Passive Investing and the Rise of Mega-Firms

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Abstract

We study how passive investing affects asset prices. Flows into passive funds raise disproportionately the stock prices of the economy's largest firms, and especially those large firms that the market overvalues. These effects are sufficiently strong to cause the aggregate market to rise even when flows are entirely due to investors switching from active to passive. Our results arise because flows create idiosyncratic volatility for large firms, which discourages investors from correcting the flows' effects on prices. Consistent with our theory, the largest firms in the S&P500 experience the highest returns and increases in volatility following flows into that index.

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1 Introduction

One of the most important capital-market developments of the past thirty years has been the growth of passive investing. Passive funds track market indices and charge lower fees than active funds. In 1993, passive funds invested in US stocks managed $23 billion of assets. That was 3.7% of the combined assets managed by active and passive funds, and 0.44% of the US stock market. By 2021, passive assets had risen to $8.4 trillion. That was 53% of combined active and passive, and 16% of the stock market.[1] The growth of passive investing has been estimated to be more than twice as high when accounting for the increasing tendency by active funds and other investors to stay close to their benchmark indices.[2]

The growth of passive investing has stimulated academic and policy interest in how it affects asset prices and the real economy. One effect that has been emphasized, drawing on the literature on rational expectations equilibria (REE) with asymmetric information (Grossman (1976), Grossman and Stiglitz (1980)), is that with fewer active funds, individual stocks become less liquid and their prices less informative. Another effect, drawing on the literature on index additions (Harris and Gurel (1986), Shleifer (1986)), is that the stock prices of firms included in the indices tracked by passive funds rise, while the prices of non-index firms do not.

In this paper we show that the growth of passive investing raises disproportionately the stock prices of the economy's largest firms, and especially those large firms that the market overvalues. Passive investing thus reduces primarily the financing costs of the largest firms and makes the size distribution of firms more skewed. These effects are generated by a different mechanism than in the REE and index-addition literatures because information in our model is symmetric and the effects arise even when indices include all firms. We also show that the effects are sufficiently strong to cause the aggregate market to rise even when the growth of passive comes entirely from investors switching from active (and not from new investors entering into stocks). Passive investing thus

[1]The data come from the 2022 Investment Company Institute (ICI) Factbook (Figure 2.9 and Tables 11 and 42), and from `https://data.worldbank.org/indicator/CM.MKT.LCAP.CD?locations=US`. We identify passive funds with index mutual funds and index exchange-traded funds (ETFs), and identify more generally passive investing with indexing throughout this paper.

[2]A measure of how far active funds stray from their benchmark indices is active share, defined in Cremers and Petajisto (2009). Petajisto (2013) finds that active share has been declining over time. Chinco and Sammon (2023) estimate that passive investing under its broader definition comprised 33.3% of the US stock market in 2021.

biases the stock market towards overvaluation. Consistent with our theory, we find that the largest firms in the S&P500 index experience the highest returns following flows into that index.

The intuition for our results is easiest to convey in the case where noise traders are present. Suppose that the stock of a large firm is in high demand by noise traders, and that active investors accommodate this demand by short-selling the stock in equilibrium. A switch by some investors from active to passive generates additional demand for the stock because passive investors hold the stock with its weight in the market index while active investors hold it with negative weight. Active investors can accommodate the additional demand by scaling up their short position. This renders them, however, more exposed to the stock's idiosyncratic risk, which is non-negligible because the firm is large. The stock price must then rise to induce active investors to take on the additional risk. Crucially, because the stock price rises, the stock's idiosyncratic price movements become larger in absolute terms. This gives rise to an amplification loop: the short position of active investors becomes even riskier, causing the stock's price to rise even further, and the stock's idiosyncratic price movements to become even larger.

The amplification loop explains why passive flows have their largest effects on the stocks of large firms that the market overvalues. It also explains why the effects of passive flows on those firms are sufficiently strong so that a switch by some investors from active to passive causes the aggregate market to rise, despite the stocks of undervalued firms dropping in price. The amplification loop relates to two additional results that we show theoretically and confirm empirically. First, passive flows raise the stock return volatility of large firms, by causing idiosyncratic price movements to become larger, but do not affect the volatility of smaller firms. Second, when firms are added to indices tracked by passive funds, the resulting stock price increase is larger for large firms.

Our results imply that large firms are less liquid than smaller firms, in the sense that an increase in demand proportional to firms' market capitalization causes the stocks of large firms to rise the most. Within the intuitive argument described above, large firms are less liquid because the increase in demand creates idiosyncratic volatility, which discourages investors from correcting the effects of the increased demand on prices. While this intuitive argument is based on noise traders and short positions, it can be generalized by dropping both elements, as we explain below.

We study the effects of passive flows on equilibrium stock prices in a tractable dynamic model that can handle multiple stocks, systematic and idiosyncratic risk, and a general size distribution of firms. We present and solve the general model in Sections 2 and 3, respectively. We calibrate the model using data on moments of asset returns and the size distribution of firms in Section 4.

In our model, agents can invest in a constant riskless rate and in multiple stocks, over an infinite horizon. Each stock's dividend flow per share is the sum of a constant and of a systematic and an idiosyncratic component that follow independent square-root processes. Some agents, the experts, can invest in all assets without constraints. They can be interpreted as investors who follow active strategies using stocks, mutual funds or hedge funds. Other agents, the non-experts, can only invest in the riskless asset and in a capitalization-weighted index. They can be interpreted as investors in passive funds. Experts and non-experts maximize a mean-variance objective over instantaneous changes in wealth. Noise traders can also be present, and hold a number of shares of each stock that is constant over time. In equilibrium, an increase in the demand of non-experts or of noise traders for a stock not only raises its price, but also renders price movements (caused by dividend shocks) larger in absolute terms. Key to this result is the realistic property of the squared-root process that the volatility of dividends per share increases with the level of dividends per share.

Our calibration assumes approximately 1700 firms sorted into five size groups, with a size distribution that conforms to a power law with exponent one, consistent with the empirical evidence (Axtell (2001)). The systematic component of dividends is assumed to decrease with firm size, so that CAPM beta decreases with size, consistent with the evidence (Fama and French (1992)).

We derive results from our calibration in Section 5 and extend them analytically to more general parameter values in Appendix A. When passive flows are due to entry by new investors into stocks, they raise the prices of all firms, with the percentage price increase being J-shaped with size: larger for small firms than for medium-size firms, and largest for the largest firms. The decreasing part of the J-shape arises from basic CAPM logic: investor entry causes the market risk premium to drop, and the resulting drop in expected returns and rise in prices is largest for the smallest firms because they have the highest CAPM beta. The increasing part of the J-shape arises because of a generalized version of the amplification loop. When experts hold long positions in all stocks, as is the case without noise traders, the increase in risk attenuates the rise in price caused by passive flows. Crucially, the attenuation effect for large firms' idiosyncratic risk is weaker than for systematic risk (and turns to an amplification effect when experts hold short positions in some stocks).

When passive flows are due to a switch from active to passive, they have no effect on stock prices in the absence of noise traders and when the index includes all firms. This is because experts and non-experts hold the same portfolio, which is the index. When noise traders are present or the index includes only firms in the larger size groups, the effect of passive flows on prices is increasing with size, negative for small firms, and disproportionately positive for large firms.

We present tests of our theory in Section 6, where we also relate our results to empirical findings in the literature. We take the index to be the S&P500, the most tracked index in the US stock market, and flows to be into index mutual funds and index ETFs tracking it. Our flow data are quarterly, from 1996 to 2020. During quarters when index funds receive high inflows, the largest firms in the index outperform the index. During the same quarters, index concentration, as measured by, e.g., the combined portfolio weight of the ten largest firms, increases. Following the same quarters, the idiosyncratic stock return volatility increases for large firms, and does so twice as much as for smaller firms. Finally, large firms experience higher stock returns than smaller firms when they are added to the index.

The effects of passive investing have mainly been analyzed within the framework proposed by Grossman and Stiglitz (1980, GS), in which informed and uninformed investors trade with noise traders. Informed and uninformed investors in GS can be interpreted as active and passive fund managers, respectively. A switch from active to passive reduces market efficiency and can exacerbate the mispricing caused by noise traders.[3] The interpretation of GS investors as fund managers is developed in Garleanu and Pedersen (2018), in which investors search for informed managers, and the efficiency of the search market for managers affects the efficiency of the asset market. In Subrahmanyam (1991), the introduction of a market index facilitates passive investing and lowers liquidity for the assets that comprise the index.[4]

[3] Pastor and Stambaugh (2012) and Stambaugh (2014) explain an increase in market efficiency, as reflected in a decline in active funds' expected returns, by the increase in the assets that active funds manage and by the decline in noise trading, respectively.

[4] Related mechanisms are at play in Bhattacharya and O'Hara (2018) and Cong and Xu (2020) who study how ETFs affect market efficiency and liquidity, Bond and Garcia (2022) who study the effects of lowering the costs of passive investing, and Haddad, Huebner, and Loualiche (2022) who study how passive investing affects the elasticity of asset demand curves. Buss and Sundaresan (2023) show that passive investing can increase market efficiency when

A different literature studies how constraints or incentives of fund managers to not deviate from their benchmark indices affect asset prices. Brennan (1993), Kapur and Timmermann (2005), Cuoco and Kaniel (2011) and Basak and Pavlova (2013) show that compensating managers based on their performance relative to indices induces them to buy index assets, causing their prices to rise. Davies (2024) shows that passive flows have their strongest positive effects on the prices of firms with high CAPM beta (the same effect that causes the decreasing part of our J-shape) or in high demand by noise traders.[5] Our model is technically closest to Buffa, Vayanos, and Woolley (2022, BVW), who examine how constraints on managers' deviations from indices affect asset prices. We depart from BVW by introducing systematic risk and a size distribution of firms. Cheng, Jondeau, Mojon, and Vayanos (2023) use our general model to study how the rise of green indices that exclude polluting firms progressively over time affects equilibrium stock prices.

Our theory has implications for recent macroeconomic trends such as the rise in industry concentration and the decline in corporate investment. Autor, Dorn, Katz, Patterson, and van Reenen (2020) show that the rise of superstar firms can account for the rise in concentration (Grullon, Larkin, and Michaely (2019)) and the decline in the labor share (Elsby, Hobijn, and Sahin (2013), Karabarbounis and Neiman (2014)). Our theory suggests that the growth of passive investing can be one factor behind the rise of superstar firms, through the steeper decline of their financing costs. Alexander and Eberly (2018) and Crouzet and Eberly (2021) attribute the decline in corporate investment (Hall (2014), Fernald, Hall, Stock, and Watson (2017)) to intangible capital, while Gutiérrez and Philippon (2017) and Covarrubias, Gutiérrez, and Philippon (2019) show that the rise in concentration and changes in corporate governance are additional causes. Our theory suggests that the growth of passive investing may also have played a role because large overvalued firms experience the steepest decline in their financing costs but may not have the best investment projects.[6] The misallocation of capital due to such financial distortions can feed into low aggregate productivity, as recent papers have shown.[7]

corporate investment responds to stock prices. Coles, Heath, and Ringgenberg (2022) and Koijen, Richmond, and Yogo (forthcoming) find that the growth of passive investing did not have a significant impact on market efficiency.

[5] Chabakauri and Rytchkov (2021) show that passive flows cause market volatility to decrease when they are due to a switch from active to passive, and to increase when they are due to entry by new investors in the stock market.

[6] Gutiérrez and Philippon (2017) find that firms with a large share of ownership by passive funds invest less. They emphasize governance-based explanations rather than valuation-based ones.

[7] Gopinath, Kalemli-Ozcan, Karabarbounis, and Villegas-Sanchez (2017) link capital misallocation due to size-

2 Model

Time t is continuous and goes from zero to infinity. The riskless rate is exogenous and equal to $r > 0$. There are N firms indexed by $n = 1, .., N$. The stock of firm n, also referred to as stock n, pays dividend flow D_{nt} per share and is in supply of $\eta_n > 0$ shares. The dividend flow of stock n is

$$D_{nt} = \bar{D}_n + b_n D_t^s + D_{nt}^i, \tag{2.1}$$

the sum of a constant component $\bar{D}_n \geq 0$, a systematic component $b_n D_t^s$ and an idiosyncratic component D_{nt}^i. The systematic component is the product of a systematic factor D_t^s times a factor loading $b_n \geq 0$. The systematic factor follows the square-root process

$$dD_t^s = \kappa^s \left(\bar{D}^s - D_t^s\right) dt + \sigma^s \sqrt{D_t^s} dB_t^s, \tag{2.2}$$

where $(\kappa^s, \bar{D}^s, \sigma^s)$ are positive constants and B_t^s is a Brownian motion. The idiosyncratic component follows the square-root process

$$dD_{nt}^i = \kappa_n^i \left(\bar{D}_n^i - D_{nt}^i\right) dt + \sigma_n^i \sqrt{D_{nt}^i} dB_{nt}^i, \tag{2.3}$$

where $\{\kappa_n^i, \bar{D}_n^i, \sigma_n^i\}_{n=1,..,N}$ are positive constants and $\{B_{nt}^i\}_{n=1,..,N}$ are Brownian motions that are mutually independent and independent of B_t^s. By possibly redefining factor loadings, we set the long-run mean $\bar{D}^s$ of the systematic factor to one. By possibly redefining the supply η_n, we set the long-run mean $\bar{D}_n + b_n + \bar{D}_n^i$ of the dividend flow of stock n to one for all n. With these normalizations, we can write the dividend flow of stock n as

$$D_{nt} = 1 + b_n(D_t^s - 1) + (D_{nt}^i - \bar{D}_n^i). \tag{2.4}$$

The square-root specification (2.2) and (2.3) ensures that dividends always stay positive. This

dependent borrowing constraints in Southern Europe to low productivity growth. Using a general decomposition, David and Venkateswaran (2019) find that financial constraints can be an important cause of capital misallocation.

is because when D_t^s or D_{nt}^i approach zero, changes in these variables have a volatility ($\sigma^s\sqrt{D_t^s}$ or $\sigma_n^i\sqrt{D_{nt}^i}$) that goes to zero but a mean ($\kappa^s\left(\bar{D}^s - D_t^s\right)$ or $\kappa_n^i\left(\bar{D}_n^i - D_{nt}^i\right)$) that remains positive and bounded away from zero. The square-root specification also implies that the volatility of changes in dividends per share increases with the level of dividends per share. This property is realistic, because when a firm grows the volatility of its cashflow shocks rises in absolute terms (i.e., not as fraction of firm size), and is key for our results as we explain in Sections 3 and 5. The square-root specification does not imply that because dividends mean-revert, prices mean-revert predictably. Indeed, in Section 3 we show that stocks' expected returns are given by a conditional CAPM, in which the only predictability arises because the time-varying volatility of dividends implies time-varying compensation for risk. A geometric Brownian motion specification for dividends, which is commonly used in the asset pricing literature, shares the same above properties with the square-root specification. We adopt the square-root specification because it yields closed-form solutions.

Denoting by S_{nt} the price of stock n, the stock's return per share in excess of the riskless rate is

$$dR_{nt}^{sh} \equiv D_{nt}dt + dS_{nt} - rS_{nt}dt, \tag{2.5}$$

and the stock's return per dollar in excess of the riskless rate is

$$dR_{nt} \equiv \frac{dR_{nt}^{sh}}{S_{nt}} = \frac{D_{nt}dt + dS_{nt}}{S_{nt}} - rdt. \tag{2.6}$$

We refer to dR_t^{sh} as share return, omitting that it is in excess of the riskless rate. We refer to dR_t as return, omitting that it is per dollar and in excess of the riskless rate.

Agents are competitive and form overlapping generations living over infinitesimal time intervals. Each generation includes experts and non-experts. Experts observe dividend flows, and can invest in the riskless asset and in the stocks without constraints. These agents can be interpreted as investors who follow active strategies using stocks, mutual funds or hedge funds. Non-experts do not observe dividend flows, and can invest in the riskless asset and in a stock portfolio that tracks an index. These agents can be interpreted as investors in passive funds.[8]

[8]Investors' choice to invest in active or passive funds can result from trading off the superior returns of active

In addition to experts and non-experts, noise traders can be present. These agents generate an exogenous demand for each stock, which is smaller than the supply coming from the issuing firm. For tractability, we take the demand by noise traders to be constant over time when expressed in number of shares. A constant demand can capture slowly mean-reverting market sentiment. When noise traders are absent, or when their demand is proportional to the firm-issued supply in the cross-section of firms, experts and non-experts hold the same portfolio of stocks in equilibrium. When instead noise-trader demand is non-proportional to the firm-issued supply, experts hold a superior portfolio. Our main result that flows into passive funds raise disproportionately the stock prices of the largest firms in the economy does not require noise traders. The presence of noise traders strengthens that result and yields additional implications.

The index includes all firms or a subset of them. It is capitalization-weighted over the firms that it includes, i.e., weights them proportionately to their market capitalization. We refer to the included and the non-included firms as index and non-index firms, respectively. We denote by $\mathcal{I}$ the subset of index firms, by $\mathcal{I}^c$ its complement and by η'_n the number of shares of firm n included in the index. Since the index is capitalization-weighted over the firms that it includes, η'_n for $n \in \mathcal{I}$ is proportional to the number of shares η_n issued by firm n. By possibly rescaling the index, we set $\eta'_n = \eta_n$ for $n \in \mathcal{I}$. For $n \in \mathcal{I}^c$, $\eta'_n = 0$.

We denote by W_{1t} and W_{2t} the wealth of an expert and a non-expert, respectively, by z_{1nt} and z_{2nt} the number of shares of firm n that these agents hold, and by μ_1 and μ_2 these agents' measure. A non-expert thus holds $z_{2nt} = \lambda\eta'_n$ shares of firm n, where λ is a proportionality coefficient that the agent chooses optimally. We denote by $u_n < \eta_n$ the number of shares of firm n held by noise traders. The special case where noise traders are absent corresponds to $u_n = 0$ for all n.

Experts and non-experts born at time t are endowed with wealth W. Their budget constraint is

$$dW_{it} = \left(W - \sum_{n=1}^{N} z_{int}S_{nt}\right) rdt + \sum_{n=1}^{N} z_{int}(D_{nt}dt + dS_{nt}) = Wrdt + \sum_{n=1}^{N} z_{int}dR^{sh}_{nt}, \tag{2.7}$$

where dW_{it} is the infinitesimal change in wealth over their life, $i = 1$ for experts, and $i = 2$ for

funds with their higher fees, in the spirit of Grossman and Stiglitz (1980).

non-experts. They have mean-variance preferences over dW_{it}. These preferences can be derived from any VNM utility u, as can be seen from the second-order Taylor expansion

$$u(W + dW_{it}) = u(W) + u'(W)dW_{it} + \frac{1}{2}u''(W)dW_{it}^2 + o(dW_{it}^2). \tag{2.8}$$

Experts, who observe $\{D_{nt}\}_{n=1,..,N}$, maximize the conditional expectation of (2.8). This is equivalent to maximizing

$$\mathbb{E}_t(dW_{1t}) - \frac{\rho}{2}\mathbb{V}\text{ar}_t(dW_{1t}) \tag{2.9}$$

with $\rho = -\frac{u''(W)}{u'(W)}$, because infinitesimal dW_{1t} implies that $\mathbb{E}_t(dW_{1t}^2)$ is equal to $\mathbb{V}\text{ar}_t(dW_{1t})$ plus smaller-order terms. Non-experts, who do not observe $\{D_{nt}\}_{n=1,..,N}$, maximize the unconditional expectation of (2.8). This is equivalent to maximizing

$$\mathbb{E}(dW_{2t}) - \frac{\rho}{2}\mathbb{V}\text{ar}(dW_{2t}), \tag{2.10}$$

because infinitesimal dW_{2t} implies that $\mathbb{E}(dW_{2t}^2)$ is equal to $\mathbb{V}\text{ar}(dW_{2t})$ plus smaller-order terms.

