ΔH-style									0.035***	0.046***
									(3.09)	(4.03)
ΔH-style*size									-0.043**	-0.044**
									(-2.30)	(-2.41)
Extra controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Contemp. ΔIO	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes

Note: Column group (1) presents results from the main regressions with annual returns, with the first difference ΔB_{style} of style-level breadth rather than ΔB. In column groups (2) and (3), returns are quarterly and bi-annual, respectively. In column group (4), the level of B_{style} is used rather than the first difference ΔB_{style}. In column group (5), the first difference ΔH_{style} of the style-level Herfindahl index is used instead of ΔB_{style}. All columns present results from pooled OLS regressions of stock returns from end of quarter t to end of quarter $t+1$ or $t+4$ on B_{style}, ΔB_{style} or ΔH_{style} in quarter t and the interaction between B_{style}, ΔB_{style} or ΔH_{style} with stock size in the same quarter. The regressions additionally include size, IO, quarterly dummies and the extra controls in Table II. The regressions are run with or without ΔIO, as indicated. Variables are transformed into deciles and normalized, as described in Table II. The t-statistics, in parentheses, are computed using robust standard errors double-clustered by stock and by quarter.

positive for large stocks and negative for small stocks. That relationship cannot arise when beliefs are described only by the intensity of disagreement, as is common in the literature.

Besides testing for the relationship between breadth and expected returns, we test for the relationship that each of the two dimensions of beliefs should have with each of breadth and expected returns. The size-dependent breadth-return relationship in our model arises because stocks for which beliefs are more polarized should (i) earn lower expected returns and (ii) have a broader investor base if they are small and a narrower base if they are large. Proxying investor beliefs by analyst forecasts, we find empirical support for both predictions. We also find that the effect of kurtosis on expected returns dominates that of range at horizons of one year and longer. That finding is consistent with the empirical breadth-return relationship turning negative for small stocks.

Our finding that range has a short-lived effect on expected returns while kurtosis has a significantly more persistent effect is intriguing. A difference between the effects of range and kurtosis might arise if polarization is a more persistent characteristic of beliefs than the intensity of disagreement. Beliefs in our model are exogenous and our model is static. Modelling the dynamics of beliefs and relating them to the different effects of range and kurtosis that we find seems an interesting direction for future research.

References

Admati, Anat, Paul Pfleiderer, and Josef Zechner, 1994, Large shareholder activism, risk sharing, and financial market equilibrium, *Journal of Political Economy* 102, 1097–1130.

Atkins, Allen, and Edward Dyl, 1997, Transactions costs and holding periods for common stocks, *Journal of Finance* 52, 309–25.

Baker, Malcolm, and Jeffrey Wurgler, 2006, Investor sentiment and the cross-section of stock returns, *Journal of Finance* 61, 1645–1680.

Barberis, Nicholas, and Andrei Shleifer, 2003, Style investing, *Journal of Financial Economics* 68, 261–199.

Bolton, Patrick, and Ernst-Ludwig Von Thadden, 1998, Blocks, liquidity, and corporate control, *Journal of Finance* 53, 1–25.

Burkart, Mike, Denis Gromb, and Fausto Panunzi, 1997, Large shareholders, monitoring, and the value of the firm, *The Quarterly Journal of Economics* 112, 693–728.

Cao, Zhiqi, and Wenfeng Wu, 2022, Ownership breadth: Investor recognition or short-sale constraints?, *Finance Research Letters* 47, 102847.

Carhart, Mark, 1997, On persistence in mutual fund performance, *Journal of Finance* 52, 57–82.

Cen, Ling, Hai Lu, and Liyan Yang, 2013, Investor sentiment, disagreement, and the breadth–return relationship, *Management Science* 59, 1076–1091.

Chen, Joseph, Harrison Hong, and Jeremy Stein, 2002, Breadth of ownership and stock returns, *Journal of Financial Economics* 66, 171–205.

Choi, James, Li Jin, and Hongjun Yan, 2012, What does stock ownership breadth measure?, *Review of Finance* 17, 1239–1278.

Diether, Karl, Christopher Malloy, and Anna Scherbina, 2002, Differences of opinion and the cross section of stock returns, *Journal of Finance* 57, 2113–2141.

Fama, Eugene, and Kenneth French, 2015, A five-factor asset pricing model, *Journal of Financial Economics* 116, 1–22.

Fama, Eugene, and James MacBeth, 1973, Risk, return, and equilibrium: Empirical tests, *Journal of Political Economy* 81, 607–636.

Harrison, Michael, and David Kreps, 1978, Speculative investor behavior in a stock market with heterogeneous expectations, *Quarterly Journal of Economics* 92, 323–336.

Hong, Harrison, and Marcin Kacperczyk, 2009, The price of sin: The effects of social norms on markets, *Journal of Financial Economics* 93, 15–36.

Hong, Harrison, Jose Scheinkman, and Wei Xiong, 2006, Asset float and speculative bubbles, *Journal of Finance* 61, 1073–1117.

Hong, Harrison, and David Sraer, 2016, Speculative betas, *Journal of Finance* 71, 2095–2144.

Hong, Harrison, and Jeremy Stein, 2007, Disagreement and the stock market, *Journal of Economic Perspectives* 21, 109–128.