3 Equilibrium

We look for an equilibrium where the price S_{nt} of stock n is

$$S_{nt} = \bar{S}_n + b_n S^s(D_t^s) + S_n^i(D_{nt}^i), \tag{3.1}$$

the sum of the present value $\bar{S}_n$ of dividends from the constant component, the present value $b_n S^s(D_t^s)$ of dividends from the systematic component, and the present value $S_n^i(D_{nt}^i)$ of dividends from the idiosyncratic component. Assuming that the functions $(S^s(D_t^s), S_n^i(D_{nt}^i))$ are twice continuously differentiable, we can write the share return dR_{nt}^{sh} of stock n as

$$\begin{aligned} dR_{nt}^{sh} &= (\bar{D}_n + b_n D_t^s + D_{nt}^i)dt + (b_n dS^s(D_t^s) + dS_n^i(D_{nt}^i)) - r\left(\bar{S}_n + b_n S^s(D_t^s) + S_n^i(D_{nt}^i)\right)dt \\ &= \mu_{nt}dt + b_n\sigma^s\sqrt{D_t^s}(S^s)'(D_t^s)dB_t^s + \sigma_n^i\sqrt{D_{nt}^i}(S_n^i)'(D_{nt}^i)dB_{nt}^i, \end{aligned} \tag{3.2}$$

where

$$\mu_{nt} \equiv \frac{\mathbb{E}_t(dR^{sh}_{nt})}{dt} = \bar{D}_n - r\bar{S}_n$$
$$+ b_n \left[D^s_t + \kappa^s(1-D^s_t)(S^s)'(D^s_t) + \frac{1}{2}(\sigma^s)^2 D^s_t (S^s)''(D^s_t) - rS^s(D^s_t) \right]$$
$$+ D^i_{nt} + \kappa^i_n(\bar{D}^i_n - D^i_{nt})(S^i_n)'(D^i_{nt}) + \frac{1}{2}(\sigma^i_n)^2 D^i_{nt}(S^i_n)''(D^i_{nt}) - rS^i_n(D^i_{nt}) \tag{3.3}$$

is the instantaneous expected share return of stock n, and the second step in (3.2) follows from (2.2), (2.3) and Ito's lemma.

Using (2.7) and (3.2), we can write the objective (2.9) of experts as

$$\sum_{n=1}^{N} z_{1nt}\mu_{nt} - \frac{\rho}{2}\left[\left(\sum_{n=1}^{N} z_{1nt} b_n \right)^2 (\sigma^s)^2 D^s_t [(S^s)'(D^s_t)]^2 + \sum_{n=1}^{N} z^2_{1nt} (\sigma^i_n)^2 D^i_{nt} [(S^i_n)'(D^i_{nt})]^2 \right]. \tag{3.4}$$

Experts maximize (3.4) over positions $\{z_{1nt}\}_{n=1,..,N}$. Their first-order condition is

$$\mu_{nt} = \rho \left[b_n \left(\sum_{m=1}^{N} z_{1mt} b_m \right) (\sigma^s)^2 D^s_t [(S^s)'(D^s_t)]^2 + z_{1nt} (\sigma^i_n)^2 D^i_{nt} [(S^i_n)'(D^i_{nt})]^2 \right]. \tag{3.5}$$

Equation (3.5) can be interpreted as a conditional CAPM. It equates the instantaneous expected share return μ_{nt} of stock n to the risk-aversion coefficient ρ times stock n's contribution to the instantaneous volatility of the return of the experts' portfolio.

Using (2.7), (3.2) and $z_{2nt} = \lambda\eta'_n$, we can write the objective (2.10) of non-experts as

$$\sum_{n=1}^{N} \lambda\eta'_n\mu_n - \frac{\rho}{2}\lambda^2 \left[\left(\sum_{n=1}^{N} \eta'_n b_n \right)^2 (\sigma^s)^2 \mathbb{E}\left[D^s_t [(S^s)'(D^s_t)]^2 \right] + \sum_{n=1}^{N} \left(\eta'_n\right)^2 (\sigma^i_n)^2 \mathbb{E}\left[D^i_{nt} [(S^i_n)'(D^i_{nt})]^2 \right] \right], \tag{3.6}$$

where $\mu_n \equiv \frac{\mathbb{E}(dR^{sh}_{nt})}{dt} = \mathbb{E}(\mu_{nt})$. Non-experts maximize (3.6) over λ. The first-order condition is

$$\sum_{n=1}^{N} \eta'_n\mu_n = \rho\lambda \left[\left(\sum_{n=1}^{N} \eta'_n b_n \right)^2 (\sigma^s)^2 \mathbb{E}\left[D^s_t [(S^s)'(D^s_t)]^2 \right] + \sum_{n=1}^{N} \left(\eta'_n\right)^2 (\sigma^i_n)^2 \mathbb{E}\left[D^i_{nt} [(S^i_n)'(D^i_{nt})]^2 \right] \right]. \tag{3.7}$$

Market clearing requires that the demand of experts, non-experts and noise traders equals the supply coming from the issuing firm:

$$\mu_1 z_{1nt} + \mu_2 \lambda \eta'_n + u_n = \eta_n. \tag{3.8}$$

Solving for $z_{1nt} = \frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1}$, and substituting into the first-order condition (3.5) of experts, we find

$$\mu_{nt} = \rho \left[b_n \left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s)^2 D^s_t [(S^s)'(D^s_t)]^2 + \frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma^i_n)^2 D^i_{nt} [(S^i_n)'(D^i_{nt})]^2 \right]. \tag{3.9}$$

We look for functions $(S^s(D^s_t), S^i_n(D^i_{nt}))$ that are affine in their arguments,

$$S^s(D^s_t) = a^s_0 + a^s_1 D^s_t, \tag{3.10}$$

$$S^i_n(D^i_{nt}) = a^i_{n0} + a^i_{n1} D^i_{nt}, \tag{3.11}$$

for positive constants $(a^s_0, a^s_1, \{a^i_{n0}, a^i_{n1}\}_{n=1,..,N})$. Substituting (3.3), (3.10) and (3.11) into (3.9), we can write (3.9) as

$$\begin{aligned} &\bar{D}_n - r\bar{S}_n + b_n \left[D^s_t + \kappa^s a^s_1 (1 - D^s_t) - r(a^s_0 + a^s_1 D^s_t) \right] + D^i_{nt} + \kappa^i_n a^i_{n1} (\bar{D}^i_n - D^i_{nt}) - r(a^i_{n0} + a^i_{n1} D^i_{nt}) \\ &= \rho \left[b_n \left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s a^s_1)^2 D^s_t + \frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma^i_n a^i_{n1})^2 D^i_{nt} \right]. \end{aligned} \tag{3.12}$$

Identifying terms in D^s_t yields a quadratic equation that determines a^s_1. Identifying terms in D^i_{nt} yields a quadratic equation that determines a^i_{n1}. Identifying the remaining terms yields $\bar{S}_n + b_n a^s_0 + a^i_{n0}$. Substituting $(a^s_1, \{a^i_{n1}\}_{n=1,..,N})$ into the first-order condition (3.7) of non-experts yields an equation for λ, whose solution completes our characterization of the equilibrium. Proposition 3.1 characterizes the equilibrium. The proposition's proof is in Appendix B, where the proofs of all analytical results in this paper are gathered.

Proposition 3.1. *In equilibrium, the price of stock n is*

$$S_{nt} = \frac{\bar{D}_n + b_n \kappa^s a_1^s + \kappa_n^i a_{n1}^i \bar{D}_n^i}{r} + b_n a_1^s D_t^s + a_{n1}^i D_{nt}^i, \tag{3.13}$$

where

$$a_1^s = \frac{2}{r + \kappa^s + \sqrt{(r+\kappa^s)^2 + 4\rho \left(\sum_{m=1}^N \frac{\eta_m - \mu_2 \lambda \eta_m' - u_m}{\mu_1} b_m\right)(\sigma^s)^2}}, \tag{3.14}$$

$$a_{n1}^i = \frac{2}{r + \kappa_n^i + \sqrt{(r+\kappa_n^i)^2 + 4\rho \frac{\eta_n - \mu_2 \lambda \eta_n' - u_n}{\mu_1}(\sigma_n^i)^2}}, \tag{3.15}$$

and $\lambda > 0$ solves

$$\begin{aligned}&\left(\sum_{m=1}^N \eta_m' b_m\right)\left(\sum_{m=1}^N (\eta_m - u_m) b_m\right)(\sigma^s a_1^s)^2 + \sum_{m=1}^N \eta_m'(\eta_m - u_m)(\sigma_m^i a_{m1}^i)^2 \bar{D}_m^i \\ &= \lambda(\mu_1 + \mu_2)\left[\left(\sum_{m=1}^N \eta_m' b_m\right)^2 (\sigma^s a_1^s)^2 + \sum_{m=1}^N \left(\eta_m'\right)^2 (\sigma_m^i a_{m1}^i)^2 \bar{D}_m^i\right].\end{aligned} \tag{3.16}$$

The price depends on $(\mu_1, \mu_2, \sigma^s, \{b_m, \sigma_m^i, \eta_m, \eta_m', u_m\}_{m=1,..,M})$ only through $\left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m\right)(\sigma^s)^2$ and $\frac{\eta_n - \mu_2\lambda\eta_n' - u_n}{\mu_1}(\sigma_n^i)^2$, and is decreasing and convex in the latter two variables.

The price of stock n depends on two measures of supply: systematic supply and idiosyncratic supply. Systematic supply is $\left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m\right)(\sigma^s)^2$, the aggregate risk-adjusted supply of all stocks that each expert holds in equilibrium. The supply of stock m held by all experts combined is equal to the supply η_m coming from the issuing firm, minus the demand $\mu_2\lambda\eta_m'$ and u_m coming from non-experts and noise traders, respectively. It is expressed in per-expert terms by dividing by the measure μ_1 of experts, is risk-adjusted by multiplying by the factor loading b_m of stock m and by the diffusion parameter σ^s of the systematic factor, and is aggregated across all stocks. Idiosyncratic supply is $\frac{\eta_n - \mu_2\lambda\eta_n' - u_n}{\mu_1}(\sigma_n^i)^2$, the risk-adjusted supply of stock n that each expert holds in equilibrium. Risk adjustment is made by multiplying by the diffusion parameter σ_n^i of the idiosyncratic component of stock n's dividends.

A reduction in systematic or idiosyncratic supply raises the price of stock n. This is the usual risk-premium channel. A reduction in systematic or idiosyncratic supply also renders the price of stock n more sensitive, in absolute terms, to shocks to the respective component of dividends. Thus, not only the price per share goes up, but also the volatility of the changes in the price per share goes up. This natural link between price level and volatility is key for our main results in Section 5. It does not arise in CARA-normal models, in which changes in supply affect the price per share but not the price volatility per share. It arises in our model because under the squared-root process for dividends, the volatility of changes in dividends per share increases with the level of dividends per share.

The intuition is as follows. A positive shock to dividends raises not only the expectation of future dividends per share but also their volatility. If the supply held by experts is positive, i.e., experts hold a long position, then the increase in volatility makes them more willing to unwind their position by selling stock n. This results in a smaller price increase, and thus in lower price volatility, compared to the case where supply is zero. If supply is negative, i.e., experts hold a short position, then the increase in volatility makes them more willing to unwind their position by buying stock n. This results in a larger price increase, and thus in higher volatility, compared to the case where supply is zero.

4 Calibration

The model parameters are the riskless rate r, the number N of firms, the parameters $(\kappa^s, \bar{D}^s, \sigma^s)$ and $(b_n, \kappa^i_n, \bar{D}^i_n, \sigma^i_n)_{n=1,..,N}$ of the dividend processes, the supply parameters $(\eta_n, \eta'_n, u_n)_{n=1,..,N}$, the measures (μ_1, μ_2) of experts and non-experts, and the risk-aversion coefficient ρ.

We set the sum $\mu_1 + \mu_2$ to one in the baseline. This is a normalization because we can redefine ρ. We set ρ to one. This is also a normalization because we can redefine the numeraire in the units of which wealth is expressed. Since the dividend flow is normalized by $\bar{D}_n + b_n + \bar{D}^i_n = 1$, redefining the numeraire amounts to rescaling the numbers of shares $(\eta_n, \eta'_n, u_n)_{n=1,..,N}$. We set the riskless rate r to 3%.

We assume that in the baseline $\mu_1 = 0.9$ and $\mu_2 = 0.1$, i.e., non-experts hold 10% of total wealth. We examine how stock prices change when μ_2 is raised to 0.6, i.e., non-experts' wealth rises six-fold.

We consider two polar cases for experts' wealth. The first polar case is when flows into passive funds are entirely due to entry by new investors into the stock market. In that case, experts' wealth does not change and μ_1 remains equal to 0.9. Non-experts' wealth becomes two-thirds ($\frac{0.6}{0.9}$) of experts' wealth, and total investable wealth rises by 50% ($\frac{0.9+0.6}{1}$). The second polar case is when flows into passive funds are entirely due to a switch by investors from active to passive. In that case, total investable wealth does not change and $\mu_1 + \mu_2$ remains equal to one. Non-experts' wealth becomes 50% larger than experts' wealth ($\frac{0.6}{0.4}$). We can derive all cases in-between the two polar cases by setting $\mu_1 = 0.9 - \zeta \times 0.5$, where $\zeta \in [0, 1]$ is equal to zero in the first polar case and to one in the second polar case.

We calibrate the number N of firms and the number η_n of shares that they issue based on the number and size distribution of publicly listed US firms. Axtell (2001) finds that the size distribution of all US firms, with size measured by sales or number of employees, is well approximated by a power law with exponent one.[9] Under that power law, if an interval $[x, \phi x]$ with $\phi > 1$ includes a fraction f of firms and their average size is s, then the adjacent interval $\left[\phi x, \phi^2 x\right]$ includes a fraction $\frac{f}{\phi}$ of firms and their average size is ϕs. Motivated by this scaling property, we set $\phi = 5$ and assume five size groups. Size group 5, the top group, includes six firms, each of which issues $625 \times \eta$ shares. Size group 4 includes 30 ($= 5 \times 6$) firms, each of which issues $125 \times \eta$ ($= \frac{1}{5} \times 625 \times \eta$) shares. Size group 3 includes 150 ($= 5 \times 30$) firms, each of which issues $25 \times \eta$ ($= \frac{1}{5} \times 125 \times \eta$) shares. Size group 2 includes 750 ($= 5 \times 150$) firms, each of which issues $5 \times \eta$ ($= \frac{1}{5} \times 25 \times \eta$) shares. Size group 1, the bottom group, includes 750 firms, each of which issues η ($= \frac{1}{5} \times 5 \times \eta$) shares. We do not assume the scaling property for size group 1 to better fit the size distribution of publicly listed US firms, with size measured by market capitalization (number of shares times price). The ratio of market capitalization between two firms in consecutive size groups is close to five, both in the data and in the model. If size group 1 is enlarged to include smaller firms, then the ratio of market capitalization between two firms in size groups 2 and 1 becomes significantly larger than five. The firms in our size groups 1, 2, 3, 4 and 5 account for more than 98.5% of the market capitalization of all publicly listed US firms.[10]

[9] For a survey on power laws and their relevance to Economics, see Gabaix (2016).

[10] As of 2 April 2024, average market capitalization was \$2.175tn for the top six US firms (Microsoft, Apple, NVIDIA, Alphabet, Amazon, Meta), \$379.2bn for the next 30 firms, \$94.14bn for the next 150 firms, \$16.06bn for

We consider two cases for noise-trader demand u_n. The baseline is that u_n is equal to zero for all firms and thus there are no noise traders. The second case is that u_n is equal to zero for one-half of the firms in each size group, and to 30% of the shares issued for the remaining half ($u_n = 30\% \times \eta_n$). The former firms are the low-demand ones and the latter firms are the high-demand ones.

We consider two cases for index composition. The baseline is that the index includes all firms and is thus the true market portfolio, i.e., $\eta'_n = \eta_n$ for all n. The second case is that the index includes only the firms in our top three size groups, i.e., $\eta'_n = \eta_n$ for the 186 firms in size groups 3, 4 and 5, and $\eta'_n = 0$ for the 1500 firms in size groups 1 and 2. Under the second assumption, the index is a large-firm index such as the Russell 200 or the S&P500.[11]

We set the mean-reversion parameters κ^s and $\{\kappa^i_n\}_{n=1,..,N}$ to a common value κ. We set the long-run means $\{\bar{D}^i_n\}_{n=1,..,N}$ and diffusion parameters $\{\sigma^i_n\}_{n=1,..,N}$ of the idiosyncratic components to common values $\bar{D}^i$ and σ^i, respectively. The stationary distribution of D^i_{nt} generated by the square-root process (2.3) is gamma with support $(0, \infty)$ and density

$$f(D^i_{nt}) = \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i - 1} e^{-\beta^i D^i_{nt}}, \tag{4.1}$$

where $\alpha^i \equiv \frac{2\kappa \bar{D}^i}{(\sigma^i)^2}$, $\beta^i \equiv \frac{2\kappa}{(\sigma^i)^2}$ and Γ is the Gamma function. The stationary distribution of D^s_t generated by the square-root process (2.2) is also gamma, with density given by (4.1) in which D^i_{nt} is replaced by D^s_t, α^i by $\alpha^s \equiv \frac{2\kappa \bar{D}^s}{(\sigma^s)^2} = \frac{2\kappa}{(\sigma^s)^2}$, and β^i by $\beta^s \equiv \frac{2\kappa}{(\sigma^s)^2}$. We set $\frac{\sigma^i}{\sqrt{\bar{D}^i}} = \frac{\sigma^s}{\sqrt{\bar{D}^s}} = \sigma^s$. This ensures that the distributions of D^s_t and D^i_{nt} are the same when scaled by their long-run means: $\frac{D^i_{nt}}{\bar{D}^i}$ has the same distribution as $\frac{D^s_t}{\bar{D}^s} = D^s_t$.

We allow for correlation between size and systematic risk. We assume that the value of the loading b_n on the systematic factor for firms in size group $m = 1, .., 5$ is $b_n = \bar{b} - (m-3)\Delta b \geq 0$.

the next 750 firms and \$2.887bn for the next 750 firms. The combined market capitalization of all 3605 publicly listed US firms was \$53.54tn. The combined market capitalization of the 1686 (=6+30+150+750+750) firms in our size groups 1, 2, 3, 4 and 5 was \$52.76tn. The market capitalization ratios are 5.74 ($= \frac{2175}{379.2}$) between size groups 5 and 4, 4.03 ($= \frac{379.2}{94.14}$) between size groups 4 and 3, 5.86 ($= \frac{94.14}{16.06}$) between size groups 3 and 2, and 5.56 ($= \frac{16.06}{2.887}$) between size groups 2 and 1. The counterparts of these ratios generated by our model are 4.92, 5.19, 5.55 and 5.79 in the baseline. If size group 1 is enlarged to include the remaining 1919 (=3605-1686) publicly listed firms, then the ratio between size groups 2 and 1 in the data jumps up to 14.53. All market capitalization data come from `https://companiesmarketcap.com/usa/largest-companies-in-the-usa-by-market-cap/`.

[11]While the S&P500 accounts for a larger fraction of market capitalization than an index made of the firms in our size groups 3, 4 and 5, it leaves out a non-negligible fraction. As of 2 April 2024, the S&P500 accounted for 81.5% of the combined market capitalization of all publicly listed US firms. Our size groups 3, 4 and 5 accounted for 72.0%.

The relationship between size and systematic risk is negative when Δb is positive, and vice-versa.

The parameters left to calibrate are $(\kappa, \bar{D}^i, \bar{b}, \Delta b, \sigma^s, \eta)$. We calibrate them based on stocks' expected return, return variance, CAPM beta and CAPM R-squared (fraction of return variance explained by index movements). We compute unconditional versions of these moments. We use the values of the moments in the baseline as calibration targets. The formulas for the moments are in Appendix C and the values of the moments in the baseline are in Table 1.

The effects of changing κ on return moments and other numerical results are similar to those of changing the remaining parameters. We set $\kappa = 4\%$.

The values of $(\bar{D}^i, \bar{b}, \Delta b)$ must satisfy $\bar{b} + (m-3)\Delta b + \bar{D}^i \leq 1$ for all $m = 1, .., 5$ because of $\bar{D}_n \geq 0$ and the normalization $\bar{D}_n + b_n + \bar{D}^i = 1$. Inequality $\bar{b} + (m-3)\Delta b + \bar{D}^i \leq 1$ for all $m = 1, .., 5$ is equivalent to $\bar{b} + 2|\Delta b| + \bar{D}^i \leq 1$. We assume that the latter inequality holds as an equality for the firms with largest b_n. This minimizes the constant component $\bar{D}_n \geq 0$ (which becomes zero for the largest b_n firms). Minimizing $\bar{D}_n$ maximizes return variances, bringing them closer to their empirical counterparts as we explain below.

We choose Δb to be positive, consistent with the empirical negative relationship between size and CAPM beta (Fama and French (1992)). We set $\Delta b = 0.025$, which generates a CAPM beta of 1.28 for the firms in size group 1 and 0.94 for the firms in size group 5. These values align closely with the data: constructing the same size groups in the data as in our model, we find that average CAPM beta is 1.27 for size group 1 and 0.92 for size group 5 when firms within groups are weighed equally, and is 1.26 for size group 1 and 0.93 for size group 5 when firms within groups are weighted according to their market capitalization.[12] Our results become significantly stronger when Δb takes zero or negative values, as we explain in Section 5.

We determine the relative size of $\bar{b}$ and $\bar{D}^i$ based on CAPM R-squared. We set $\bar{b} = 0.85$ and $\bar{D}^i = 0.10$, which generates a CAPM R-squared that averages to 22.19% across the firms in all size groups, and to 28.68% when the firms are weighted according to their market capitalization. The counterparts of these R-squared in the data are 25.79% and 29.71%.[13] Our results become stronger

[12]In each quarter during the sample period of our empirical exercise in Section 6, we sort the 1686 largest firms into five size groups as in our model. We regress the quarterly equal- and value-weighted excess returns of the resulting five portfolios on the excess return of the market (CRSP index) to compute equal- and value-weighted CAPM betas.

[13]We construct the five size groups as in the CAPM beta exercise, compute R-squared for each firm from a CAPM regression with monthly returns and a five-year lookback window, and average across firms using equal weights or

when R-squared takes smaller values, through smaller $\bar{b}$ and larger $\bar{D}^i$.

We determine the supply parameter η based on stocks' expected returns (in excess of the riskless rate). We set $\eta = 0.00005$, which generates expected returns across size groups that lie between 4-6%. Expected return ranges from 5.61% for the firms in size group 1 to 4.23% for the firms in size group 5.

We determine the diffusion parameter σ^s based on stocks' return variances. Raising σ^s (and σ^i through $\frac{\sigma^i}{\sqrt{\bar{D}^i}} = \sigma^s$) has a non-monotone effect on variances. For given values of D_t^s and $\{D_{nt}^i\}_{n=1,..,N}$, variances rise. At the same time, the stationary distributions of D_t^s and $\{D_{nt}^i\}_{n=1,..,N}$ shift more weight towards very small or very large values, for which variances are low. We choose σ^s to maximize return variances. Return volatility ranges from 21.12% for firms in size group 1 to 11.13% for firms in size group 5. These values are about one-half of their counterparts in the data. The discrepancy is partly due to discount-rate shocks in our model being perfectly correlated with cashflow shocks and attenuating them. (Following a positive shock to the systematic component of dividends, the instantaneous variance of changes in dividends per share increases, rendering experts less willing to hold the stocks.)

5 Passive Flows and Stock Prices

We show our main results on how flows into passive funds affect stock prices in the context of the calibration described in Section 4. In Appendix A we derive analytical results for more general parameter values, which parallel the numerical results shown in this section.

5.1 No Noise Traders

Table 1 shows price and return moments in the baseline, in which there are no noise traders, the index includes all firms, and non-experts hold 10% of total wealth. We report the unconditional average of the price and the unconditional moments of returns. Since the price is linear in D_t^s and D_{nt}^i, we can compute its unconditional average by setting the systematic component D_t^s and the idiosyncratic component D_{nt}^i of dividends to their long-run means, $\bar{D}^s = 1$ and $\bar{D}_n^i$.

market-capitalization weights.

When moving from the smallest to the largest size group, expected return, return volatility, and CAPM beta decline, while price rises. The decline in CAPM beta is built into our calibration because we set Δb to a positive value. The decline in expected return and rise in price reflect the decline in CAPM beta because without noise traders the conditional CAPM holds in our model. The decline in return volatility reflects partly the decline in CAPM beta. It also reflects that shocks to the idiosyncratic component of dividends have larger effects on the stock prices of small firms. This is because Proposition 3.1 implies that the price is more sensitive to idiosyncratic dividend shocks when idiosyncratic supply is small. Because idiosyncratic dividend shocks have larger effects on the stock prices of small firms, CAPM R-squared rises when moving from small to large firms, consistent with the empirical evidence.

Table 1: Price and Return Moments.

Size Group	**Price**	**Expected Return** (%)	**Return Volatility** (%)	**CAPM Beta**	**CAPM** R^2 (%)
1 (Smallest)	4.72	5.61	21.12	1.28	22.01
2	5.47	4.95	18.18	1.10	21.89
3	6.08	4.49	15.93	0.98	22.67
4	6.32	4.26	13.71	0.93	27.53
5 (Largest)	6.23	4.23	11.13	0.94	42.68

Table 2 shows how flows into passive funds affect stock prices. We compute the percentage change in the unconditional average of the price. Computing instead the unconditional average of the percentage change in the price yields similar results. We use the percentage change in the unconditional average of the price because the equations ((5.3) and (5.4)) are simpler. The percentage price change is the realized return.

The second and third columns of Table 2 report the percentage price change when μ_2 is raised to 0.6 and μ_1 is held equal to 0.9. This corresponds to entry by new investors into the stock market through passive funds. The second column assumes that the index includes all firms, and the third column assumes that only size groups 3, 4 and 5 are included. The fourth and fifth columns are counterparts of the second and third columns when μ_2 is raised to 0.6 and μ_1 is lowered to 0.4.

This corresponds to a switch by investors from active to passive.

Table 2: Percentage Price Change Following Flows into Passive Funds.

Size Group	**Entry into the Stock Market**		**Switch from Active to Passive**	
	All Firms in Index	**Size Groups 3-5 in Index**	**All Firms in Index**	**Size Groups 3-5 in Index**
1 (Smallest)	6.55	6.34	0	-0.75
2	5.74	5.32	0	-1.62
3	5.92	6.35	0	1.70
4	7.36	8.85	0	5.20
5 (Largest)	8.03	10.39	0	7.45

Table 2 shows our main results. Consider first the case where flows into passive funds are due to entry and where the index includes all firms. As shown in the second column of Table 2, stock prices increase. Moreover, the effect is J-shaped with size: the percentage price increase becomes smaller when moving from size group 1 to size group 2, becomes larger when moving from size group 2 to size group 5, and is largest for size group 5.

To explain the intuition for the J-shape, we return to the general formulas derived in Section 3. Passive flows amount to raising $\mu_2\lambda$ (the measure μ_2 of non-experts times the fraction λ of the index that they hold). Equation (3.1) implies that the percentage price change of stock n due to passive flows is

$$\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial(\mu_2\lambda)} = \frac{b_n\frac{\partial S^s(\bar{D}^s)}{\partial(\mu_2\lambda)} + \frac{\partial S^i_n(\bar{D}^i_n)}{\partial(\mu_2\lambda)}}{\bar{S}_n + b_n S^s(\bar{D}^s) + S^i_n(\bar{D}^i_n)}. \tag{5.1}$$

For small and medium-size firms (size groups 1 and 2), the present value $S^i_n(\bar{D}^i_n)$ of the idiosyncratic component of dividends is almost insensitive to passive flows, i.e., $\frac{\partial S^i_n(\bar{D}^i_n)}{\partial(\mu_2\lambda)} \approx 0$. Indeed, since shocks to idiosyncratic dividends of small and medium-size firms account for a negligible fraction of market movements, these dividends are discounted at the riskless rate r independently of passive flows. Passive flows affect small and medium-size firms because they raise the present value $b_n S^s(\bar{D}^s)$ of the systematic component of dividends. Since that present value rises more for firms with higher b_n and thus with higher CAPM beta, and since b_n decreases with size ($\Delta b > 0$), (5.1) implies that passive flows have a smaller effect on the stocks of medium-size than of small firms. This explains the decreasing part of the J-shape. It also explains why our results become significantly stronger when b_n is independent of size ($\Delta b = 0$) or increases with size ($\Delta b < 0$), as in those cases the effect causing the decreasing part of the J-shape disappears or changes sign. The explanation for the increasing part of the J-shape is that since shocks to idiosyncratic dividends of large firms account for a non-negligible fraction of market movements, these dividends are discounted at a rate higher than r. Passive flows lower that discount rate, thus raising the present value $S_n^i(\bar{D}_n^i)$ of idiosyncratic dividends. Since that effect is absent for medium-size firms, (5.1) implies that passive flows can have a smaller effect on the stocks of medium-size than of large firms.

The above explanation leaves two questions open. First, why is the effect of passive flows on the present value of idiosyncratic dividends of the stocks of large firms so sizeable as to overcome the effect of CAPM beta? Second, since idiosyncratic dividends of stocks of large firms contribute to these stocks' CAPM beta, why is the effect of passive flows not subsumed by beta? In particular, why is the effect of passive flows J-shaped with size, while beta decreases with size? According to the basic CAPM logic in the Introduction, the effect of passive flows should depend only on beta and be an increasing function of it.[14]

To answer both questions, we distinguish between a partial effect of passive flows that holds price volatility constant, and a total effect that includes the change in volatility. We compute the partial effect on the price of stock n by calculating how (a_1^s, a_{n1}^i) in the left-hand side of (3.12) change when $\mu_2\lambda$ changes and (a_1^s, a_{n1}^i) in the right-hand side remains constant. This yields the partial effect because the left-hand side of (3.12) corresponds to expected return and the right-hand

[14] Formally, consider a two-period CAPM world, in which stock n pays expected dividend $\bar{D}_n$ and has CAPM beta β_n. The stock's expected return is $r + \beta_n\text{MRP}$, where r and MRP are the riskless rate and market risk premium, respectively. The price of stock n is

$$S_n = \frac{\bar{D}_n}{1 + r + \beta_n\text{MRP}}.$$

Since flows into passive funds lower MRP, their effect is proportional in the cross-section to

$$\frac{1}{S_n}\frac{\partial S_n}{\partial(-\text{MRP})} = \frac{\beta_n}{1 + r + \beta_n\text{MRP}}, \tag{5.2}$$

and is increasing in β_n.

side to volatility. Using (3.1), (3.12) and (3.13), we find

$$\frac{1}{S_{nt}}\left.\frac{\partial S_{nt}}{\partial(\mu_2\lambda)}\right|_{\substack{\text{constant}\\\text{volatility}}} = \frac{\rho}{\mu_1 S_{nt}}\left[b_n\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2\bar{D}^s + \eta'_n(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i\right]$$
$$= \frac{\rho}{\mu_1}\mathbb{C}\text{ov}_t\left(dR_{nt}, \sum_{m=1}^N \eta'_m dR_{mt}^{sh}\right), \tag{5.3}$$

where the second step follows from (3.2), (3.10) and (3.11), and the covariance is evaluated for $(D_t^s, D_{nt}^i) = (\bar{D}^s, \bar{D}_n^i)$. Equation (5.3) shows that the partial effect of passive flows is equal to the covariance between the return of the stock n and the return of the index. That covariance is, in turn, proportional to stock n's conditional CAPM beta. The partial effect of passive flows thus depends only on beta, as per the simple CAPM argument in the Introduction.

The partial effect of passive flows can be further decomposed into an effect due to the reduction in systematic supply and an effect due to the reduction in idiosyncratic supply. The reduction in systematic supply raises the present value $b_n S^s(\bar{D}^s)$ of the systematic component of dividends, with the effect being proportional to the covariance between stock n and the index that arises because of that component. Likewise, the reduction in idiosyncratic supply raises the present value $S_n^i(\bar{D}_n^i)$ of the idiosyncratic component of dividends, with the effect being proportional to the covariance arising because of that component. The systematic and idiosyncratic covariance correspond to the first and second term, respectively, in the square bracket in (5.3). The idiosyncratic covariance is much smaller than the systematic covariance, even for the largest firms: in Table 2, it is smaller by a factor of approximately fifteen for the firms in size group 5.

We next turn to the total effect of passive flows, which includes the change in volatility. We compute the total effect on the price of stock n by allowing (a_1^s, a_{n1}^i) in the left-hand side of (3.12) to change when $\mu_2\lambda$ changes. Using (3.13)-(3.15), we find

$$\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial(\mu_2\lambda)} = \frac{\rho}{\mu_1 S_{nt} r}\left[\frac{b_n\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{\sqrt{1+\frac{4\rho(\sigma^s)^2}{(r+\kappa^s)^2}\left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1}b_m\right)}} + \frac{\eta'_n(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i}{\sqrt{1+\frac{4\rho(\sigma_n^i)^2}{(r+\kappa_n^i)^2}\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}}}\right]. \tag{5.4}$$

The systematic and the idiosyncratic covariance are present in the numerator of the first and second term, respectively, in the square bracket in (5.4). They receive different weights, however, as shown in the denominator, with the weight given to the idiosyncratic covariance being larger. This explains formally why the effect of passive flows is not subsumed into CAPM beta, and why the flows' effect on the present value of idiosyncratic dividends can overcome the effect of beta.

The intuition is as follows. Since passive flows reduce systematic and idiosyncratic supply, they render the price of stock n more sensitive to dividend shocks (Proposition 3.1). The resulting increase in stock n's volatility lowers the experts' willingness to hold the stock and attenuates the stock's price rise. Crucially, the attenuation effect is weaker for idiosyncratic supply than for systematic supply. This is because the increase in idiosyncratic price volatility pertains to a long position in one stock rather than in the aggregate market. Because of the weaker attenuation, the weighted idiosyncratic covariance in Table 2 is smaller than the weighted systematic covariance by a factor of only 1.2 for the stocks in size group 5.

Consider next the case where the index includes only firms in size groups 3, 4 and 5 (and flows into passive funds are still due to entry by new investors into the stock market). As shown in the third column of Table 2, the percentage price increase remains J-shaped when moving across size groups. Relative to the case where the index includes all stocks, the effect rises more sharply with size when moving from size group 2 to size group 5. This is because non-experts establish larger positions in the more restricted set of stocks, causing the reduction in idiosyncratic supply to be larger.

Consider finally the case where flows into passive funds are due to a switch from active to passive. When the index includes all firms, stock prices do not change. This is because experts and non-experts hold the same portfolio, which is the index. When instead the index includes only firms in size groups 3, 4 and 5, prices drop for size groups 1 and 2, and rise for size groups 3, 4 and especially 5. Moreover, the effect is asymmetric in the sense that the price rises exceed the price drops in absolute value, and the aggregate market rises. The asymmetry is surprising. Indeed, non-experts hold a portfolio (the index) that approximates the aggregate market, and are equally risk-averse as experts. Therefore, a substitution of experts by non-experts should have almost no effect on the exposure of each expert to the aggregate market. As a result, the compensation that experts require to hold aggregate-market risk should remain approximately the same, and the aggregate market should not rise.

The aggregate market rises for the same reason as why the effect of passive flows is not subsumed by CAPM beta. A switch from active to passive raises the present value of idiosyncratic dividends for firms in size groups 3, 4 and especially 5, and the effect is larger than the contribution of idiosyncratic covariance to beta. Therefore, even though the exposure of each expert to the aggregate market remains approximately the same, the aggregate market rises.

Flows into passive funds affect not only stock prices but also return volatilities. Volatilities, shown in Table 3, do not change when the index includes all firms and flows into passive funds are due to a switch from active to passive. In all other cases, volatilities do not change for size groups 1 and 2, but rise for size groups 3, 4 and especially 5. The volatility of the aggregate market rises as well in these cases. As we show in Appendix A (Proposition A.2), key to the rise in the return volatility of large firms is that passive flows render their stock prices more sensitive to shocks to the idiosyncratic component of dividends, while price sensitivity does not change for small firms.

Table 3: Change in Return Volatility Following Flows into Passive Funds.

Size Group	**Baseline Return Volatility**	**Change in Return Volatility**			
		Entry into the Stock Market		**Switch from Active to Passive**	
		All Firms in Index	**Size Groups 3-5 in Index**	**All Firms in Index**	**Size Groups 3-5 in Index**
1 (Smallest)	21.12	-0.04	-0.04	0	0
2	18.18	0.12	0.11	0	-0.05
3	15.93	0.24	0.26	0	0.09
4	13.71	0.47	0.56	0	0.38
5 (Largest)	11.13	0.72	0.91	0	0.71

5.2 Noise Traders

Table 4 is the counterpart of Table 1 with noise traders. Firms within each size group are split equally across those without noise traders and those for which noise traders hold 30% of the shares issued. This yields ten groups of firms. The effects across size groups are similar to those in Table

1. The effects within size groups depend on size. Within size groups 1 and 2, price, expected return and volatility are independent of noise-trader demand. By contrast, within size groups 3, 4 and especially 5, price and volatility rise and expected return declines when moving from low to high noise-trader demand.

Table 4: Price and Return Moments with Noise Traders.

Size Group	**Noise-Trader Demand**	**Price**	**Expected Return (%)**	**Return Volatility (%)**	**Market Beta**	**CAPM R^2 (%)**
1 (Smallest)	Low	4.84	5.29	21.11	1.28	23.59
	High	4.84	5.29	21.11	1.28	23.57
2	Low	5.59	4.69	18.22	1.10	23.52
	High	5.60	4.68	18.23	1.10	23.42
3	Low	6.19	4.27	16.00	0.99	24.38
	High	6.26	4.24	16.06	0.98	23.92
4	Low	6.42	4.07	13.81	0.94	29.41
	High	6.60	4.00	14.01	0.93	27.96
5 (Largest)	Low	6.32	4.05	11.27	0.94	44.61
	High	6.54	3.95	11.61	0.94	42.13

Table 4 implies that the risk-return relationship is positive across size groups but is negative within size groups involving large firms. The negative risk-return relationship is driven by noise-trader demand, which affects a firm's stock price through the present value of the idiosyncratic component of dividends. High demand lowers idiosyncratic supply, raising the present value and the price, and lowering expected return. High demand also raises return volatility because it renders the price more sensitive to shocks to idiosyncratic dividends. The effects of demand are present only for large firms because idiosyncratic dividends for small firms are discounted at the riskless rate r regardless of demand.