Kyle, Albert, 1985, Continuous auctions and insider trading, *Econometrica* 53, 1315–35.

Lou, Dong, 2014, Attracting investor attention through advertising, *Review of Financial Studies* 27, 1797–1829.

Merton, Robert, 1987, A simple model of capital market equilibrium with incomplete information, *Journal of Finance* 42, 483–510.

Miller, Edward, 1977, Risk, uncertainty, and divergence of opinion, *Journal of Finance* 32, 1151–1168.

Nagel, Stefan, 2005, Short sales, institutional investors and the cross-section of stock returns, *Journal of Financial Economics* 78, 277–309.

Petersen, Mitchell, 2009, Estimating standard errors in finance panel data sets: Comparing approaches, *Review of Financial Studies* 22, 435–480.

Scheinkman, Jose, and Wei Xiong, 2003, Overconfidence and speculative bubbles, *Journal of Political Economy* 111, 1183–1219.

Vayanos, Dimitri, and Paul Woolley, 2013, An institutional theory of momentum and reversal, *Review of Financial Studies* 26, 1087–1145.

Appendix

A Proofs

Proof of Proposition 2.1. Suppose $\lambda(n) = \lambda$ for all n. Setting

$$S_i \equiv b_i \bar{D} + \bar{D}_i - b_i \lambda + a\theta_i \sigma_i^2 \phi_i, \tag{A.1}$$

we can write (2.8) as (2.10). Equation (2.10) has a unique solution because when the integrand is positive, it is decreasing in ϕ_i. Integrating $\lambda(n) = a\sigma^2 \sum_{j=1}^{I} b_j x_j(n)$ over n and using (2.7) and $\lambda(n) = \lambda$ for all n, we find

$$\lambda = a\sigma^2 \sum_{j=1}^{I} b_j \theta_j. \tag{A.2}$$

Substituting (A.2) into (A.1), we find (2.9). Substituting (A.1) into (2.6) and using $\lambda(n) = \lambda$ for all n, we find (2.11). □

Proposition A.1 derives conditions for the factor premia $\{\lambda(n)\}_{n\in[0,1]}$ to be equal across investors

Proposition A.1. *The factor premia $\{\lambda(n)\}_{n\in[0,1]}$ are equal across investors under either one of the following sufficient conditions:*

(A) *In addition to trading assets $i = 0, 1, .., I$, investors can trade an asset $I + 1$ that is in zero supply and pays dividend D per share in period 1. Agents can trade asset $I + 1$ without short-sale constraints.*

(B) *For each stock i, the following conditions hold:*

  (i) *The function $n \to \epsilon_i(n)$ takes values in a finite set $Z_i = \{z_{i1}, .., z_{ik_i}\}$.*

  (ii) *For each permutation P of the set $\{1, .., k_i\}$, there exists one stock i' such that $(\theta_i, b_i, \sigma_i) = (\theta_{i'}, b_{i'}, \sigma_{i'})$ and $\{n \in [0, 1] : \epsilon_i(n) = z_{ik}\} = \{n \in [0, 1] : \epsilon_{i'}(n) = z_{iP(k)}\}$.*

Proof of Proposition A.1. Under Condition (A), we can write the maximization problem of investor n as

$$\max_{\substack{\{x_i(n)\}_{i=1,..,I},x(n)\\ x_i(n)\geq 0}} \sum_{i=1}^{I}\left(b_i\bar{D}+\bar{D}_i+\epsilon_i(n)-S_i\right)x_i(n)+(\bar{D}-S)x(n)$$
$$-\frac{a}{2}\left[\sigma^2\left(\sum_{i=1}^{I}b_ix_i(n)+x(n)\right)^2+\sum_{i=1}^{I}\sigma_i^2x_i(n)^2\right],$$

where S denotes the price of asset $I+1$ and $x(n)$ denotes the number of shares of the asset that the investor holds. The first-order condition for stock i is (2.4) and (2.5), where $\lambda(n)\equiv a\sigma^2\left(\sum_{j=1}^{I}b_jx_j(n)+x(n)\right)$. The first-order condition for asset $I+1$ is

$$\bar{D}-S-\lambda(n)=0$$

and implies that $\lambda(n)$ is equal to the common value $\bar{D}-S$ for all n. Following the same steps as in Proposition 2.1 and using that asset $I+1$ is in zero supply, we can then show that equilibrium prices of stocks $i=1,..,I$ are given by (2.9) and equilibrium holdings of these stocks by investors are given by (2.11).

Under Condition (B), (2.6) and $\lambda(n)=a\sum_{i=1}^{I}b_ix_i(n)$ imply

$$\begin{aligned}\lambda(n)&=\sum_{i=1}^{I}b_j\max\left\{\frac{b_i\bar{D}+\bar{D}_i+\epsilon_i(n)-S_i-b_i\lambda(n)}{\sigma_i^2},0\right\}\\&=\sum_{\mathcal{I}}\sum_{i\in\mathcal{I}}b_i\max\left\{\frac{b_i\bar{D}+\bar{D}_i+\epsilon_i(n)-S_i-b_i\lambda(n)}{\sigma_i^2},0\right\},\end{aligned}\tag{A.3}$$

where $\mathcal{I}$ denotes a set formed all stocks i with the same characteristics $(\theta_i,b_i,\sigma_i^2)$. Suppose that S_i is given by (2.9) for all $i=1,..,I$, in which case Condition (B) implies that $\bar{D}_i-S_i$ is equal across $i\in\mathcal{I}$. Suppose also, proceeding by contradiction, that $\lambda(n)>\lambda(n')$ for some $n,n'\in[0,1]$. Condition (B), equality of $\bar{D}_i-S_i$ across $i\in\mathcal{I}$, and $b_i\geq 0$ for all $i\in\mathcal{I}$, imply

$$\sum_{i\in\mathcal{I}}b_i\max\left\{\frac{b_i\bar{D}+\bar{D}_i+\epsilon_i(n)-S_i-b_i\lambda(n)}{\sigma_i^2},0\right\}\leq\sum_{i\in\mathcal{I}}b_i\max\left\{\frac{b_i\bar{D}+\bar{D}_i+\epsilon_i(n')-S_i-b_i\lambda(n')}{\sigma_i^2},0\right\}.\tag{A.4}$$

Summing (A.4) over $\mathcal{I}$ and using (A.3), we find $\lambda(n)\leq\lambda(n')$, which contradicts $\lambda(n)>\lambda(n')$. Therefore, $\lambda(n)$ is equal across n. □

Proof of Lemma 2.1. We denote the supremum and the infimum of the support of the distribution of investor beliefs for stock i by $\bar{\epsilon}_i$ and $\underline{\epsilon}_i$, respectively. We first consider comparative static

(I). Setting ϵ such that

$$\bar{\epsilon}_i = \mu_i + \frac{1}{\chi}(\epsilon - \mu_i) \Leftrightarrow \epsilon = \mu_i + \chi(\bar{\epsilon}_i - \mu_i)$$

in (2.12), we find that the supremum of the support of the distribution for stock i' is not larger than $\mu_i + \chi(\bar{\epsilon}_i - \mu_i)$. Setting $\epsilon + \Delta\epsilon$ such that

$$\bar{\epsilon}_i = \mu_i + \frac{1}{\chi}(\epsilon + \Delta\epsilon - \mu_i) \Leftrightarrow \epsilon + \Delta\epsilon = \mu_i + \chi(\bar{\epsilon}_i - \mu_i)$$

in (2.12), we find that the supremum is not smaller than $\mu_i + \chi(\bar{\epsilon}_i - \mu_i)$. Therefore, the supremum is $\mu_i + \chi(\bar{\epsilon}_i - \mu_i)$. The same argument implies that the infimum of the support of the distribution for stock i' is $\mu_i + \chi(\underline{\epsilon}_i - \mu_i)$. Therefore, the range for stock i' is $\chi(\bar{\epsilon}_i - \underline{\epsilon}_i)$, which is χ times the range $\bar{\epsilon}_i - \underline{\epsilon}_i$ for stock i. The variance for stock i' is

$$\int_{\mu_i+\chi(\underline{\epsilon}_i-\mu_i)}^{\mu_i+\chi(\bar{\epsilon}_i-\mu_i)} (\epsilon - \mu_i)^2 dF_{i'}(\epsilon).$$

Using (2.12) and then making the change of variable $\hat{\epsilon} = \mu_i + \frac{1}{\chi}(\epsilon - \mu_i)$, we can write that variance as

$$\int_{\mu_i+\chi(\underline{\epsilon}_i-\mu_i)}^{\mu_i+\chi(\bar{\epsilon}_i-\mu_i)} (\epsilon - \mu_i)^2 dF_i\left(\mu_i + \frac{1}{\chi}(\epsilon - \mu_i)\right)$$
$$= \chi^2 \int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\hat{\epsilon} - \mu_i)^2 dF_i\left(\hat{\epsilon}\right),$$

which is χ^2 times the variance for stock i. Therefore, the standard deviation for stock i' is χ times that for stock i. The kurtosis for stock i' is

$$\frac{\int_{\mu_i+\chi(\underline{\epsilon}_i-\mu_i)}^{\mu_i+\chi(\bar{\epsilon}_i-\mu_i)} (\epsilon - \mu_i)^4 dF_{i'}(\epsilon)}{\left[\int_{\mu_i+\chi(\underline{\epsilon}_i-\mu_i)}^{\mu_i+\chi(\bar{\epsilon}_i-\mu_i)} (\epsilon - \mu_i)^2 dF_{i'}(\epsilon)\right]^2}.$$

Using (2.12) and then making the change of variable $\hat{\epsilon} = \mu_i + \frac{1}{\chi}(\epsilon - \mu_i)$, we can write that kurtosis as

$$\frac{\chi^4 \int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\hat{\epsilon} - \mu_i)^4 dF_i\left(\hat{\epsilon}\right)}{\left[\chi^2 \int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\hat{\epsilon} - \mu_i)^2 dF_i\left(\hat{\epsilon}\right)\right]^2} = \frac{\int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\hat{\epsilon} - \mu_i)^4 dF_i\left(\hat{\epsilon}\right)}{\left[\int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\hat{\epsilon} - \mu_i)^2 dF_i\left(\hat{\epsilon}\right)\right]^2},$$

which is the kurtosis for stock i.