Table 5 is the counterpart of Table 2 with noise traders. When flows into passive funds are due to entry by new investors into the stock market, their effect varies across size groups in a manner similar to Table 2. The effect within size groups 1 and 2 is independent of noise-trader demand. By contrast, within size groups 3, 4 and especially 5, flows have a larger effect on the stock prices of high-demand firms. The partial effect that flows have holding volatility constant does not depend on noise-trader demand. (CAPM beta is approximately independent of demand.) Demand influences instead the total effect of flows, which includes the change in volatility. Because stocks of high-demand firms are in low idiosyncratic supply, the attenuation effect caused by the increase in price sensitivity is weaker.

Table 5: Percentage Price Change Following Flows into Passive Funds with Noise Traders.

Size Group	**Noise-Trader Demand**	**Entry into the Stock Market**		**Switch from Active to Passive**	
		All Firms in Index	**Size Groups 3-5 in Index**	**All Firms in Index**	**Size Groups 3-5 in Index**
1 (Smallest)	Low	6.88	6.67	-0.07	-0.99
	High	6.87	6.67	0.03	-0.90
2	Low	6.00	5.62	-0.21	-1.80
	High	5.99	5.61	0.18	-1.45
3	Low	6.00	6.32	-0.69	0.32
	High	6.03	6.39	0.75	2.12
4	Low	7.08	8.20	-1.49	1.55
	High	7.63	9.21	2.14	8.80
5 (Largest)	Low	7.54	9.20	-1.63	2.02
	High	9.13	12.49	3.14	22.59

When flows into passive funds are due to a switch by investors from active to passive, they affect prices even in the case where the index includes all firms. Stock prices drop for low-demand firms and rise for high-demand firms. Moreover, the effect is asymmetric in the sense that the price rises exceed the price drops in absolute value, within size groups 3, 4 and 5, and across the aggregate market. Stocks in low noise-trader demand drop in price because they are undervalued and attractive to experts, so a substitution of experts by non-experts lowers their net demand. Conversely, stocks in high noise-trader demand rise because they are unattractive to experts, so a substitution raises their net demand. The asymmetry arises for a similar reason as in Table 2. A switch from active to passive raises the present value of idiosyncratic dividends of stocks of high-demand firms in size groups 3, 4 and especially 5. Experts either hold a small long position in these stocks, in which case the attenuation effect is weak, or a short position, in which case attenuation turns into amplification. In our calibrated example, amplification arises in the case where the index includes only size groups 3, 4 and 5. That case corresponds to the fifth column in

Table 5, which shows a particularly large price rise for stocks of high-demand firms in size group 5.

Using Tables 2 and 5, we can compare the size-dependent effect of passive flows to the effect that flows raise the stock prices of index firms relative to non-index firms (Harris and Gurel (1986), Shleifer (1986)). We focus on the case where the index includes only size groups 3, 4 and 5, and compute the within-index effect, defined as the average stock price rise for firms in size group 5 minus that for firms in size groups 3 and 4, to the across-index effect, defined as the average stock price rise for firms in size groups 3, 4 and 5 minus that for firms in size groups 1 and 2. When passive flows are due to entry, the within-index effect always exceeds the across-index effect, and can be twice as large (Table 5, all firms in index). When passive flows are due to a switch from active to passive, the within-index effect ranges from two-thirds of the across-index effect (Table 2, size groups 3-5 in index) to one-and-a-half times it (Table 5, all firms in index).

The effect of passive flows on the aggregate market in Tables 2 and 5 translates into a demand elasticity that is at the high end of the estimates in the literature. Suppose that the measure μ_2 of non-experts increases from 0.1 to 0.6, holding the measure μ_1 of experts equal to 0.9. When there are no noise traders and the index includes all firms, the aggregate market rises by 6.79% and non-experts' holdings (equal to $\mu_2\lambda$ times the value of the market) increase by 27.86% of the market's initial value. The resulting elasticity is 4.00 ($=\frac{27.86}{6.79}$). Allowing for noise traders or for a narrow index yields similar elasticities. By contrast, Gabaix and Koijen (2021) estimate an elasticity of 0.2 for the aggregate market, while the literature on index additions estimates elasticities ranging from 0.4 to 4 for individual firms. Our model might be generating an elasticity at the high end of these estimates for two reasons. First, the fraction of truly active investors might be smaller than in our calibration because many active funds in practice have constraints limiting their deviations from benchmark indices. Second, the elasticity estimates in the literature mostly concern short-run elasticities, while the elasticities in our model are long-run.

5.3 Index Additions

We next compute the change in a firm's stock price and return volatility when the firm is added to the index. Passive flows in that case are only into that firm, while passive flows in Sections 5.1 and 5.2 are into each firm in the index. Table 6 reports the percentage price change of stock n and the change in the stock's return volatility when firm n is added to the index. We assume that there are noise traders, the measure of experts is 0.9 and the measure of non-experts is 0.6. We consider both the case where the index before the addition includes all firms except firm n, and the case where the index before the addition includes all firms in size groups 3, 4 and 5 except firm n.

Table 6: Effects of Index Additions.

Size Group	**Noise-Trader Demand**	**Percentage Price Change**		**Change in Return Volatility**	
		All Firms in Index	**Size Groups 3-5 in Index**	**All Firms in Index**	**Size Groups 3-5 in Index**
1 (Smallest)	Low	0.08	0.11	0.00	0.00
	High	0.08	0.11	0.00	0.00
2	Low	0.32	0.45	0.01	0.01
	High	0.33	0.46	0.01	0.01
3	Low	1.18	1.68	0.06	0.08
	High	1.28	1.83	0.06	0.09
4	Low	2.75	4.10	0.19	0.28
	High	3.57	5.43	0.24	0.36
5 (Largest)	Low	2.59	4.16	0.23	0.37
	High	4.63	7.89	0.40	0.67

Adding a firm to the index raises the firm's stock price and return volatility. The effect is almost zero for small firms, but grows with size and becomes significant for large firms. The intuition is the same as for the effect of noise-trader demand (Table 4). Index additions lower idiosyncratic supply, raising the present value of idiosyncratic dividends and the price. Index additions also raise return volatility because they render the price more sensitive to shocks to idiosyncratic dividends.

Holding size constant, index additions have larger effects on the price and return volatility of stocks in high noise-trader demand. The intuition is similar as why passive flows have a larger effect on these stocks (Table 5). Index additions raise the price and render it more sensitive to shocks to idiosyncratic dividends. Because stocks of high noise-trader demand are in low idiosyncratic supply, the attenuation effect caused by the increase in price sensitivity is weaker.

6 Empirical Evidence

In this section we present tests of our theory and relate our results to empirical findings in the literature. We take the index to be the S&P500, and passive flows to be into US listed index mutual funds and index ETFs tracking it. The S&P500 index accounts for the bulk of passive investing in US stocks: index mutual funds tracking the S&P500 index account for 47% to 87% of assets of all index mutual funds invested in US stocks in our sample. We refer to index mutual funds and index ETFs tracking the S&P500 index as S&P500 index funds.

6.1 Data and Descriptive Statistics

Our data on stock returns, market capitalization, and the composition of the S&P500 index come from the Center for Research in Security Prices (CRSP). Our data on net assets of S&P500 index mutual funds come from the Investment Company Institute (ICI). Our data on net assets of S&P500 index ETFs come from CRSP. Our data on the announcement and effective dates of index changes come from Siblis Research. We include in our analysis only plain-vanilla ETFs, excluding alternative ETFs such as leveraged ETFs, inverse ETFs and buffered ETFs. Our ETF sample consists of the SPDR S&P500 ETF Trust, the iShares Core S&P500 ETF, and the Vanguard S&P500 Index Fund ETF, which collectively account for almost all of the plain-vanilla S&P500 ETF market. Our sample begins in the second quarter of 1996 and ends in the fourth quarter of 2020.

Table 7 reports descriptive statistics. The descriptive statistics in Panel A concern aggregate variables sourced at a quarterly frequency. The descriptive statistics in Panel B concern firm-level variables pertaining to all S&P500 firms and sourced at a quarterly frequency. The descriptive statistics in Panel C concern firm-level variables pertaining to episodes where firms were added to the index. There are 426 index-addition episodes during our sample period. All variables in Panel A except the last (VIX) and all variables in Panel C are multiplied by 100.

The first two rows in Panel A concern the return of a portfolio of stocks of large firms in the index in excess of the index return. The large-firm portfolio consists of the top decile of S&P500 firms based on market capitalization. Deciles are formed at the end of any given quarter (implying that the large-firm portfolio is rebalanced quarterly). The first row concerns the equal-weighted

Table 7: Descriptive Statistics

	Mean	Standard Deviation	25th Percentile	Median	75th Percentile	Skewness	Kurtosis
Panel A: Aggregate Variables							
$R^{ew}_{Large-Index}$	-0.15	1.56	-1.06	-0.39	0.72	0.52	0.96
$R^{vw}_{Large-Index}$	-0.17	1.86	-1.27	-0.20	1.12	-0.17	0.50
$PassiveFlow$	0.05	0.09	0.01	0.05	0.10	0.33	3.62
$\Delta Top10$	0.51	3.87	-1.98	0.41	3.17	0.20	0.69
$\Delta Dispersion$	0.51	3.58	-1.90	0.45	2.37	0.41	1.23
ΔH	0.81	5.61	-2.85	0.58	3.55	0.52	1.57
VIX	20.36	7.59	14.57	19.31	24.92	1.80	6.03
Panel B: Firm-Level Variables for All Firms							
$TotVol$	-4.01	0.50	-4.36	-4.05	-3.71	0.43	0.70
$IdioVol$	-4.28	0.50	-4.64	-4.31	-3.95	0.34	0.40
Panel C: Firm-Level Variables for Index-Addition Episodes							
$CAR^{m}_{a,e-1}$	3.66	7.67	-0.84	2.53	7.09	1.28	9.92
$CAR^{m}_{e-1,e}$	1.04	4.36	-1.31	0.41	2.42	1.70	8.38
$CAR^{m}_{e,e+5}$	-1.12	5.65	-3.32	-0.66	1.69	-0.89	5.27
$CAR^{FFm}_{a,e-1}$	3.50	7.08	-0.83	2.56	6.72	0.81	8.28
$CAR^{FFm}_{e-1,e}$	0.88	4.36	-1.28	0.39	2.19	1.54	8.43
$CAR^{FFm}_{e,e+5}$	-0.98	5.09	-3.23	-0.71	1.73	-0.93	4.01
$Cap/\$SP500IndexCap$	0.08	0.08	0.05	0.06	0.09	8.41	106.10

quarterly excess return of the large-firm portfolio, $R^{ew}_{Large-Index}$, and the second row concerns the value-weighted quarterly excess return, $R^{vw}_{Large-Index}$. The means of $R^{ew}_{Large-Index}$ and $R^{vw}_{Large-Index}$ are -0.15% and -0.17%, respectively, implying that stocks of large firms underperformed the index over our sample period. The standard deviations are 1.56% and 1.86%, respectively.

The third row in Panel A concerns passive flows. We measure flows into S&P500 index funds in any given quarter by the ratio of S&P500 index fund net assets to index market capitalization (i.e., combined capitalization of all S&P500 stocks) minus the same ratio in the previous quarter:

$$PassiveFlow_t = \frac{\$SP500IndexAssets_t}{\$SP500IndexCap_t} - \frac{\$SP500IndexAssets_{t-1}}{\$SP500IndexCap_{t-1}}.$$

The mean of passive flow is 0.05% quarterly, implying that the ratio of S&P500 index fund net assets to S&P500 market capitalization grew by approximately 5% during our sample period. The standard deviation of passive flow is 0.09%.

The fourth, fifth and sixth rows in Panel A concern three measures of index concentration:

the combined portfolio weight of the stocks of the top ten firms in the index, denoted by $Top10$, the standard deviation of index weights across all S&P500 firms, denoted by $Dispersion$, and the Herfindahl index of index weights across all S&P500 firms, denoted by H. The descriptive statistics concern the first difference of the logarithm of the three variables. Index concentration has been growing during our sample period, by rates ranging from 0.51% to 0.81% per quarter. Thus, large firms have been becoming a larger fraction of the index over time. The seventh row in Panel A concerns VIX, the CBOE volatility index.

The first row in Panel B concerns total volatility ($TotVol$) and the second row concerns idiosyncratic volatility ($IdioVol$). Total volatility is the standard deviation of daily stock returns in any given quarter. Idiosyncratic volatility is the standard deviation of daily residual returns from the Fama-French three-factor model. The descriptive statistics concern the logarithm of the two variables.

The first six rows in Panel C concern the stock returns of the firms that are added to the index. We partition the period around each addition episode into three sub-periods. Two dates defining the sub-periods are the announcement date, after the market close of which the addition is announced, and the effective date, after the market close of which the addition is implemented. The first sub-period ranges from the announcement date to one trading day before the effective date. The second sub-period ranges from one trading day before the effective date to the effective date. The third sub-period ranges from the effective date to five trading days after that date. We compute cumulative abnormal return (CAR) within each sub-period, from the close of the starting date to the close of the ending date.

The first, second and third rows in Panel C concern returns during the first, second, and third sub-period, respectively, adjusted for market movements by subtracting the market return. The fourth, fifth and sixth rows in Panel C concern returns during the same sub-periods adjusted using the Fama-French three-factor model augmented with a momentum factor (FFm). The two adjustment methods yield similar results.

During the first sub-period, the stock price of the firm that is added to the index rises on average, in anticipation of the demand by index funds. The mean abnormal return is 3.66% using the market adjustment and 3.50% using the FFm adjustment. During the second sub-period, the stock price rises further on average, as index funds buy the stock. The mean abnormal return is 1.04% using the market adjustment and 0.88% using the FFm adjustment. During the third subperiod, the stock price drops on average, as the market absorbs the demand imbalance. The mean abnormal return is -1.12% using the market adjustment and -0.98% using the FFm adjustment.

The seventh row in Panel C concerns the market capitalization of the firms that are added to the index. To make capitalization comparable across index-addition episodes, we divide the capitalization of the added firm by the capitalization of the index at the end of the month before the addition announcement. The mean of the resulting variable, $Cap/\$SP500IndexCap$, is 0.08% and the standard deviation is 0.08%. The kurtosis is high (106.10) because while most firms that are added to the index have capitalizations similar to the smaller firms in the index, a few firms are large.[15] Our tests on index additions account for the high kurtosis.

6.2 Tests

6.2.1 Passive Flows and Excess Returns

Table 8 reports results from regressing the excess return of the large-firm portfolio on passive flows. For ease of interpretation, we standardize $PassiveFlow$ to a mean of zero and a standard deviation of one. We denote the resulting variable by $\widehat{PassiveFlow}$. We use the same notation for VIX and for the three measures of S&P500 index concentration in Table 9. In both Tables 8 and 9, the t-statistics, in parentheses, are based on Newey-West heteroskedasticity- and autocorrelation-consistent standard errors with three lags. Our findings are robust to increasing the number of lags.

Consistent with our model, the relationship between passive flows and excess returns of large firms is positive and significant economically and statistically. In the univariate regressions in Columns (1) and (2), a one-standard-deviation increase in $PassiveFlow$ is associated with an increase in the quarterly excess returns of large firms by 0.55%. This is approximately one-third of the quarterly excess returns' standard deviation in Table 7. The t-statistics are around 3.60.

[15] An example of a large addition is Tesla. Its capitalization was $387 billion on the announcement date, 16 November 2020, and rose to $659 billion on the effective date, 18 December 2020. On the effective date, Tesla had the sixth largest capitalization among S&P500 firms. The large rise in Tesla's capitalization between announcement and effective date is consistent with our model.

Economic and statistical significance are similar when controlling for the contemporaneous and one-quarter lagged return of the S&P500 index and for VIX, in Columns (3) and (4).[16]

Table 8: Passive Flows and Excess Returns on Large Stocks

	(1)	(2)	(3)	(4)
	$R^{ew}_{Large-Index}$	$R^{vw}_{Large-Index}$	$R^{ew}_{Large-Index}$	$R^{vw}_{Large-Index}$
$\widehat{PassiveFlow}$	0.00549	0.00550	0.00523	0.00525
	(3.60)	(3.67)	(4.14)	(3.64)
R_{Index}			-0.0374	-0.0203
			(-1.69)	(-0.70)
$L.R_{Index}$			-0.0104	0.00773
			(-0.41)	(0.36)
$\widehat{VIX}$			0.00201	0.00271
			(1.35)	(1.31)
Constant	-0.00146	-0.00166	-0.000197	-0.00134
	(-0.90)	(-0.79)	(-0.10)	(-0.52)
Observations	99	99	99	99
Adjusted R^2	0.124	0.087	0.206	0.123

We can convert the 0.55% estimate into a cumulative effect over the length of our sample. Recall from Table 7 that the mean and standard deviation of $PassiveFlow$ are 0.05% and 0.09%, respectively. Since our sample comprises 99 quarters, the cumulative effect of passive flows is $0.55\% \times \frac{0.05\%}{0.09\%} \times 99 = 30.25\%$. Our estimate thus suggests that the rise in passive investing over the past 25 years caused firms that were in the top decile of the S&P500 index during that period to rise by 30% more than the index.

The estimated 30% effect of passive flows in Table 8 is larger than in our calibration. For example, the difference between the return of size group 5 and the average return of size groups 3, 4 and 5 in Tables 2 and 5 ranges from zero (Table 2, switch from active to passive, all firms in index) to 6% (Table 5, switch from active to passive, size groups 3-5 in index). The discrepancy might be arising for the same two reasons mentioned in the context of elasticities in Section 5. First, the fraction of truly active investors might be smaller than in our calibration. Second, the 30% estimate concerns a contemporaneous effect of passive flows, which can partly mean-revert.

The finding in Table 8 that passive flows raise the stock prices of the largest firms the most is

[16]Instrumental variable regressions using fund flows at the fund family level (Blackrock, Vanguard and State Street) and changes in the VIX as instruments for passive flows yield results of similar economic and statistical significance.

consistent with other findings in the literature. Ben-David, Franzoni, and Moussawi (2018) find that increases in a firm's ownership by ETFs have significantly larger effects on the firms in the S&P500 than on the smaller firms in the Russell 3000 (Table IV). Haddad, Huebner, and Loualiche (2022) find that demand elasticities are smaller for large firms than for smaller firms (Figure 3), implying that an increase in demand proportional to firms' market capitalization causes the stocks of large firms to rise the most.

Table 9 reports results from regressing changes in S&P500 index concentration on passive flows. Consistent with our model, the relationship between passive flows and changes in all three measures of concentration is positive and significant economically and statistically. In the univariate regressions in Columns (1)–(3), a one-standard-deviation increase in $PassiveFlow$ is associated with an increase in the concentration measures by 0.23-0.24 standard deviations. Economic and statistical significance are similar when controlling for the contemporaneous and lagged return of the S&P500 index and for VIX, in Columns (4)–(6).

Table 9: Passive Flows and Index Concentration

	(1)	(2)	(3)	(4)	(5)	(6)
	$\widehat{\Delta Top10}$	$\Delta\widehat{Dispersion}$	$\widehat{\Delta H}$	$\widehat{\Delta Top10}$	$\Delta\widehat{Dispersion}$	$\widehat{\Delta H}$
$\widehat{PassiveFlow}$	0.244	0.239	0.230	0.235	0.233	0.224
	(2.30)	(2.02)	(1.92)	(2.74)	(2.46)	(2.35)
R_{Index}				-0.520	0.0127	0.0905
				(-0.34)	(0.01)	(0.06)
$L.R_{Index}$				0.476	0.508	0.574
				(0.43)	(0.53)	(0.60)
$\widehat{VIX}$				0.239	0.274	0.283
				(1.73)	(1.98)	(2.01)
Observations	99	99	99	99	99	99
Adjusted R^2	0.060	0.057	0.053	0.121	0.126	0.125

6.2.2 Passive Flows and Return Volatility

Our model predicts that passive flows should raise the stock return volatility of the largest firms in the S&P500 index, while the effect should be weaker or negative for smaller firms. To test for this prediction, we perform panel regressions of volatility on one-quarter lagged $PassiveFlow$ interacted with a $Large$ firm indicator. The indicator is equal to one if the firm belongs to the top decile of

S&P500 firms based on market capitalization, and to zero otherwise. The regression results are reported in Table 10.

Table 10: Passive Flows and Return Volatility

	(1)	(2)	(3)	(4)
	$TotVol$	$IdioVol$	$TotVol$	$IdioVol$
L.$PassiveFlow \times Large$	21.66	19.30	22.34	18.41
	(2.33)	(2.52)	(2.26)	(2.44)
L.$PassiveFlow$	20.51	20.64		
	(0.83)	(1.21)		
L.$Large$	-0.0354	-0.0471	-0.0401	-0.0668
	(-2.38)	(-2.84)	(-3.26)	(-4.81)
L.R_{Index}	-0.350	-0.356		
	(-1.41)	(-1.93)		
L.$TotVol$	0.610		0.530	
	(15.33)		(29.59)	
L.$IdioVol$		0.628		0.456
		(22.88)		(28.33)
Observations	45,737	45,737	45,737	45,737
Firm fixed effects	Yes	Yes	Yes	Yes
Time fixed effects	No	No	Yes	Yes
Adjusted R^2	0.559	0.600	0.777	0.712

The dependent variable in the regressions is the logarithm of total volatility or of idiosyncratic volatility. The independent variables in Columns (1) and (2) are the interaction term, its two constituents separately, the one-quarter lagged index return, the logarithm of one-quarter lagged total or idiosyncratic volatility (to control for serial dependence in volatility), and firm fixed effects. In Columns (3) and (4), we introduce additionally time fixed effects to absorb the time-series variation, and drop lagged $PassiveFlow$ and index return. We conservatively double-cluster standard errors by firm and time.

Consistent with our model, passive flows impact more strongly the stock return volatility of the largest firms, and this effect is significant economically and statistically. In Column (1), a one-standard-deviation increase in $PassiveFlow$ is associated with a percentage increase in total volatility by 1.85% (=$20.51 \times 0.09\%$) for firms outside the top decile, and this effect approximately doubles to 3.80% (=$(20.51 + 21.66) \times 0.09\%$) for firms in the top decile. Moreover, the incremental effect for large firms is statistically significant while the effect for other firms is not. Also consistent with our model, the effect of passive flows is not confined to total volatility but extends to idiosyncratic volatility: the coefficients of $PassiveFlow$ and of the interaction term in Column (2) are approximately equal to their counterparts in Column (1). Statistical significance is similar when adding time fixed effects, in Columns (3) and (4).

6.2.3 Index Additions

Table 11 reports results from regressing stock returns during index-addition episodes on firm size. The t-statistics, in parentheses, are based on White heteroskedasticity-robust standard errors. Consistent with our model and earlier findings by Wurgler and Zhuravskaya (2002), size is positively related to CAR during the first and second sub-periods, and negatively during the third sub-period.[17] These relationships are significant economically and statistically. A one-standard deviation increase in $Cap/\$SP500IndexCap$, our measure of firm size, is associated with an increase in market-adjusted CAR during the first sub-period by 2.23% (=27.92 × 0.08%). This is almost two-thirds of the mean CAR during that sub-period. The corresponding increase in CAR during the second sub-period is 0.65% (=8.066 × 0.08%), almost two-thirds of the mean CAR during that sub-period, and the corresponding decrease in CAR during the third sub-period is 0.50% (=−6.234 × 0.08%). Results for the FFm-adjusted CAR are similar. Results are also similar when performing robust regression to account for the high kurtosis of $Cap/\$SP500IndexCap$.

Table 11: Index Additions and Firm Size

	(1)	(2)	(3)	(4)	(5)	(6)
	$CAR^{m}_{a,e-1}$	$CAR^{m}_{e-1,e}$	$CAR^{m}_{e,e+5}$	$CAR^{FFm}_{a,e-1}$	$CAR^{FFm}_{e-1,e}$	$CAR^{FFm}_{e,e+5}$
$Cap/\$SP500IndexCap$	27.92	8.066	-6.234	23.52	6.501	-7.433
	(7.28)	(2.38)	(-2.62)	(7.28)	(2.22)	(-2.47)
Constant	1.383	0.388	-0.610	1.588	0.346	-0.374
	(2.84)	(1.19)	(-1.74)	(3.62)	(1.12)	(-1.14)
Observations	426	426	426	426	426	426
Adjusted R^2	0.092	0.022	0.006	0.076	0.013	0.013

[17] Table 6 in the calibrated example does not report the change in expected return following index additions. Expected return declines, and more so for larger stocks.

7 Conclusion

The growth of passive investing over the past thirty years and its effects on asset prices and the real economy have attracted attention by academics and policy-makers. In this paper we show that flows into passive funds raise disproportionately the stock prices of the economy's largest firms, and especially those large firms that the market overvalues. These effects arise even when the indices tracked by the funds include all firms, and are sufficiently strong to cause the aggregate market to rise even when flows are entirely due to investors switching from active to passive. Our model implies additionally that passive flows raise the stock return volatility of large firms but do not affect that of smaller firms. A key intuition behind our results is that flows create idiosyncratic volatility for large firms, which discourages investors from correcting the flows' effects on prices. Consistent with our theory, we find that the largest firms in the S&P500 index experience the highest returns and increases in volatility following flows into that index.

Our theory implies that passive investing reduces primarily the financing costs of the largest firms in the economy and makes the size distribution of firms more skewed. Quantifying these effects is a natural extension of our research. A quantification exercise would also determine the contribution of the rise in passive investing to recent macroeconomic trends such as the rise in industry concentration and the decline in corporate investment. Some papers quantifying these trends emphasize heterogeneity in financing costs, which they often model through borrowing constraints. Our theory links this heterogeneity to stock-market distortions, which can be a more relevant channel for large firms.

An additional extension of our research concerns the design of indices. Passive funds in our model track capitalization-weighted indices. While such indices are the most common in practice, other types of indices, such as price-weighted or equal-weighted, also exist. It would be interesting to determine how indices should be designed to achieve welfare objectives. If the growth of passive funds reduces primarily the financing costs of the largest or overvalued firms, and this leads to welfare-reducing industry concentration or capital misallocation, then should capitalization-weighting be moderated? Should upper bounds be imposed on weights, as is the case for some sovereign-bond indices? Is capitalization-weighting the best solution despite its drawbacks?

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Appendix – For Online Publication

A General Results

We first examine how the effect of passive flows on stock prices depends on firm characteristics.

Proposition A.1. *Consider firms n and n' with $(\kappa_n^i, \bar{D}_n^i, \sigma_n^i) = (\kappa_{n'}^i, \bar{D}_{n'}^i, \sigma_{n'}^i)$ and $(n, n') \in \mathcal{I} \times \mathcal{I}$ or $(n, n') \in \mathcal{I}^c \times \mathcal{I}^c$. When μ_2 increases holding μ_1 constant, firm n experiences a larger percentage stock price increase than firm n', for all D_t^s and $D_{nt}^i = D_{n't}^i$, in the following cases:*

(i) $b_n > b_{n'}$ and $(\eta_n, u_n) = (\eta_{n'}, u_{n'})$, under the sufficient condition

$$\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2 - (r + \kappa^s) a_1^s} > \frac{\eta_n'(\sigma_n^i a_{n1}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i}. \tag{A.1}$$

(ii) $\eta_n > \eta_{n'}$ and $(b_n, \frac{u_n}{\eta_n}) = (b_{n'}, \frac{u_{n'}}{\eta_{n'}})$, under the sufficient condition $(n, n') \in \mathcal{I}^c \times \mathcal{I}^c$; or $(n, n') \in \mathcal{I} \times \mathcal{I}$, $\mu_2 \lambda + \frac{u_n}{\eta_n} \leq 1$ and

$$(r + \kappa_n^i)^2 \geq 2(\sqrt{2} - 1)\rho \frac{\eta_n(1 - \mu_2 \lambda) - u_n}{\mu_1} (\sigma_n^i)^2; \tag{A.2}$$

or $(n, n') \in \mathcal{I} \times \mathcal{I}$, $\mu_2 \lambda + \frac{u_n}{\eta_n} > 1$ and

$$1 + \frac{\rho(\mu_2 \lambda + \frac{u_n}{\eta_n} - 1)}{\mu_1} \left(\frac{3\eta_{n'}(\sigma_n^i a_{n'1}^i)^2}{2 - (r + \kappa_n^i) a_{n'1}^i} - \frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2 - (r + \kappa^s) a_1^s} \right) \geq 0. \tag{A.3}$$

(iii) $u_n > u_{n'}$ and $(b_n, \eta_n) = (b_{n'}, \eta_{n'})$, under the sufficient condition $(n, n') \in \mathcal{I} \times \mathcal{I}$ and

$$\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2 - (r + \kappa^s) a_1^s} \leq \frac{3\eta_n(\sigma_n^i a_{n'1}^i)^2}{2 - (r + \kappa_n^i) a_{n'1}^i}. \tag{A.4}$$

When μ_2 increases and μ_1 decreases by a corresponding fraction $\phi \in (0, 1]$, the results in Case (ii) with $(n, n') \in \mathcal{I} \times \mathcal{I}$ and $\mu_2 \lambda + \frac{u_n}{\eta_n} > 1$, and in Case (iii), hold, provided additionally that $n \in \operatorname{argmax}_m \frac{u_m}{\eta_m}$ and that $\phi < 1$ or $\mathcal{I} \subsetneq \{1, .., N\}$ or $\#\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\} > 1$.

When passive flows are due to entry by new investors into the stock market, Proposition A.1 shows the following results. First, flows generate a larger percentage price increase for firms that load more on the systematic factor (larger b_n), holding all else (including size) constant. This result requires that flows impact the present value of the systematic component of dividends more, in percentage terms, than the present value of the idiosyncratic component. This condition is intuitive because the discount rate is larger for the systematic component, and holds for all firms in our calibration. Second, passive flows generate a larger percentage price increase for larger firms (larger η_n), holding all else constant. This result requires an upper bound on firm size when experts hold a long position in equilibrium ($\mu_2\lambda + \frac{u_n}{\eta_n} > 1$), and a lower bound on size when experts hold a short position, with the requirements being always satisfied when the position of experts approaches zero. Third, passive flows generate a larger percentage price increase for firms whose stocks that are in higher demand by noise traders (larger u_n), holding all else constant. This result requires that firms are sufficiently large. Table 5 shows that the result can indeed reverse for small firms.

The effects of size and noise-trader demand carry through to the case where passive flows are partly due to a switch by investors from active to passive ($0 < \phi < 1$), or are purely due to such a switch but the index does not include all firms ($\mathcal{I} \subsetneq \{1, .., N\}$) or noise traders do not hold the market portfolio (the set $\{\frac{u_m}{\eta_m} : m \in \{1, .., N\}\}$ is not a singleton). The effects of passive flows increase with size for firms that are in the index ($n \in \mathcal{I}$) and whose stocks are in high demand by noise-traders ($n \in \text{argmax}_m \frac{u_m}{\eta_m}$). Holding size constant, the effects of passive flows are higher for high-demand firms ($n \in \text{argmax}_m \frac{u_m}{\eta_m}$). We next examine how the effect of passive flows on return volatilities depends on firm characteristics.

Proposition A.2. *Consider firms n and n' with $(b_n, \kappa_n^i, \bar{D}_n^i, \sigma_n^i) = (b_{n'}, \kappa_{n'}^i, \bar{D}_{n'}^i, \sigma_{n'}^i)$ and $(n, n') \in \mathcal{I} \times \mathcal{I}$. When μ_2 increases holding μ_1 constant, firm n experiences a rise in stock return volatility and firm n' experiences a decline for $D_t^s = \bar{D}^s = 1$, $D_{nt}^i = D_{n't}^i = \bar{D}_n^i$ and an interval of values of $\bar{D}_n = 1 - b_n - \bar{D}_n^i$, in the following cases:*

(i) $\eta_{n'} \approx 0$, η_n *satisfies (A.4) for* $n' = n$, *and* $\frac{u_n}{\eta_n} = \frac{u_{n'}}{\eta_{n'}}$, *under the sufficient condition*

$$\max\left\{\frac{2\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s}-1\right)(r+\kappa_n^i)a_{n1}^i\bar{D}_n^i}{3+\frac{(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i}{(b_n\sigma^s a_1^s)^2}},0\right\} < \left(\frac{(r+\kappa^s)(\sigma_n^i)^2}{(r+\kappa_n^i)^2 b_n(\sigma^s)^2 a_1^s}-1\right)\bar{D}_n^i. \quad \text{(A.5)}$$

(ii) $u_n > u_{n'}$ *and* $\eta_n = \eta_{n'}$, *under the sufficient conditions*

$$\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} \geq \frac{\psi\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}, \quad \text{(A.6)}$$

$$(r+\kappa^s)b_n a_1^s < \frac{2(\psi-1)\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n'1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s}-1\right)(r+\kappa_n^i)a_{n1}^i\bar{D}_n^i}{\psi+\frac{(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i}{(b_n\sigma^s a_1^s)^2}} \quad \text{(A.7)}$$

and (A.4), for some scalar $\psi > 1$.

Return volatility of both stocks rises for values of $\bar{D}_n$ *above the interval, and declines for values below the interval. When* μ_2 *increases and* μ_1 *decreases by a corresponding fraction* $\phi \in (0,1]$, *the result in Case (ii) holds, provided additionally that* $n \in \operatorname{argmax}_m \frac{u_m}{\eta_m}$ *and that* $\phi < 1$ *or* $\mathcal{I} \subsetneq \{1,..,N\}$ *or* $\#\{\frac{u_m}{\eta_m} : m \in \{1,..,N\}\} > 1$.

When passive flows are due to entry, Proposition A.2 shows that they are more likely to raise stock return volatility for larger or high-demand firms. Higher likelihood is in the sense that there exists a parameter interval within which volatility rises for large or high-demand firms and declines for small or low-demand firms. Outside that interval, volatility moves in the same direction for all firms.

The parameter that defines the interval is the constant component $\bar{D}_n$ of dividends. When $\bar{D}_n$ is large, passive flows raise stock return volatilities of all firms. Intuitively, flows raise the present value of the systematic and the idiosyncratic component of dividends and render them more sensitive to shocks. Volatility rises if the sensitivity of the price divided by the price increases. If $\bar{D}_n$ is large, then the percentage change in the price is small since the present value of the constant component of dividends does not rise in response to flows. Volatility rises because the percentage change in price sensitivity, which does not involve $\bar{D}_n$, is larger.

For large firms, the effect of flows on the present value of the idiosyncratic component of dividends, and on that component's sensitivity to shocks, is significant enough to cause volatility to rise even for smaller values of $\bar{D}_n$. The same is true for high-demand firms, provided that these stocks are also large ((A.4) is met). In both cases, the return volatility caused by the idiosyncratic component must be large enough relative to that caused by the systematic component ((A.5) and (A.7) are met). Unlike the price level results, the volatility results can fail to hold for extreme values of D_t^s and D_{nt}^i, and are shown when D_t^s and D_{nt}^i are equal to their long-run means. The effect of size carries through to the case where passive flows are due to a switch from active to passive, under the same conditions as in Proposition A.1.

We finally examine how the effect of index additions on stock prices depends on firm characteristics. We assume that when a firm n is added to the index, a firm $\hat{n}$ with identical characteristics $(b_n, \kappa_n^i, \bar{D}_n^i, \sigma_n^i, \eta_n, u_n) = (b_{\hat{n}}, \kappa_{\hat{n}}^i, \bar{D}_{\hat{n}}^i, \sigma_{\hat{n}}^i, \eta_{\hat{n}}, u_{\hat{n}})$ is taken out of the index. This ensures that aggregate quantities, such as the discount rate of the systematic component of dividends, do not change after the addition.[18] Proposition A.3 shows that index additions generate a larger percentage price increase for firms that are larger or whose stocks are in higher demand by noise traders.

Proposition A.3. *Consider firms n and n' with $(b_n, \kappa_n^i, \bar{D}_n^i, \sigma_n^i) = (b_{n'}, \kappa_{n'}^i, \bar{D}_{n'}^i, \sigma_{n'}^i)$. Firm n experiences a larger percentage stock price increase than firm n' when it is added to the index, for all D_t^s and $D_{nt}^i = D_{n't}^i$, in the following cases:*

(i) $\eta_n > \eta_{n'}$ and $\frac{u_n}{\eta_n} = \frac{u_{n'}}{\eta_{n'}}$, under the sufficient condition

$$(r + \kappa_n^i)^2 \geq 2(\sqrt{2} - 1)\rho \frac{\eta_n - u_n}{\mu_1} (\sigma_n^i)^2. \tag{A.8}$$

(ii) $u_n > u_{n'}$ and $\eta_n = \eta_{n'}$.

[18] Our assumption also ensures that the number of firms in the index does not change after the addition. By contrast, Table 6 in the calibrated example assumes that when a firm is added to the index, no firm is taken out, so the number of firms in the index increases by one. The data in Table 6 remain almost the same if the number of firms in the index is assumed to not change after the addition.

B Proofs

Proof of Proposition 3.1. The quadratic equation derived from (3.12) by identifying terms in D_t^s is

$$\rho \left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s a_1^s)^2 + (r + \kappa^s) a_1^s - 1 = 0. \tag{B.1}$$

The quadratic equation derived by identifying terms in D_{nt}^i is

$$\rho \frac{\eta_n - \mu_2 \lambda \eta'_n - u_n}{\mu_1} (\sigma_n^i a_{n1}^i)^2 + (r + \kappa_n^i) a_{n1}^i - 1 = 0. \tag{B.2}$$

The equation derived by identifying the remaining terms is

$$\bar{D}_n - r\bar{S}_n + b_n \left(\kappa^s a_1^s - r a_0^s \right) + \kappa_n^i a_{n1}^i \bar{D}_n^i - r a_{n0}^i = 0. \tag{B.3}$$

When $\sum_{m=1}^N (\eta_m - \mu_2 \lambda \eta'_m - u_m) b_m \geq 0$, the left-hand side of (B.1) is increasing for positive values of a_1^s, and (B.1) has a unique positive solution, given by (3.14). When $\sum_{n=1}^N (\eta_m - \mu_2 \lambda \eta'_m - u_m) b_n < 0$, the left-hand side of (B.1) is hump-shaped for positive values of a_1^s, and (B.1) has either two positive solutions (including one double positive solution) or no solution. When two solutions exist, (3.14) gives the smaller of them, which is the continuous extension of the unique positive solution when $\sum_{m=1}^N (\eta_m - \mu_2 \lambda \eta'_m - u_m) b_m > 0$. Equation (3.15) gives the analogous solution of (B.2). Equation (B.3) yields

$$\bar{S}_n + b_n a_0^s + a_{n0}^i = \frac{\bar{D}_n + b_n \kappa^s a_1^s + \kappa_n^i a_{n1}^i \bar{D}_n^i}{r}. \tag{B.4}$$

Substituting (3.10), (3.11) and (B.4) into (3.1), we find (3.13).