We next consider comparative static (P). The same argument as for comparative static (I) implies that the supremum of the support of the distribution for stock i' is $\bar{\epsilon}_i$ and the infimum is

$\underline{\epsilon}_i$. Therefore, the range for stock i' is $\bar{\epsilon}_i - \underline{\epsilon}_i$, which is the range for stock i. Using (2.13), we can write the variance for stock i' as

$$\psi \int_{\bar{\epsilon}_i}^{\underline{\epsilon}_i} (\epsilon - \mu_i)^2 dF_i(\epsilon),$$

which is ψ times the variance for stock i. Therefore, the standard deviation for stock i' is $\sqrt{\psi}$ times that for stock i. Using (2.13), we can write the kurtosis for stock i' as

$$\frac{\psi \int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\epsilon - \mu_i)^4 dF_i(\epsilon)}{\left[\psi \int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\epsilon - \mu_i)^2 dF_i(\epsilon)\right]^2} = \frac{1}{\psi} \frac{\int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\epsilon - \mu_i)^4 dF_i(\epsilon)}{\left[\int_{\underline{\epsilon}_i}^{\bar{\epsilon}_i} (\epsilon - \mu_i)^2 dF_i(\epsilon)\right]^2},$$

which is $\frac{1}{\psi}$ times the kurtosis for stock i. □

Proof of Proposition 2.2. We first consider comparative static (I). When (2.14) is violated, the argument that follows the proposition's statement implies that the short-sale constraint for stock i' is not binding for any investor, and the unique solution of (2.10) for stock i' satisfies

$$\phi_{i'} = \frac{\mu_i}{a\theta_i \sigma_i^2} - 1.$$

When instead (2.14) holds, the short-sale constraint for stock i' is binding for some investors, and the unique solution of (2.10) for stock i' satisfies

$$\phi_{i'} > \frac{\mu_i}{a\theta_i \sigma_i^2} - 1$$

and is given by

$$\int_{a\theta_i \sigma_i^2 \phi_{i'}}^{\bar{\epsilon}_{i'}} \left(\frac{\epsilon}{a\theta_i \sigma_i^2} - \phi_{i'} \right) dF_{i'}(\epsilon) = 1. \tag{A.5}$$

Using (2.12) and then making the change of variable $\hat{\epsilon} = \mu_i + \frac{1}{\chi}(\epsilon - \mu_i)$, we can write (A.5) as

$$\begin{aligned} &\int_{a\theta_i \sigma_i^2 \phi_{i'}}^{\bar{\epsilon}_{i'}} \left(\frac{\epsilon}{a\theta_i \sigma_i^2} - \phi_{i'} \right) dF_i \left(\mu_i + \frac{1}{\chi}(\epsilon - \mu_i) \right) = 1 \\ &= \int_{\mu_i + \frac{1}{\chi}(a\theta_i \sigma_i^2 \phi_{i'} - \mu_i)}^{\bar{\epsilon}_i} \left(\frac{\mu_i + \chi(\hat{\epsilon} - \mu_i)}{a\theta_i \sigma_i^2} - \phi_{i'} \right) dF_i\left(\hat{\epsilon}\right) = 1. \end{aligned} \tag{A.6}$$

Differentiating implicitly (A.6) with respect to χ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, we find

$$\int_{\mu_i+\frac{1}{\chi}(a\theta_i\sigma_i^2\phi_{i'}-\mu_i)}^{\overline{\epsilon}_i} \left(\frac{\hat{\epsilon}-\mu_i}{a\theta_i\sigma_i^2} - \frac{\partial\phi_{i'}}{\partial\chi}\right) dF_i\left(\hat{\epsilon}\right) = 0$$

$$\Rightarrow \frac{\partial\phi_{i'}}{\partial\chi} = \frac{\int_{\mu_i+\frac{1}{\chi}(a\theta_i\sigma_i^2\phi_{i'}-\mu_i)}^{\overline{\epsilon}_i} \frac{\hat{\epsilon}-\mu_i}{a\theta_i\sigma_i^2} dF_i\left(\hat{\epsilon}\right)}{\int_{\mu_i+\frac{1}{\chi}(a\theta_i\sigma_i^2\phi_{i'}-\mu_i)}^{\overline{\epsilon}_i} dF_i\left(\hat{\epsilon}\right)} = \frac{1}{a\theta_i\sigma_i^2}\left[\mathbb{E}\left(\hat{\epsilon}|\hat{\epsilon} \in \left[\mu_i + \frac{1}{\chi}(a\theta_i\sigma_i^2\phi_{i'} - \mu_i), \overline{\epsilon}_i\right]\right) - \mu_i\right] > 0, \tag{A.7}$$

where the positive sign follows because $\mu_i + \frac{1}{\chi}(a\theta_i\sigma_i^2\phi_{i'} - \mu_i) > \underline{\epsilon}_i$. Therefore, $\phi_{i'}$, viewed as function of χ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, increases in χ when (2.14) holds and is independent of χ when (2.14) is violated. Since there exists a threshold χ^* such that (2.14), viewed as function of χ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, holds for all $\chi > \chi^*$ and is violated for all $\chi \leq \chi^*$, $\phi_{i'} > \phi_i$ when (2.14) holds and $\phi_{i'} = \phi_i$ when (2.14) is violated. The statement in the proposition for comparative static (I) then follows from (2.9) and because expected return $\frac{b_i\bar{D}+\bar{D}-S_i}{S_i}$ is decreasing in the price.