Substituting $\mu_n = \mathbb{E}(\mu_{nt})$ and (3.9)-(3.11) into (3.7), we find

$$\left(\sum_{m=1}^{N} \eta'_m b_m \right) \left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m \right) (\sigma^s a_1^s)^2 + \sum_{m=1}^{N} \eta'_m \left(\frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} \right) (\sigma_m^i a_{m1}^i)^2 \bar{D}_m^i$$

$$= \lambda \left[\left(\sum_{m=1}^{N} \eta'_m b_m \right)^2 (\sigma^s a_1^s)^2 + \sum_{m=1}^{N} \left(\eta'_m\right)^2 (\sigma_m^i a_{m1}^i)^2 \bar{D}_m^i \right], \tag{B.5}$$

which we can rewrite as (3.16). Since $\eta_m > u_m$ for all m, (3.16) implies $\lambda > 0$.

Equations (3.13)-(3.15) imply that the price depends on $(\mu_1, \mu_2, \sigma^s, \{b_m, \sigma_m^i, \eta_m, \eta'_m, u_m\}_{m=1,..,M})$ only through $\left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s)^2$ and $\frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma_n^i)^2$. The price is decreasing and convex in the latter two variables if a_1^s is decreasing and convex in $\left(\sum_{m=1}^{N} \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m\right)(\sigma^s)^2$, and a_{n1}^i is decreasing and convex in $\frac{\eta_n - \mu_2\lambda\eta'_n - u_n}{\mu_1}(\sigma_n^i)^2$. These properties hold if the function

$$\Psi(z) \equiv \frac{1}{A + \sqrt{B + Cz}}$$

is decreasing and convex for $z \geq -\frac{B}{C}$, where (A, B, C) are positive constants. The function $\Psi(z)$ is decreasing because its derivative

$$\Psi'(z) = -\frac{C}{2\sqrt{B + Cz}} \frac{1}{\left(A + \sqrt{B + Cz}\right)^2}$$

is negative. Since, in addition, $\Psi'(z)$ is increasing, $\Psi(z)$ is convex.

An equilibrium exists if (B.5), in which a_1^s and $\{a_{n1}^i\}_{n=1,..,N}$ are implicit functions of λ defined by (3.14) and (3.15), respectively, has a solution. For all non-positive values of λ, both sides of (B.5) are well-defined because the non-negativity of $\sum_{m=1}^{N}(\eta_m - \mu_2\lambda\eta'_m - u_m)b_m$ and $\eta_n - \mu_2\lambda\eta'_n - u_n$ ensures that (3.14) and (3.15) have a solution for a_1^s and $\{a_{n1}^i\}_{n=1,..,N}$, respectively. Moreover, the right-hand side of (B.5) is positive, and exceeds the left-hand side which is non-positive. An equilibrium exists if both sides of (B.5) remain well-defined for a sufficiently large positive value of λ that renders them equal.

If an equilibrium exists, then it is unique. To show uniqueness, we treat (B.5) as an equation in the unknown $\mu_2\lambda$ instead of λ. Since the function $\Psi(z)$ is decreasing, (3.14) and (3.15) imply that the right-hand side of (B.5) is increasing in $\mu_2\lambda$. Equations (3.14) and (3.15) also imply that the left-hand side of (B.5) is decreasing in $\mu_2\lambda$ if the function

$$\Phi(z) \equiv \frac{z}{\left(A + \sqrt{B + Cz}\right)^2}$$

is increasing for $z \geq -\frac{B}{C}$, where (A, B, C) are positive constants. Showing that $\Phi(z)$ is increasing is equivalent to showing that

$$\hat{\Phi}(y) \equiv \frac{y^2 - B}{(A+y)^2}$$

is increasing for $y \equiv \sqrt{B + Cz} \geq 0$. The latter property follows because the functions $\hat{\Phi}_1(y) \equiv \frac{y}{A+y}$ and $\hat{\Phi}_2(y) \equiv -\frac{B}{(A+y)^2}$ are increasing for $y \geq 0$. Since the right-hand side of (B.5) is increasing in $\mu_2\lambda$ and the left-hand side is increasing, a solution $\mu_2\lambda$ of (B.5) is unique. □

Proof of Proposition A.1. We first consider the case where μ_2 increases holding μ_1 constant. Since the right-hand side of (B.5) is increasing in $\mu_2\lambda$ and the left-hand side is decreasing, and since the right-hand side is decreasing in μ_2, $\mu_2\lambda$ increases in μ_2. Differentiating, we find

$$\begin{aligned}
\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial \mu_2} &= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial(\mu_2\lambda)} \\
&= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\frac{b_n\frac{\partial a_1^s}{\partial(\mu_2\lambda)}(\kappa^s + rD_t^s) + \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)}(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)} \\
&= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\frac{\frac{b_n\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2(\kappa^s + rD_t^s)}{\sqrt{(r+\kappa^s)^2 + 4\rho\left(\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1}b_m\right)(\sigma^s)^2}} + \frac{\eta_n'(\sigma_n^i a_{n1}^i)^2(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{\sqrt{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - \mu_2\lambda\eta_n' - u_n}{\mu_1}(\sigma_n^i)^2}}}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)} \\
&= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\frac{\frac{b_n\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s} + \frac{\eta_n'(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)} \equiv \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{\rho}{\mu_1}\mathcal{Z}.
\end{aligned}$$

where the second step follows from (3.13), the third step from (3.14) and (3.15), and the fourth step again from (3.14) and (3.15).

The result in Case (i) will follow if we show that $\mathcal{Z}$ increases in b_n holding $\sum_{m=1}^N \eta_m' b_m$ and $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1}b_m$ constant. The derivative of $\mathcal{Z}$ with respect to b_n has the same sign as

$$\begin{aligned}
\mathcal{Z}_b \equiv &\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s}\left[\bar{D}_n + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)\right] \\
&- \frac{\eta_n'(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}a_1^s(\kappa^s + rD_t^s).
\end{aligned}$$

Since $\bar{D}_n \geq 0$, $\mathcal{Z}_b$ is positive if (A.1) holds.

The result in Case (ii) will follow if we show that $\mathcal{Z}$ increases in η_n holding $\sum_{m=1}^N \eta'_m b_m$, $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m$ and $\frac{u_n}{\eta_n} \equiv \hat{u}$ constant, and setting $\eta'_n = \eta_n$ if $(n, n') \in \mathcal{I} \times \mathcal{I}$ and $\eta'_n = 0$ if $(n, n') \in \mathcal{I}^c \times \mathcal{I}^c$. In the case $(n, n') \in \mathcal{I}^c \times \mathcal{I}^c$, (3.15), $\eta'_n = 0$ and $\eta_n > u_n$ imply that the denominator of $\mathcal{Z}$ decreases in η_n, and $\eta'_n = 0$ implies that the numerator is independent of η_n. Therefore, $\mathcal{Z}$ increases in η_n. In the case $(n, n') \in \mathcal{I} \times \mathcal{I}$, the derivative of $\mathcal{Z}$ with respect to η_n has the same sign as $\mathcal{Z}_{\eta 1} + \mathcal{Z}_{\eta 2}$, where

$$\begin{aligned}\mathcal{Z}_{\eta 1} \equiv & \frac{\partial}{\partial \eta_n}\left[\frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)}{2 - (r + \kappa_n^i) a_{n1}^i}\right]\left[\bar{D}_n + b_n a_1^s (\kappa^s + rD_t^s)\right] \\ & - \frac{b_n \left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s)^2 (a_1^s)^3 (\kappa^s + rD_t^s)}{2 - (r + \kappa^s) a_1^s} \frac{\partial a_{n1}^i}{\partial \eta_n}(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)\end{aligned} \tag{B.6}$$

and

$$\begin{aligned}\mathcal{Z}_{\eta 2} \equiv & \left[\frac{\partial}{\partial \eta_n}\left[\frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i}\right] a_{n1}^i - \frac{\eta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i} \frac{\partial a_{n1}^i}{\partial \eta_n}\right] \\ = & (a_{n1}^i)^2 \frac{\partial}{\partial \eta_n}\left[\frac{\eta_n (\sigma_n^i a_{n1}^i)^2 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2 - (r + \kappa_n^i) a_{n1}^i}\right].\end{aligned}$$

Equation (3.15) implies that $\mathcal{Z}_{\eta 2}$ is positive if the function

$$\Xi(z) \equiv \frac{z}{\left(A + \sqrt{B + Cz}\right)\sqrt{B + Cz}}$$

is increasing for $B + Cz \geq 0$, where (A, B) are positive constants. If $C \leq 0$, then $\Xi(z)$ is increasing because the numerator is increasing and the denominator is decreasing. If $C > 0$, then showing that $\Xi(z)$ is increasing is equivalent to showing that

$$\hat{\Xi}(y) \equiv \frac{y^2 - B}{(A + y)y}$$

is increasing for $y \equiv \sqrt{B + Cz} \geq 0$. The latter property follows because the functions $\hat{\Xi}_1(y) \equiv \frac{y}{A+y}$ and $\hat{\Xi}_2(y) \equiv -\frac{B}{(A+y)y}$ are increasing for $y \geq 0$. To show that $\mathcal{Z}_{\eta 1}$ is non-negative, we distinguish the cases $\mu_2\lambda + \hat{u} \leq 1$ and $\mu_2\lambda + \hat{u} > 1$.

In the case $\mu_2\lambda + \hat{u} \leq 1$, (3.15) and $\eta'_n = \eta_n$ imply that the partial derivative in the second line of (B.6) is non-positive. Equations (3.15) and $\eta'_n = \eta_n$ imply that the partial derivative in the first line of (B.6) is non-negative if the function

$$\Theta(z) \equiv \frac{z}{\left(A + \sqrt{B + Cz}\right)^2 \sqrt{B + Cz}}$$

is non-decreasing in z for $A \equiv r + \kappa_n^i$, $B \equiv (r + \kappa_n^i)^2$, $C \equiv 4\rho\frac{1-\mu_2\lambda-\hat{u}}{\mu_1}(\sigma_n^i)^2$ and $z = \eta_n$. The derivative $\Theta'(z)$ has the same sign as

$$\begin{aligned}
&\left(A + \sqrt{B + Cz}\right)^2 \sqrt{B + Cz} - \frac{Cz}{\sqrt{B + Cz}}\left(A + \sqrt{B + Cz}\right)\sqrt{B + Cz} - \frac{Cz}{2\sqrt{B + Cz}}\left(A + \sqrt{B + Cz}\right)^2 \\
&= \frac{A + \sqrt{B + Cz}}{\sqrt{B + Cz}}\left[A\left(B + \frac{Cz}{2}\right) + \left(B - \frac{Cz}{2}\right)\sqrt{B + Cz}\right]. \qquad \text{(B.7)}
\end{aligned}$$

The sign of (B.7) is non-negative if

$$\begin{aligned}
&A^2\left(B + \frac{Cz}{2}\right)^2 \geq \left(B - \frac{Cz}{2}\right)^2 (B + Cz) \\
&\Leftrightarrow B\left(B + \frac{Cz}{2}\right)^2 \geq \left(B - \frac{Cz}{2}\right)^2 (B + Cz) \\
&\Leftrightarrow B^2 + BCz - \frac{(Cz)^2}{4} \geq 0 \\
&\Leftrightarrow B \geq \frac{\sqrt{2} - 1}{2} Cz. \qquad \text{(B.8)}
\end{aligned}$$

Substituting for (B, C, z) and using $\hat{u} = \frac{u_n}{\eta_n}$, (B.8) becomes (A.2). Therefore, $Z_{\eta 1}$ is positive if (A.2) holds.

In the case $\mu_2\lambda + \hat{u} > 1$, (3.15) and $\eta'_n = \eta_n$ imply that a_{n1}^i increases in η_n and the partial derivative in the first line of (B.6) is positive. Since $\bar{D}_n \geq 0$, $\mathcal{Z}_{\eta 1}$ is positive if

$$\frac{\partial}{\partial \eta_n}\left[\frac{\eta_n(\sigma_n^i)^2(a_{n1}^i)^3}{2 - (r + \kappa_n^i)a_{n1}^i}\right] - \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2 - (r + \kappa^s)a_1^s}\frac{\partial a_{n1}^i}{\partial \eta_n} > 0. \qquad \text{(B.9)}$$

Equation (B.9) holds under the sufficient condition

$$\begin{aligned}
&\frac{(\sigma_n^i)^2(a_{n1}^i)^3}{2-(r+\kappa_n^i)a_{n1}^i} + \left[\frac{3\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} - \frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}\right]\frac{\partial a_{n1}^i}{\partial \eta_n} \geq 0 \\
&\Leftrightarrow \frac{(\sigma_n^i)^2(a_{n1}^i)^3}{2-(r+\kappa_n^i)a_{n1}^i} \\
&\quad + \left[\frac{3\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} - \frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}\right]\frac{(\sigma_n^i)^2(a_{n1}^i)^3}{2-(r+\kappa_n^i)a_{n1}^i}\frac{\rho(\mu_2\lambda+\hat{u}-1)}{\mu_1} \geq 0 \\
&\Leftrightarrow 1 + \frac{\rho(\mu_2\lambda+\hat{u}-1)}{\mu_1}\left[\frac{3\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} - \frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}\right] \geq 0. \qquad \text{(B.10)}
\end{aligned}$$

Equation (B.10) coincides with (A.3) for $\eta_{n'}$ instead of η_n. Since a_{n1}^i increases in η_n, (A.3) ensures that (B.10) holds for all $\eta \in [\eta_{n'}, \eta_n]$, which in turn ensures that $Z_{\eta 1}$ is non-negative for all $\eta \in [\eta_{n'}, \eta_n]$.

The result in Case (iii) will follow if we show that $\mathcal{Z}$ increases in u_n holding $\sum_{m=1}^N \eta_m' b_m$ and $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m$ constant. Since $(n, n') \in \mathcal{I} \times \mathcal{I}$, the derivative of $\mathcal{Z}$ with respect to u_n has the same sign as $\mathcal{Z}_{u1} + \mathcal{Z}_{u2}$, where

$$\begin{aligned}
\mathcal{Z}_{u1} \equiv\ & \frac{\partial}{\partial u_n}\left[\frac{\eta_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}\right]\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s)\right] \\
& - \frac{b_n\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s}\frac{\partial a_{n1}^i}{\partial u_n}(\kappa_n^i \bar{D}_n^i + rD_{nt}^i) \qquad \text{(B.11)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Z}_{u2} &\equiv \left[\frac{\partial}{\partial u_n}\left[\frac{\eta_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right]a_{n1}^i - \frac{\eta_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\frac{\partial a_{n1}^i}{\partial u_n}\right] \\
&= (a_{n1}^i)^2\frac{\partial}{\partial u_n}\left[\frac{\eta_n(\sigma_n^i)^2(a_{n1}^i)^2(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right].
\end{aligned}$$

Since (3.15) implies that a_{n1}^i increases in u_n, $\mathcal{Z}_{u2}$ is positive. To show that $\mathcal{Z}_{u1}$ is non-negative, we follow the same argument as when showing that $\mathcal{Z}_{\eta 1}$ is positive in the case $\mu_2\lambda + \hat{u} > 1$. The counterpart of (B.9) is

$$\frac{\partial}{\partial u_n}\left[\frac{\eta_n(\sigma_n^i)^2(a_{n1}^i)^3}{2-(r+\kappa_n^i)a_{n1}^i}\right]-\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}\frac{\partial a_{n1}^i}{\partial u_n}\geq 0,$$

and the counterpart of (B.10) is

$$\frac{3\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}-\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}\geq 0. \tag{B.12}$$

Since a_{n1}^i increases in u_n, (A.4) ensures that (B.12) holds for all $u\in[u_{n'},u_n]$, which in turn ensures that Z_{u1} is non-negative for all $u\in[u_{n'},u_n]$.

We next consider the case where μ_2 increases and μ_1 decreases by a corresponding fraction $\phi\in(0,1]$. We begin with a lemma.

Lemma B.1. *The following inequality holds*

$$\frac{\partial(\mu_2\lambda)}{\partial\mu_2}+\phi\frac{\mu_2\lambda+\max_m\frac{u_m}{\eta_m}-1}{\mu_1}\equiv\Delta\geq 0 \tag{B.13}$$

and is strict when $\phi<1$ or $\mathcal{I}\subsetneq\{1,..,N\}$ or $\#\{\frac{u_m}{\eta_m}:m\in\{1,..,N\}\}>1$.