We next consider comparative static (P). The same argument as for comparative static (I) implies that when (2.14) is violated, the short-sale constraint for stock i' is not binding for any investor, and $\phi_{i'} = \frac{\mu_i}{a\theta_i\sigma_i^2} - 1$. When instead (2.14) holds, the short-sale constraint for stock i' is binding for some investors, and $\phi_{i'}$ satisfies

$$\phi_{i'} > \frac{\mu_i}{a\theta_i\sigma_i^2} - 1$$

and is given by

$$\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_i} \left(\frac{\epsilon}{a\theta_i\sigma_i^2} - \phi_{i'}\right) dF_{i'}(\epsilon) = 1. \tag{A.8}$$

Using (2.13), we can write (A.8) as

$$\psi \int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_i} \left(\frac{\epsilon}{a\theta_i\sigma_i^2} - \phi_{i'}\right) dF_i(\epsilon) + (1-\psi)\left(\frac{\mu_i}{a\theta_i\sigma_i^2} - \phi_{i'}\right) = 1 \tag{A.9}$$

when $\phi_{i'} < \frac{\mu_i}{a\theta_i\sigma_i^2}$, and as

$$\psi \int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_i} \left(\frac{\epsilon}{a\theta_i\sigma_i^2} - \phi_{i'}\right) dF_i(\epsilon) = 1 \tag{A.10}$$

when $\phi_{i'} \geq \frac{\mu_i}{a\theta_i\sigma_i^2}$. Differentiating implicitly (A.9) with respect to ψ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$

constant, we find

$$\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_i}\left(\frac{\epsilon}{a\theta_i\sigma_i^2}-\phi_{i'}\right)dF_i(\epsilon)-\left(\frac{\mu_i}{a\theta_i\sigma_i^2}-\phi_{i'}\right)-\frac{\partial\phi_{i'}}{\partial\psi}\left(\psi\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_{i'}}dF_i(\epsilon)+(1-\psi)\right)=0$$

$$\Rightarrow\frac{\partial\phi_{i'}}{\partial\psi}=-\frac{\int_{\underline{\epsilon}_i}^{a\theta_i\sigma_i^2\phi_{i'}}\left(\frac{\epsilon}{a\theta_i\sigma_i^2}-\phi_{i'}\right)dF_i(\epsilon)}{\psi\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_{i'}}dF_i(\epsilon)+(1-\psi)}>0. \tag{A.11}$$

Differentiating implicitly (A.10) with respect to ψ holding $(\mu_i,\theta_i,b_i,\sigma_i,\underline{\epsilon}_i,\overline{\epsilon}_i)$ constant, we find

$$\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_i}\left(\frac{\epsilon}{a\theta_i\sigma_i^2}-\phi_{i'}\right)dF_i(\epsilon)-\frac{\partial\phi_{i'}}{\partial\psi}\psi\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_{i'}}dF_i(\epsilon)=0$$

$$\Rightarrow\frac{\partial\phi_{i'}}{\partial\psi}=\frac{\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_i}\left(\frac{\epsilon}{a\theta_i\sigma_i^2}-\phi_{i'}\right)dF_i(\epsilon)}{\psi\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_{i'}}dF_i(\epsilon)}>0. \tag{A.12}$$

Therefore, $\phi_{i'}$, viewed as function of ψ holding $(\mu_i,\theta_i,b_i,\sigma_i,\underline{\epsilon}_i,\overline{\epsilon}_i)$ constant, increases in ψ when (2.14) holds and is independent of ψ when (2.14) is violated. Since (2.14), viewed as function of ψ holding $(\mu_i,\theta_i,b_i,\sigma_i,\underline{\epsilon}_i,\overline{\epsilon}_i)$ constant, is independent of ψ, $\phi_{i'}<\phi_i$ when (2.14) holds and $\phi_{i'}=\phi_i$ when (2.14) is violated. The statement in the proposition for comparative static (P) then follows from (2.9) and because expected return is decreasing in the price. □

Proof of Proposition 2.3. We first consider comparative static (I). When (2.14) is violated, the argument that follows the statement of Proposition 2.2 implies that the short-sale constraint for stock i' is not binding for any investor. Therefore, $B_{i'}=1$. When instead (2.14) holds, the short-sale constraint for stock i' is binding for some investors and

$$\begin{aligned}B_{i'}&=\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_{i'}}dF_{i'}(\epsilon)\\&=F_{i'}(\overline{\epsilon}_{i'})-F_{i'}\left(a\theta_i\sigma_i^2\phi_{i'}\right)\\&=F_i(\overline{\epsilon}_i)-F_i\left(\mu_i+\frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'}-\mu_i\right)\right)\\&=1-F_i\left(\mu_i+\frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'}-\mu_i\right)\right),\end{aligned} \tag{A.13}$$

where the third step follows from (2.12) and $\overline{\epsilon}_{i'}=\mu_i+\chi(\overline{\epsilon}_i-\mu_i)$. Differentiating $\mu_i+\frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'}-\mu_i\right)$

with respect to χ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, and using (A.7), we find

$$\begin{aligned}
&\frac{\partial}{\partial \chi}\left(\mu_i + \frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'} - \mu_i\right)\right) \\
&= \frac{1}{\chi}\left(-\frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'} - \mu_i\right) + \left[\mathbb{E}\left(\hat{\epsilon}|\hat{\epsilon} \in \left[\mu_i + \frac{1}{\chi}(a\theta_i\sigma_i^2\phi_{i'} - \mu_i), \overline{\epsilon}_i\right]\right) - \mu_i\right]\right) \\
&> \frac{1}{\chi}\left(-\frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'} - \mu_i\right) + \left[\left[\mu_i + \frac{1}{\chi}(a\theta_i\sigma_i^2\phi_{i'} - \mu_i), \overline{\epsilon}_i\right] - \mu_i\right]\right) = 0,
\end{aligned}$$

where the inequality is strict because (A.13) implies $\mu_i + \frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'} - \mu_i\right) < \overline{\epsilon}_i$. Therefore, $\mu_i + \frac{1}{\chi}\left(a\theta_i\sigma_i^2\phi_{i'} - \mu_i\right)$ increases in χ and (A.13) implies that $B_{i'}$, viewed as function of χ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, is non-increasing in χ. Since $B_{i'}$ is non-increasing in χ when (2.14) holds and is independent of χ when (2.14) is violated, and since there exists a threshold χ^* such that (2.14) holds for all $\chi > \chi^*$ and is violated for all $\chi \leq \chi^*$, $B_{i'} \leq B_i$ when (2.14) holds and $B_{i'} = B_i$ when (2.14) is violated. When $F_i(\epsilon)$ increases in ϵ, (A.13) implies that $B_{i'}$ decreases in χ. Therefore, $B_{i'} < B_i$ when (2.14) holds and $B_{i'} = B_i$ when (2.14) is violated.

We next consider comparative static (P). When (2.14) is violated, the argument that follows the statement of Proposition 2.2 implies that the short-sale constraint for stock i' is not binding for any investor. Therefore, $B_{i'} = 1$. Since (2.14), viewed as function of ψ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, is independent of ψ, $B_{i'} = B_i$. When instead (2.14) holds, the short-sale constraint for stock i' is binding for some investors and

$$B_{i'} = \int_{a\theta_i\sigma_i^2\phi_{i'}}^{\overline{\epsilon}_{i'}} dF_{i'}(\epsilon) = F_{i'}(\overline{\epsilon}_{i'}) - F_{i'}\left(a\theta_i\sigma_i^2\phi_{i'}\right). \tag{A.14}$$

Using (2.13) and $\overline{\epsilon}_{i'} = \overline{\epsilon}_i$, we can write (A.14) as

$$B_{i'} = \psi\left[F_i(\overline{\epsilon}_i) - F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right)\right] + (1-\psi) = 1 - \psi F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right) \tag{A.15}$$

when (2.18) is violated, and as

$$B_{i'} = \psi\left[F_i(\overline{\epsilon}_i) - F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right)\right] = \psi\left[1 - F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right)\right] \tag{A.16}$$

when (2.18) holds. We first consider the case where (2.19) holds. Since (A.11) implies that $\phi_{i'}$ increases in ψ, (2.18) is violated for all $\hat{\psi} \in [\psi, 1]$ and (A.15) implies that $B_{i'}$, viewed as function of ψ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, decreases in ψ. Therefore, $B_{i'} > B_i$. We next consider the case where (2.18) holds and $F_i(\epsilon)$ has a density $f_i(\epsilon)$ in $[\mu_i, \overline{\epsilon}_i]$ that is positive and non-decreasing in ϵ. Differentiating (A.16) with respect to ψ holding $(\mu_i, \theta_i, b_i, \sigma_i, \underline{\epsilon}_i, \overline{\epsilon}_i)$ constant, and using (A.12), we find

$$\begin{aligned}
\frac{\partial B_{i'}}{\partial \psi} &= 1 - F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right) - \psi a\theta_i\sigma_i^2 f_i\left(a\theta_i\sigma_i^2\phi_{i'}\right)\frac{\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\bar{\epsilon}_i}\left(\frac{\epsilon}{a\theta_i\sigma_i^2}-\phi_{i'}\right)dF_i(\epsilon)}{\psi\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\bar{\epsilon}_{i'}}dF_i(\epsilon)} \\
&= 1 - F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right) - \frac{a\theta_i\sigma_i^2 f_i\left(a\theta_i\sigma_i^2\phi_{i'}\right)}{\psi\left[1-F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right)\right]} \\
&\geq 1 - F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right) - \frac{a\theta_i\sigma_i^2}{\psi\left(\bar{\epsilon}_i - a\theta_i\sigma_i^2\phi_{i'}\right)} \\
&= \frac{a\theta_i\sigma_i^2}{\psi\left(\bar{\epsilon}_i - a\theta_i\sigma_i^2\phi_{i'}\right)}\left[\psi\left(\frac{\bar{\epsilon}_i}{a\theta_i\sigma_i^2}-\phi_{i'}\right)\left[1-F_i\left(a\theta_i\sigma_i^2\phi_{i'}\right)\right]-1\right] \\
&> \frac{a\theta_i\sigma_i^2}{\psi\left(\bar{\epsilon}_i - a\theta_i\sigma_i^2\phi_{i'}\right)}\left[\psi\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\bar{\epsilon}_i}\left(\frac{\epsilon}{a\theta_i\sigma_i^2}-\phi_{i'}\right)dF_i(\epsilon)-1\right] = 0,
\end{aligned}$$