Proof of Lemma B.1. We proceed by contradiction, assuming that $\Delta\leq 0$ and that this inequality is strict except when $\phi<1$ or $\mathcal{I}\subsetneq\{1,..,N\}$ or $\#\{\frac{u_m}{\eta_m}:m\in\{1,..,N\}\}>1$. Differentiating with respect to μ_2 and using $\frac{\partial\mu_1}{\partial\mu_2}=-\phi$, we find

$$\begin{aligned}\frac{\partial}{\partial\mu_2}\left(\frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1}\right)&=-\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\frac{\eta_n'}{\mu_1}+\phi\frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1^2}\\&=\begin{cases}-\left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}+\phi\frac{\mu_2\lambda+\frac{u_n}{\eta_n}-1}{\mu_1}\right)\frac{\eta_n}{\mu_1} & \text{for} \quad n\in\mathcal{I},\\ \phi\frac{\eta_n-u_n}{\mu_1^2} & \text{for} \quad n\notin\mathcal{I}.\end{cases}\end{aligned} \tag{B.14}$$

Consider first the case where $\mathcal{I}=\{1,..,N\}$ and $\#\{\frac{u_m}{\eta_m}:m\in\{1,..,N\}\}=1$. Equation (B.14) and our assumption on the sign of Δ imply that the derivative of $\frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1}$ with respect to μ_2 is positive when $\phi=1$ and is non-negative when $\phi<1$. Since the function $\Phi(z)$ is increasing, (3.14) and (3.15) imply that the derivative with respect to μ_2 of the left-hand side of (B.5) is positive when $\phi = 1$ and is non-negative when $\phi < 1$. Likewise, since the function $\Psi(z)$ is decreasing, (3.14) and (3.15) imply that the derivative with respect to μ_2 of the term in square brackets multiplying λ in the left-hand side of (B.5) is negative when $\phi = 1$ and is non-positive when $\phi < 1$. Therefore, (B.5) implies

$$\frac{\partial \lambda}{\partial \mu_2} \geq 0 \tag{B.15}$$

and that this inequality is strict when $\phi = 1$. Equation (3.16) also implies

$$\lambda(\mu_1 + \mu_2) \geq 1 - \max_m \frac{u_m}{\eta_m}. \tag{B.16}$$

Combining (B.15) and (B.16), we find

$$\begin{aligned}\Delta = \frac{\partial(\mu_2\lambda)}{\partial \mu_2} + \phi\frac{\mu_2\lambda + \max_m \frac{u_m}{\eta_m} - 1}{\mu_1} &= \lambda + \mu_2\frac{\partial \lambda}{\partial \mu_2} + \phi\frac{\mu_2\lambda + \max_m \frac{u_m}{\eta_m} - 1}{\mu_1} \\ &= \mu_2\frac{\partial \lambda}{\partial \mu_2} + (1-\phi)\lambda,\end{aligned}$$

which is positive when $\phi = 1$ because of (B.15) and when $\phi < 1$ because $\lambda > 0$. This contradicts our assumption on the sign of Δ. Consider next the case where $\mathcal{I} \subsetneq \{1,..,N\}$ or $\#\{\frac{u_m}{\eta_m} : m \in \{1,..,N\}\} > 1$. Equations (B.14), $\eta_n > u_n$ and our assumption on the sign of Δ imply that the derivative of $\frac{\eta_n - \mu_2\lambda\eta_n' - u_n}{\mu_1}$ with respect to μ_2 is non-negative for all n and positive for some n. The same argument as in the case where $\mathcal{I} = \{1,..,N\}$ and $\#\{\frac{u_m}{\eta_m} : m \in \{1,..,N\}\} = 1$ then implies that (B.15) holds as a strict inequality and this contradicts our assumption on the sign of Δ. $\square$

Using (3.13)-(3.15) and $\frac{\partial \mu_1}{\partial \mu_2} = -\phi$, we find

$$\begin{aligned}\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial \mu_2} &= \frac{\partial(\mu_2\lambda)}{\partial \mu_2}\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial(\mu_2\lambda)} + \phi\frac{1}{S_{nt}}\frac{\partial S_{nt}}{\partial \mu_1} \\ &= \frac{\rho}{\mu_1}\frac{\frac{b_n\left(\sum_{m=1}^N \hat{\Delta}_m b_m\right)(\sigma^s)^2(a_1^s)^3(\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s} + \frac{\hat{\Delta}_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}}{\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)\right]} \equiv \frac{\rho}{\mu_1}\hat{\mathcal{Z}},\end{aligned}$$

where

$$\hat{\Delta}_n \equiv \frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta'_n + \phi\frac{\mu_2\lambda\eta'_n + u_n - \eta_n}{\mu_1}.$$

Using the definition of $\hat{\Delta}_n$, we find

$$\begin{aligned}
\frac{\sum_{m=1}^N \hat{\Delta}_m b_m}{\sum_{m=1}^N \eta'_m b_m} &= \frac{\sum_{m=1}^N \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta'_m + \phi\frac{\mu_2\lambda\eta'_m + u_m - \eta_m}{\mu_1}\right) b_m}{\sum_{m=1}^N \eta'_m b_m} \\
&= \frac{\sum_{m\in\mathcal{I}} \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_m + \phi\frac{\mu_2\lambda\eta_m + u_m - \eta_m}{\mu_1}\right) b_m + \sum_{m\notin\mathcal{I}}\left(\phi\frac{u_m-\eta_m}{\mu_1}\right) b_m}{\sum_{m\in\mathcal{I}}\eta_m b_m} \\
&\le \frac{\sum_{m\in\mathcal{I}} \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_m + \phi\frac{\mu_2\lambda\eta_m + u_m - \eta_m}{\mu_1}\right) b_m}{\sum_{m\in\mathcal{I}}\eta_m b_m} \\
&\le \frac{\sum_{m\in\mathcal{I}} \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_m + \phi\frac{\mu_2\lambda\eta_m + \left(\max_{m'}\frac{u_{m'}}{\eta_{m'}}\right)\eta_m - \eta_m}{\mu_1}\right) b_m}{\sum_{m\in\mathcal{I}}\eta_m b_m} \\
&= \frac{\partial(\mu_2\lambda)}{\partial\mu_2} + \phi\frac{\mu_2\lambda + \max_m\frac{u_m}{\eta_m} - 1}{\mu_1} = \Delta, \qquad \text{(B.17)}
\end{aligned}$$

and for $n \in \mathcal{I} \cap \operatorname{argmax}_m \frac{u_m}{\eta_m}$,

$$\begin{aligned}
\hat{\Delta}_n &= \frac{\partial(\mu_2\lambda)}{\partial\mu_2}\eta_n + \phi\frac{\mu_2\lambda\eta_n + u_n - \eta_n}{\mu_1} \\
&= \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2} + \phi\frac{\mu_2\lambda + \frac{u_n}{\eta_n} - 1}{\mu_1}\right)\eta_n \\
&= \left(\frac{\partial(\mu_2\lambda)}{\partial\mu_2} + \phi\frac{\mu_2\lambda + \max_m\frac{u_m}{\eta_m} - 1}{\mu_1}\right)\eta_n = \Delta\eta_n. \qquad \text{(B.18)}
\end{aligned}$$

Since $\phi < 1$ or $\mathcal{I} \subsetneq \{1,..,N\}$ or $\#\{\frac{u_m}{\eta_m} : m \in \{1,..,N\}\} > 1$, Lemma B.1 implies $\Delta > 0$.

The result in Case (ii) will follow if we show that $\hat{\mathcal{Z}}$ increases in η_n holding $\sum_{m=1}^N \eta'_m b_m$, $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta'_m - u_m}{\mu_1} b_m$ and $\frac{u_n}{\eta_n} \equiv \hat{u}$ constant, and setting $\eta'_n = \eta_n$. The derivative of $\hat{\mathcal{Z}}$ with respect to η_n has the same sign as $\hat{\mathcal{Z}}_{\eta 1} + \hat{\mathcal{Z}}_{\eta 2}$, where

$$\hat{\mathcal{Z}}_{\eta 1} \equiv \frac{\partial}{\partial\eta_n}\left[\frac{\hat{\Delta}_n(\sigma_n^i)^2(a_{n1}^i)^3(\kappa_n^i\bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i}\right]\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s)\right]$$

$$- \frac{\left(\sum_{m=1}^{N} \hat{\Delta}_m b_m\right)(\sigma^s)^2 (a_1^s)^3 (\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s} \frac{\partial a_{n1}^i}{\partial \eta_n} (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)$$

and

$$\begin{aligned}
\hat{\mathcal{Z}}_{\eta 2} &\equiv \left[\frac{\partial}{\partial \eta_n} \left[\frac{\hat{\Delta}_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \right] a_{n1}^i - \frac{\hat{\Delta}_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \frac{\partial a_{n1}^i}{\partial \eta_n} \right] \\
&= (a_{n1}^i)^2 \frac{\partial}{\partial \eta_n} \left[\frac{\hat{\Delta}_n (\sigma_n^i a_{n1}^i)^2 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \right].
\end{aligned}$$

Equation (B.18) implies $\hat{\mathcal{Z}}_{\eta 2} = \Delta \mathcal{Z}_{\eta 2}$. Since $\mathcal{Z}_{\eta 2}$ is positive, so is $\hat{\mathcal{Z}}_{\eta 2}$. If $\mu_2 \lambda + \frac{u_n}{\eta_n} > 1$, then (3.15) and $\eta'_n = \eta_n$ imply that a_{n1}^i increases in η_n. Combining with (B.17) and (B.18), we find $\hat{\mathcal{Z}}_{\eta 1} \geq \Delta \mathcal{Z}_{\eta 2}$. Since (A.3) ensures that $\mathcal{Z}_{\eta 1}$ is non-negative for all $\eta \in [\eta_{n'}, \eta_n]$, it also ensures that $\hat{\mathcal{Z}}_{\eta 1}$ is non-negative.

The result in Case (iii) will follow if we show that $\hat{\mathcal{Z}}$ increases in u_n holding $\sum_{m=1}^{N} \eta'_m b_m$, $\sum_{m=1}^{N} \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m$ and Δ_n constant. Holding Δ_n constant is a sufficient condition because Δ_n increases in u_n. The derivative of $\hat{\mathcal{Z}}$ with respect to u_n has the same sign as $\hat{\mathcal{Z}}_{u1} + \hat{\mathcal{Z}}_{u2}$, where

$$\begin{aligned}
\hat{\mathcal{Z}}_{u1} &\equiv \frac{\partial}{\partial u_n} \left[\frac{\Delta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)}{2-(r+\kappa_n^i)a_{n1}^i} \right] \left[\bar{D}_n + b_n a_1^s (\kappa^s + rD_t^s) \right] \\
&\quad - \frac{b_n \left(\sum_{m=1}^{N} \Delta_m b_m\right)(\sigma^s)^2 (a_1^s)^3 (\kappa^s + rD_t^s)}{2-(r+\kappa^s)a_1^s} \frac{\partial a_{n1}^i}{\partial u_n} (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathcal{Z}}_{u2} &\equiv \left[\frac{\partial}{\partial u_n} \left[\frac{\Delta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \right] a_{n1}^i - \frac{\Delta_n (\sigma_n^i)^2 (a_{n1}^i)^3 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \frac{\partial a_{n1}^i}{\partial u_n} \right] \\
&= (a_{n1}^i)^2 \frac{\partial}{\partial u_n} \left[\frac{\Delta_n (\sigma_n^i)^2 (a_{n1}^i)^2 (\kappa_n^i \bar{D}_n^i + rD_{nt}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \right].
\end{aligned}$$

Equation (B.18) implies $\hat{\mathcal{Z}}_{u2} = \Delta \mathcal{Z}_{u2}$. Since $\mathcal{Z}_{u2}$ is positive, so is $\hat{\mathcal{Z}}_{u2}$. Since a_{n1}^i increases in u_n, (B.17) and (B.18) imply $\hat{\mathcal{Z}}_{u1} \geq \Delta \mathcal{Z}_{u2}$. Since (A.3) ensures that Z_{u1} is non-negative for all $u \in [u_{n'}, u_n]$, it also ensures that $\hat{\mathcal{Z}}_{u1}$ is non-negative. □

Proof of Proposition A.2. We first consider the case where μ_2 increases holding μ_1 constant.

Equations (3.2), (3.10), (3.11) and (3.13) imply that conditional return volatility is

$$\sqrt{\frac{\mathbb{V}\mathrm{ar}_t(dR_{nt})}{dt}} = \frac{\sqrt{b_n^2(\sigma^s a_1^s)^2 D_t^s + (\sigma_n^i a_{n1}^i)^2 D_{nt}^i}}{\frac{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)}{r}}. \tag{B.19}$$

The change in volatility has the same sign as the change in variance (the square of (B.19)), which has the same sign as

$$\begin{aligned}
&\left[b_n^2(\sigma^s)^2 2a_1^s \frac{\partial a_1^s}{\partial(\mu_2\lambda)} D_t^s + (\sigma_n^i)^2 2a_{n1}^i \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)} D_{nt}^i\right]\left[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)\right] \\
&- 2\left[b_n^2(\sigma^s a_1^s)^2 D_t^s + (\sigma_n^i a_{n1}^i)^2 D_{nt}^i\right]\left[b_n \frac{\partial a_1^s}{\partial(\mu_2\lambda)}(\kappa^s + rD_t^s) + \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)}(\kappa_n^i \bar{D}_n^i + rD_{nt}^i)\right] \\
&= 2\left[b_n^2(\sigma^s)^2 a_1^s \frac{\partial a_1^s}{\partial(\mu_2\lambda)} D_t^s + (\sigma_n^i)^2 a_{n1}^i \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)} D_{nt}^i\right]\bar{D}_n \\
&\quad + 2\left[b_n^2(\sigma^s)^2 a_1^s D_t^s(\kappa_n^i \bar{D}_n^i + rD_{nt}^i) - (\sigma_n^i)^2 a_{n1}^i D_{nt}^i(\kappa^s + rD_t^s)\right] b_n \left[a_{n1}^i \frac{\partial a_s^i}{\partial(\mu_2\lambda)} - a_1^s \frac{\partial a_{n1}^i}{\partial(\mu_2\lambda)}\right].
\end{aligned} \tag{B.20}$$

Using (3.14) and (3.15), we find that (B.20) has the same sign as

$$\begin{aligned}
&\left[\frac{b_n^2\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^4 D_t^s}{2-(r+\kappa^s)a_1^s} + \frac{\eta_n'(\sigma_n^i a_{n1}^i)^4 D_{nt}^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n \\
&+ \left[b_n(\sigma^s)^2 a_1^s D_t^s(\kappa_n^i \bar{D}_n^i + rD_{nt}^i) - (\sigma_n^i)^2 a_{n1}^i D_{nt}^i(\kappa^s + rD_t^s)\right] b_n a_1^s a_{1n}^i \\
&\times \left[\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n'(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right],
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&\left[\frac{b_n^2\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s} + \frac{\eta_n'(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n \\
&+ \left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s a_{1n}^i \bar{D}_n^i \\
&\times \left[\frac{\left(\sum_{m=1}^N \eta_m' b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n'(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] \equiv \mathcal{V}
\end{aligned} \tag{B.21}$$

for $D_t^s = \bar{D}^s = 1$ and $D_{nt}^i = \bar{D}_n^i$.

Consider first Case (i). When $\eta_{n'} \approx 0$, $\frac{u_n}{\eta_n} = \frac{u_{n'}}{\eta_{n'}}$ implies $u_{n'} \approx 0$, and (3.15) implies $a_{n'1}^i = \frac{1}{r+\kappa_n^i}$. Equation (B.21) implies that volatility declines for stock n' if

$$\begin{aligned}
&\frac{b_n^2\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s}\bar{D}_n \\
&+\left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - \frac{(r+\kappa^s)(\sigma_n^i)^2}{r+\kappa_n^i}\right]\frac{b_n a_1^s \bar{D}_n^i}{r+\kappa_n^i}\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} < 0 \\
&\Leftrightarrow \bar{D}_n < \left(\frac{(r+\kappa^s)(\sigma_n^i)^2}{(r+\kappa_n^i)^2 b_n(\sigma^s)^2 a_1^s} - 1\right)\bar{D}_n^i \equiv \bar{D}_{n,\max}. \qquad \text{(B.22)}
\end{aligned}$$

Conversely, (B.21) implies that volatility rises for stock $n \in \mathcal{I}$ if

$$\bar{D}_n > \frac{\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s} - 1\right)(r+\kappa_n^i)(b_n\sigma^s a_1^s)^2 a_{1n}^i \bar{D}_n^i\left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right]}{\frac{b_n^2\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s} + \frac{\eta_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}} \equiv \bar{D}_{n,\min} \qquad \text{(B.23)}$$

Equations (B.22) and (B.23) imply that volatility rises for stock n and declines for stock n' if $\bar{D}_n \in (\min\{\bar{D}_{n,\min}, 0\}, \bar{D}_{n,\max})$. If instead $\bar{D}_n > \bar{D}_{n,\max}$ then volatility rises for both stocks, and if $\bar{D}_n \in [0, \min\{\bar{D}_{n,\min}, 0\})$ then volatility declines for both stocks. Since (A.4) for $n' = n$ implies

$$\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \le \frac{2}{3}\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} \qquad \text{(B.24)}$$

and

$$\frac{\eta_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i} \ge \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s \sigma_n^i a_1^s a_{n1}^i)^2 \bar{D}_n^i}{3\left[2-(r+\kappa^s)a_1^s\right]}, \qquad \text{(B.25)}$$

the lower bound $\bar{D}_{n,\min}$ satisfies

$$\bar{D}_{n,\min} \le \frac{2\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s} - 1\right)(r+\kappa_n^i)a_{1n}^i \bar{D}_n^i}{3 + \frac{(\sigma_n^i a_{n1}^i)^2 \bar{D}_n^i}{(b_n\sigma^s a_1^s)^2}}, \qquad \text{(B.26)}$$

and the interval $(\min\{\bar{D}_{n,\min}, 0\}, \bar{D}_{n,\max})$ is non-empty if (A.5) holds.

Consider next Case (ii). We begin by showing that if $\bar{D}_n \geq \frac{1}{2}(r+\kappa^s)b_n a_1^s$ (and the other sufficient conditions in the proposition are met) then $\mathcal{V}$ is larger for stock n than for stock n', or equivalently $\mathcal{V}$ increases in u_n holding $\sum_{m=1}^N \eta'_m b_m$ and $\sum_{m=1}^N \frac{\eta_m - \mu_2 \lambda \eta'_m - u_m}{\mu_1} b_m$ constant. Since $(n, n') \in \mathcal{I} \times \mathcal{I}$ and a_{n1}^i increases in u_n, the derivative of $\mathcal{V}$ with respect to u_n has the same sign as $\mathcal{V}_{u1} + \mathcal{V}_{u2} + \mathcal{V}_{u3} + \mathcal{V}_{u4}$, where

$$\begin{aligned}
\mathcal{V}_{u1} &\equiv \frac{\partial}{\partial a_{n1}^i}\left[\frac{\eta_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n, \\
\mathcal{V}_{u2} &\equiv -(r+\kappa^s)(\sigma_n^i)^2 b_n a_1^s a_{1n}^i \bar{D}_n^i \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right], \\
\mathcal{V}_{u3} &\equiv \left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s \bar{D}_n^i \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right], \\
\mathcal{V}_{u4} &\equiv -\left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s a_{1n}^i \bar{D}_n^i \frac{\partial}{\partial a_{n1}^i}\left[\frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right].
\end{aligned}$$

Since a_{n1}^i increases in u_n, (A.6) implies that the term in the first bracket in $\mathcal{V}_{u3}$ and $\mathcal{V}_{u4}$ is negative for all $u \in [u_{n'}, u_n]$. Therefore, $\mathcal{V}_{u3} + \mathcal{V}_{u4}$ is positive if

$$a_{n1}^i \frac{\partial}{\partial a_{n1}^i}\left[\frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] - \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] > 0. \tag{B.27}$$

Equation (B.27) holds under the sufficient condition

$$\begin{aligned}
&\frac{2\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} - \left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] \geq 0 \\
\Leftrightarrow\ &\frac{3\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i} \geq \frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s},
\end{aligned}$$

which holds for all $u \in [u_{n'}, u_n]$ if (A.4) holds. The sum $\mathcal{V}_{u1} + \mathcal{V}_{u2}$ is positive under the sufficient condition

$$\frac{4\eta_n(\sigma_n^i)^4(a_{n1}^i)^3\bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\bar{D}_n-(r+\kappa^s)(\sigma_n^i)^2 b_n a_1^s a_{1n}^i \bar{D}_n^i\left[\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}-\frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right]\geq 0. \tag{B.28}$$

If (A.4) holds, then (B.28) holds for all $u\in[u_{n'},u_n]$ if

$$\begin{aligned}&\frac{4\eta_n(\sigma_n^i)^4(a_{n1}^i)^3\bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\bar{D}_n-2(r+\kappa^s)(\sigma_n^i)^2 b_n a_1^s a_{1n}^i \bar{D}_n^i\frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\geq 0\\ &\Leftrightarrow \bar{D}_n\geq\frac{1}{2}(r+\kappa^s)b_n a_1^s.\end{aligned}$$

We next determine whether the volatilities of stocks n and n' rise or decline. Equation (B.21) implies that stock n's volatility rises if $\bar{D}_n>\bar{D}_{n,\min}$. If $\bar{D}_{n,\min}>\frac{1}{2}(r+\kappa^s)b_n a_1^s$, then $\mathcal{V}$ for stock n' and $\bar{D}_n=\bar{D}_{n,\min}$ is negative. This is because $\mathcal{V}$ for stock n and $\bar{D}_n=\bar{D}_{n,\min}$ is zero, and $\mathcal{V}$ is larger for stock n than for stock n' if $\bar{D}_n>\frac{1}{2}(r+\kappa^s)b_n a_1^s$. Therefore, the threshold $\bar{D}_{n,\max}$ below which volatility declines for stock n' exceeds $\bar{D}_{n,\min}$. Since (A.6) implies

$$\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}-\frac{\eta_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\geq\left(1-\frac{1}{\psi}\right)\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s}$$

and

$$\frac{\eta_n(\sigma_n^i a_{n1}^i)^4\bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\leq\frac{\left(\sum_{m=1}^N \eta'_m b_m\right)(\sigma^s\sigma_n^i a_1^s a_{n1}^i)^2\bar{D}_n^i}{\psi\left[2-(r+\kappa^s)a_1^s\right]},$$

the lower bound $\bar{D}_{n,\min}$ satisfies

$$\bar{D}_{n,\min}\geq\frac{(\psi-1)\left(\frac{(r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i}{(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s}-1\right)(r+\kappa_n^i)a_{1n}^i\bar{D}_n^i}{\psi+\frac{(\sigma_n^i a_{n1}^i)^2\bar{D}_n^i}{(b_n\sigma^s a_1^s)^2}}. \tag{B.29}$$

Since a_{n1}^i increases in u_n, $\bar{D}_{n,\min}>\frac{1}{2}(r+\kappa^s)b_n a_1^s$ holds if (A.7) holds. Therefore, if (A.7) holds, then volatility rises for stock n and declines for stock n' if $\bar{D}_n\in(\bar{D}_{n,\min},\bar{D}_{n,\max})$. If instead $\bar{D}_n>\bar{D}_{n,\max}$ then volatility rises for both stocks, and if $\bar{D}_n\in[0,\bar{D}_{n,\min})$ then volatility declines for both stocks.