where the second step follows from (A.10), the third step because $f_i(\epsilon)$ is non-decreasing in ϵ, the fifth step because the positive density $f_i(\epsilon)$ implies that $F_i(\epsilon)$ increases in $\epsilon \in [\mu_i, \bar{\epsilon}_i]$, and the sixth step from (A.10). Therefore, $B_{i'}$ increases in ψ. Since (A.11) implies that $\phi_{i'}$ increases in ψ, (2.18) holds for all $\hat{\psi} \in [\psi, 1]$, and $B_{i'} < B_i$.

Writing (A.8) as

$$\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\bar{\epsilon}_i}\left(\epsilon - a\theta_i\sigma_i^2\phi_{i'}\right)dF_{i'}(\epsilon) = a\theta_i\sigma_i^2,$$

and differentiating implicitly with respect to $\theta_i\sigma_i^2$ holding $(\mu_i, b_i, \underline{\epsilon}_i, \bar{\epsilon}_i)$ constant, we find

$$-a\frac{\partial\left(\theta_i\sigma_i^2\phi_{i'}\right)}{\partial\left(\theta_i\sigma_i^2\right)}\int_{a\theta_i\sigma_i^2\phi_{i'}}^{\bar{\epsilon}_i}dF_{i'}(\epsilon) = a.$$

Therefore, $\theta_i\sigma_i^2\phi_{i'}$ decreases in $\theta_i\sigma_i^2$. Since (2.18) can be written as

$$\theta_i\sigma_i^2\phi_{i'} \geq \frac{\mu_i}{a},$$

there exists a threshold Θ_ψ^* such that (2.14) holds for $\theta_i\sigma_i^2 \leq \Theta_\psi^*$. Likewise, since (2.19) can be written as

$$\theta_i\sigma_i^2\phi_i < \frac{\mu_i}{a},$$

there exists a threshold Θ_1^* such that (2.14) holds for $\theta_i\sigma_i^2 > \Theta_1^*$. Since $\phi_{i'} < \phi_i$, these thresholds satisfy $\Theta_\psi^* < \Theta_1^*$. □

Proof of Proposition 2.4. When

$$\frac{\mu_i - \underline{\epsilon}_i}{a\theta_{i'}\sigma_i^2} > 1 \tag{A.17}$$

is violated, the argument that follows the statement of Proposition 2.2 implies that the short-sale constraint for stock i' is not binding for any investor. Therefore, $B_{i'} = 1$. When instead (A.17) holds, the short-sale constraint for stock i' is binding for some investors and

$$\begin{aligned} B_{i'} &= \int_{a\theta_{i'}\sigma_i^2\phi_{i'}}^{\bar{\epsilon}_i} dF_i(\epsilon) \\ &= F_i(\bar{\epsilon}_i) - F_i\left(a\theta_{i'}\sigma_i^2\phi_{i'}\right) \\ &= 1 - F_i\left(a\theta_{i'}\sigma_i^2\phi_{i'}\right). \end{aligned} \tag{A.18}$$

Since $\theta_{i'}\sigma_i^2\phi_{i'}$ decreases in $\theta_{i'}\sigma_i^2$, as shown in the proof of Proposition 2.3, (A.18) implies that $B_{i'}$, viewed as function of $\theta_{i'}$ holding $(b_i, \bar{D}_i, \sigma_i, F_i(\epsilon))$ constant is non-decreasing in $\theta_{i'}$. Since $B_{i'}$ is non-decreasing in $\theta_{i'}$ when (A.17) holds and is independent of $\theta_{i'}$ when (A.17) is violated, and since there exists a threshold θ^* such that (A.17) holds for all $\theta_{i'} < \theta^*$ and is violated for all $\theta_{i'} \geq \theta^*$, $B_{i'} \geq B_i$ when (2.20) holds and $B_{i'} = B_i$ when (2.20) is violated. When $F_i(\epsilon)$ increases in ϵ, (A.18) implies that $B_{i'}$ increases in $\theta_{i'}$. Therefore, $B_{i'} > B_i$ when (2.20) holds and $B_{i'} = B_i$ when (2.20) is violated. □

B Investment Styles of 13-F Investors by Thomson Reuters

Table B.I presents the 32 investment styles in which Thompson Reuters (TR) classifies 13-F investors.

TR classifies 13-F investors into styles based on the characteristics of the stocks that they hold, their historical investment behavior, their current transactions and their general business type. TR first classifies each stock into a certain group or style based on its price-earnings ratio, dividend yield, and the three- to five-year projected earnings-per-share growth relative to the corresponding S&P500 or sector averages. For each 13-F investor, TR then calculates the weights of the different groups or styles of stocks. The group with the biggest weight generally characterizes the investor's style.

Some classifications are more mechanical. 13-F investors whose portfolios follow the composition of certain indices (e.g. S&P 500, Russell 1000/2000/3000, etc) are classified into the Index style. Styles such as "Broker Dealer," "Hedge Funds" and "VC/Private Equity" are assigned mainly based on the business type of the corresponding investors. Finally, some 13-F investors

Table B.I: **The 32 investment styles in which Thomson Reuters classifies 13-F investors.**

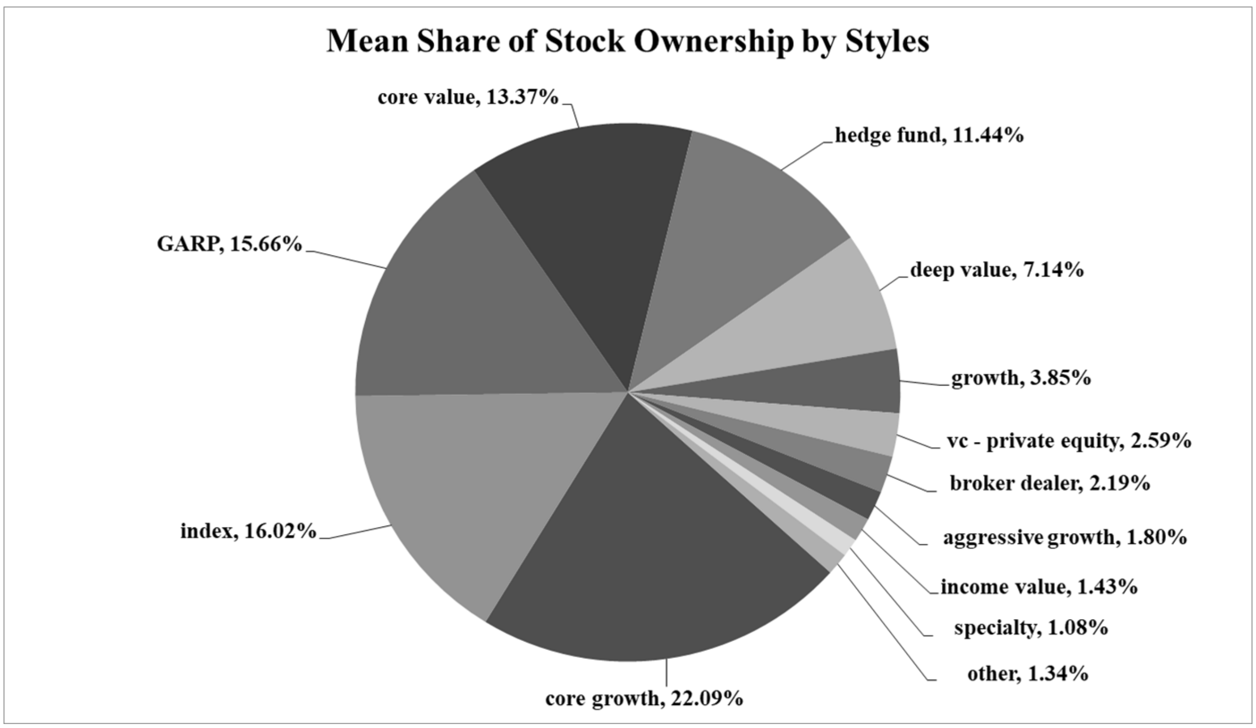
General Styles	Hedge Fund Styles
Aggressive Growth, Broker Dealer, Core Growth, Core Value, Deep Value, Emerging Markets, GARP(Growth at Reasonable Price), Growth, Hedge Fund, Income Value, Index, Mixed Style, Momentum, Sector Specific, Specialty, VC(Venture Capital)/Private Equity, Yield	Capital Structure Arbitrage, Convertible Arbitrage, CTA(Commodity Trading Advisors) Managed Futures, Distressed Securities, Emerging Markets (Hedge), Equity Hedge, Event Driven (Merger/ Risk Arbitrage), Fixed Income Arbitrage, Funds of Funds, Global Macro, Long Bias, Long-Short, Market Neutral, Multi-Strategy (Hedge), Quantitative/Statistical Arbitrage

Note: The 32 investment styles in which Thompson Reuters (TR) classifies 13-F investors. The left column reports the seventeen general styles and the right column reports the fifteen hedge fund styles. The styles are reported alphabetically in each column. The information is available on http://banker.thomsonib.com/ta/help/webhelp/Ownership_Glossary.htm

are classified into hedge-fund styles depending on their exact investment strategy (e.g. "Convertible Arbitrage," "Quantitative-Statistical Arbitrage," "Emerging Markets," "Fund of funds"). The relative importance of hedge-fund styles is small.

The pie chart in Figure B.1 shows the size of each of the 32 styles in our sample, defined as the asset value attributed to the style over the total asset value of all styles. There are twelve styles with size above 1%. The combined size of the remaining twenty styles is 1.34%.

Figure B.1: **Mean share of stock ownership by style**



Note: Mean percentage shares in our sample of the 32 investment styles in which Thompson Reuters (TR) classifies 13-F investors. The average shares above 1% are reported separately (twelve styles) and the average shares below 1% are reported together as “other” (twenty styles).