We next consider the case where μ_2 increases and μ_1 decreases by a corresponding fraction $\phi \in (0,1]$. Proceeding as in the case where μ_1 is held constant, and using the same notation as in the corresponding part of the proof of Proposition A.1, we find that the change in volatility has the same sign as

$$\left[\frac{b_n^2\left(\sum_{m=1}^N \hat{\Delta}_m b_m\right)(\sigma^s a_1^s)^4}{2-(r+\kappa^s)a_1^s} + \frac{\hat{\Delta}_n(\sigma_n^i a_{n1}^i)^4 \bar{D}_n^i}{2-(r+\kappa_n^i)a_{n1}^i}\right]\bar{D}_n$$
$$+\left[(r+\kappa_n^i)b_n(\sigma^s)^2 a_1^s - (r+\kappa^s)(\sigma_n^i)^2 a_{n1}^i\right] b_n a_1^s a_{1n}^i \bar{D}_n^i$$
$$\times\left[\frac{\left(\sum_{m=1}^N \hat{\Delta}_m b_m\right)(\sigma^s a_1^s)^2}{2-(r+\kappa^s)a_1^s} - \frac{\hat{\Delta}_n(\sigma_n^i a_{n1}^i)^2}{2-(r+\kappa_n^i)a_{n1}^i}\right] \equiv \hat{\mathcal{V}} \tag{B.30}$$

for $D_t^s = \bar{D}^s = 1$ and $D_{nt}^i = \bar{D}_n^i$. The result in Case (i) follows by proceeding as in the case where μ_1 is held constant. Equation (B.22) remains the same. Equation (B.23) remains the same except that $\sum_{m=1}^N \eta_m' b_m$ is replaced by $\sum_{m=1}^N \hat{\Delta}_m b_m$ and η_n is replaced by $\hat{\Delta}_n$. Equations (B.24) and (B.25) remain the same with the same replacements, as can be seen by combining (A.4) for $n' = n$, (B.17), (B.18) and $n \in \mathcal{I} \cap \operatorname{argmax}_m \frac{u_m}{\eta_m}$. Equation (B.26) remains the same. □

Proof of Proposition A.3. Suppose that stock n is added to the index. Since a_1^s does not change, (3.13) implies

$$\frac{S_{nt}^{\text{post}} - S_{nt}^{\text{pre}}}{S_{nt}^{\text{pre}}} = \frac{\left(a_{n1}^{i,\text{post}} - a_{n1}^{i,\text{pre}}\right)(k_n^i \bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^{i,\text{pre}}(k_n^i \bar{D}_n^i + rD_{nt}^i)}, \tag{B.31}$$

where $(S_{nt}^{\text{pre}}, a_{n1}^{i,\text{pre}})$ denote the values of (S_{nt}, a_{n1}^i) before index addition and $(S_{nt}^{\text{post}}, a_{n1}^{i,\text{post}})$ denote the values after addition. The value $a_{n1}^{i,pre}$ is obtained from (3.15) by setting $\mu_2\lambda = 0$, and the value $a_{n1}^{i,\text{post}}$ is obtained for $\mu_2\lambda$. Treating a_{n1}^i as a function of x ranging from zero to $\mu_2\lambda$, we can write (B.31) as

$$\frac{S_{nt}^{\text{post}} - S_{nt}^{\text{pre}}}{S_{nt}^{\text{pre}}} = \frac{\left(\int_0^{\mu_2\lambda} \frac{\partial a_{n1}^i(x)}{\partial x} dx\right)(k_n^i \bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i \bar{D}_n^i + rD_{nt}^i)}$$

$$= \frac{\left(\int_0^{\mu_2\lambda} \frac{\eta_n(\sigma_n^i)^2(a_{n1}^i(x))^3}{2-(r+\kappa_n^i)a_{n1}^i(x)}dx\right)(k_n^i\bar{D}_n^i + rD_{nt}^i)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i\bar{D}_n^i + rD_{nt}^i)},$$

where the second step follows from (3.15).

The result in Case (i) will follow if we show that

$$\frac{\frac{\eta_n(\sigma_n^i)^2(a_{n1}^i(x))^3}{2-(r+\kappa_n^i)a_{n1}^i(x)}}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i\bar{D}_n^i + rD_{nt}^i)} \equiv \mathcal{Y}$$

increases in η_n holding $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m$ and $\frac{u_n}{\eta_n}$ constant. Since $\eta_n > u_n$, (3.15) implies that $a_{n1}^i(0)$ decreases in η_n and so does the denominator of $\mathcal{Y}$. The numerator of $\mathcal{Y}$ is non-decreasing in η_n under the same condition that is needed for the partial derivative in the first line of (B.6) to be non-negative. That condition is (A.2), which for general x becomes

$$(r+\kappa_n^i)^2 \geq 2(\sqrt{2}-1)\rho\frac{\eta_n(1-x)-u_n}{\mu_1}(\sigma_n^i)^2. \tag{B.32}$$

Equation (A.8) ensures that (B.32) holds for all $x \in [0, \mu_2\lambda]$ and thus $\mathcal{Y}$ increases in η_n.

The result in Case (ii) will follow if we show that $\mathcal{Y}$ increases in u_n holding $\sum_{m=1}^N \frac{\eta_m - \mu_2\lambda\eta_m' - u_m}{\mu_1} b_m$ constant. Since (3.15) implies that $a_{n1}^i(0)$ increases in u_n,

$$\frac{a_{n1}^i(0)}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(0)(k_n^i\bar{D}_n^i + rD_{nt}^i)}$$

increases in u_n. Therefore, to show that $\mathcal{Y}$ increases in u_n, it suffices to show that

$$\frac{(a_{n1}^i(x))^3}{a_{n1}^i(0)[2-(r+\kappa_n^i)a_{n1}^i(x)]} = (a_{n1}^i(x))^2 \frac{r+\kappa_n^i + \sqrt{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}}{\sqrt{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - x\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}}$$

increases in u_n. That property follows from $a_{n1}^i(x)$ and

$$\frac{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}{(r+\kappa_n^i)^2 + 4\rho\frac{\eta_n - x\eta_n - u_n}{\mu_1}(\sigma_n^i)^2}$$

being increasing in u_n for $x \in [0, \mu_2\lambda]$. □

C Return Moments

To compute conditional expected return, we divide the right-hand side of (3.12) by S_{nt}. Using (3.13), and dropping the subscript n from $(\kappa_n^i, \bar{D}_n^i, \sigma_n^i)$, we find

$$\frac{\mathbb{E}_t(dR_{nt})}{dt} = \rho r \frac{\left[b_n\left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta_m'-u_m}{\mu_1}b_m\right)(\sigma^s a_1^s)^2 D_t^s + \frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1}(\sigma_n^i a_{n1}^i)^2 D_{nt}^i\right]}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)}. \tag{C.1}$$

Unconditional expected return is the expectation of (C.1)

$$\frac{\mathbb{E}(dR_{nt})}{dt} = \rho r \mathbb{E}\left\{\frac{\left[b_n\left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta_m'-u_m}{\mu_1}b_m\right)(\sigma^s a_1^s)^2 D_t^s + \frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1}(\sigma^i a_{n1}^i)^2 D_{nt}^i\right]}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)}\right\}. \tag{C.2}$$

When the stationary distribution of (D_t^s, D_{nt}^i) is gamma, the expectation in (C.2) becomes

$$\begin{aligned}&\int_{D_{nt}^i=0}^{\infty}\int_{D_t^s=0}^{\infty} \frac{\left[b_n\left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta_m'-u_m}{\mu_1}b_m\right)(\sigma^s a_1^s)^2 D_t^s + \frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1}(\sigma^i a_{n1}^i)^2 D_{nt}^i\right]}{\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)} \\ &\times \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)}(D_t^s)^{\alpha^s-1}e^{-\beta^s D_t^s}\frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D_{nt}^i)^{\alpha^i-1}e^{-\beta^i D_{nt}^i}dD_t^s dD_{nt}^i.\end{aligned} \tag{C.3}$$

Because the functions $(D_t^s)^{\alpha^s-1}$ and $(D_{nt}^i)^{\alpha^i-1}$ go to ∞ when D_t^s and D_{nt}^i, respectively, go to zero, the numerical calculation of the double integral in (C.3) becomes slow and inaccurate if the lower bounds are close to zero. We instead use a fast and accurate method by writing the double integral as a sum of four terms. We fix a small $\epsilon > 0$ and a large M. The integration domain for the first term is $(D_t^s, D_{nt}^i) \in [\epsilon, M] \times [\epsilon, M\bar{D}^i]$, and we compute that term using Matlab's double integration routine. The integration domain for the second term is $(D_t^s, D_{nt}^i) \in [0, \epsilon] \times [\epsilon, M\bar{D}^i]$, and we compute that term as

$$\begin{aligned}&\int_{D_{nt}^i=\epsilon}^{M\bar{D}^i}\int_{D_t^s=0}^{\epsilon} \frac{b_n\left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta_m'-u_m}{\mu_1}b_m\right)(\sigma^s a_1^s)^2 D_t^s}{\bar{D}_n + b_n a_1^s\kappa^s + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)}\frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)}(D_t^s)^{\alpha^s-1}\frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D_{nt}^i)^{\alpha^i-1}e^{-\beta^i D_{nt}^i}dD_t^s dD_{nt}^i \\ &+ \int_{D_{nt}^i=\epsilon}^{M\bar{D}^i}\int_{D_t^s=0}^{\epsilon} \frac{\frac{\eta_n-\mu_2\lambda\eta_n'-u_n}{\mu_1}(\sigma^i a_{n1}^i)^2 D_{nt}^i}{\bar{D}_n + b_n a_1^s\kappa^s + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)}\frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)}(D_t^s)^{\alpha^s-1}\frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D_{nt}^i)^{\alpha^i-1}e^{-\beta^i D_{nt}^i}dD_t^s dD_{nt}^i \\ &= \int_{D_{nt}^i=\epsilon}^{M\bar{D}^i} \frac{b_n\left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta_m'-u_m}{\mu_1}b_m\right)(\sigma^s a_1^s)^2}{\bar{D}_n + b_n a_1^s\kappa^s + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)}\frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)}\frac{\epsilon^{\alpha^s+1}}{\alpha^s+1}\frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)}(D_{nt}^i)^{\alpha^i-1}e^{-\beta^i D_{nt}^i}dD_{nt}^i\end{aligned}$$

$$+\int_{D^i_{nt}=\epsilon}^{M\bar{D}^i} \frac{\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}(\sigma^i a^i_{n1})^2 D^i_{nt}}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}(\kappa^i \bar{D}^i + rD^i_{nt})} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} \frac{\epsilon^{\alpha^s}}{\alpha^s} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i-1} e^{-\beta^i D^i_{nt}} dD^i_{nt}.$$

Thus, we approximate $\kappa^s + rD^s_t$ by κ^s and $e^{-\beta^s D^s_t}$ by one, then compute the exact integrals of $(D^s_t)^{\alpha^s}$ and $(D^s_t)^{\alpha^s-1}$ over $[0, \epsilon]$, and then use Matlab's integration routine to integrate with respect to D^i_{nt} over $[\epsilon, M\bar{D}_n]$. The integration domain for the third term is $(D^s_t, D^i_{nt}) \in [\epsilon, M] \times [0, \epsilon]$, and we compute that term as

$$\int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=\epsilon}^{M} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1} b_m\right) (\sigma^s a^s_1)^2 D^s_t}{\bar{D}_n + b_n a^s_1(\kappa^s + rD^s_t) + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s-1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i-1} dD^s_t dD^i_{nt}$$
$$+\int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=\epsilon}^{M} \frac{\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}(\sigma^i a^i_{n1})^2 D^i_{nt}}{\bar{D}_n + b_n a^s_1(\kappa^s + rD^s_t) + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s-1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i-1} dD^s_t dD^i_{nt}$$
$$= \int_{D^s_t=\epsilon}^{M} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1} b_m\right) (\sigma^s a^s_1)^2 D^s_t}{\bar{D}_n + b_n a^s_1(\kappa^s + rD^s_t) + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s-1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i}}{\alpha^i} dD^i_{nt}$$
$$+\int_{D^s_t=\epsilon}^{M} \frac{\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}(\sigma^i a^i_{n1})^2}{\bar{D}_n + b_n a^s_1(\kappa^s + rD^s_t) + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s-1} e^{-\beta^s D^s_t} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i+1}}{\alpha^i+1} dD^i_{nt}.$$

Thus, we approximate $\kappa^i \bar{D}^i + rD^i_{nt}$ by $\kappa^i \bar{D}^i$ and $e^{-\beta^i D^i_{nt}}$ by one, then compute the exact integrals of $(D^i_{nt})^{\alpha^i}$ and $(D^i_{nt})^{\alpha^i-1}$ over $[0, \epsilon]$, and then use Matlab's integration routine to integrate with respect to D^s_t over $[\epsilon, M]$. The integration domain for the fourth term is $(D^s_t, D^i_{nt}) \in [0, \epsilon] \times [0, \epsilon]$, and we compute that term as

$$\int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=0}^{\epsilon} \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1} b_m\right) (\sigma^s a^s_1)^2 D^s_t}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s-1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i-1} dD^s_t dD^i_{nt}$$
$$+\int_{D^i_{nt}=0}^{\epsilon} \int_{D^s_t=0}^{\epsilon} \frac{\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}(\sigma^i a^i_{n1})^2 D^i_{nt}}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} (D^s_t)^{\alpha^s-1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} (D^i_{nt})^{\alpha^i-1} dD^s_t dD^i_{nt}$$
$$= \frac{b_n \left(\sum_{m=1}^N \frac{\eta_m-\mu_2\lambda\eta'_m-u_m}{\mu_1} b_m\right) (\sigma^s a^s_1)^2}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} \frac{\epsilon^{\alpha^s+1}}{\alpha^s+1} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i}}{\alpha^i} dD^i_{nt}$$
$$+ \frac{\frac{\eta_n-\mu_2\lambda\eta'_n-u_n}{\mu_1}(\sigma^i a^i_{n1})^2}{\bar{D}_n + b_n a^s_1 \kappa^s + a^i_{n1}\kappa^i \bar{D}^i} \frac{(\beta_s)^{\alpha^s}}{\Gamma(\alpha^s)} \frac{\epsilon^{\alpha^s}}{\alpha^s} \frac{(\beta_i)^{\alpha^i}}{\Gamma(\alpha^i)} \frac{\epsilon^{\alpha^i+1}}{\alpha^i+1} dD^i_{nt}.$$

Thus, we approximate $\kappa^s + rD^s_t$ by κ^s, $\kappa^i \bar{D}^i + rD^i_{nt}$ by $\kappa^i \bar{D}^i$, and $e^{-\beta^s D^s_t}$ and $e^{-\beta^i D^i_{nt}}$ by one, and then compute the exact integrals of $(D^s_t)^{\alpha^s}$, $(D^s_t)^{\alpha^s-1}$, $(D^i_{nt})^{\alpha^i}$ and $(D^i_{nt})^{\alpha^i-1}$ over $[0, \epsilon]$. The sum of the four terms is independent of ϵ for ϵ ranging from 0.00001 to 0.01. For larger values of ϵ the approximations become inaccurate, and for smaller values of ϵ the Matlab integration routines become inaccurate.

Conditional return variance is the square of (B.19). Unconditional return variance is the expectation of conditional variance

$$\frac{\mathbb{V}\text{ar}(dR_{nt})}{dt} = r^2\mathbb{E}\left\{\frac{b_n^2(\sigma^s a_1^s)^2 D_t^s + (\sigma^i a_{n1}^i)^2 D_{nt}^i}{[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)]^2}\right\}, \tag{C.4}$$

because infinitesimal dR_{nt} implies that $\mathbb{E}(dR_{nt}^2)$ and $\mathbb{E}_t(dR_{nt}^2)$ are equal to $\mathbb{V}\text{ar}(dR_{nt})$ and $\mathbb{V}\text{ar}_t(dR_{nt})$, respectively, plus smaller-order terms. We calculate the expectation in (C.4) by writing the double integral as a sum of four terms, as in the case of expected return.

Unconditional CAPM beta is

$$\beta_{nt}^{\text{CAPM}} = \frac{\frac{\mathbb{C}\text{ov}(dR_{nt},dR_{Mt})}{dt}}{\frac{\mathbb{V}\text{ar}(dR_{Mt})}{dt}}, \tag{C.5}$$

where dR_{Mt} denotes the return of the index. The numerator of (C.5) is

$$\frac{\mathbb{C}\text{ov}(dR_{nt}, dR_{Mt})}{dt} =$$
$$r^2\mathbb{E}\left\{\frac{b_n(\sum_{m=1}^N \eta_m' b_m)(\sigma^s a_1^s)^2 D_t^s + \eta_n'(\sigma^i a_{n1}^i)^2 D_{nt}^i}{[\bar{D}_n + b_n a_1^s(\kappa^s + rD_t^s) + a_{n1}^i(\kappa^i\bar{D}^i + rD_{nt}^i)][\sum_{m=1}^N \eta_m'[\bar{D}_m + b_m a_1^s(\kappa^s + rD_t^s) + a_{m1}^i(\kappa^i\bar{D}^i + rD_{mt}^i)]}\right\}. \tag{C.6}$$

Computing the expectation in (C.6) requires integrating over $(D_t^s, \{D_{mt}^i\}_{m=1,..,N})$, i.e., $N+1$ random variables. To keep the integration manageable, we replace $\{D_{mt}^i\}_{m\neq n}$ by their expectations $\bar{D}^i$, thus applying the law of large numbers. We then calculate the expectation over (D_t^s, D_{nt}^i) by writing the double integral as a sum of four terms, as in the case of expected return. The denominator of (C.5) is

$$\frac{\mathbb{V}\text{ar}(dR_{Mt})}{dt} = r^2\mathbb{E}\left\{\frac{(\sum_{m=1}^N \eta_m' b_m)^2(\sigma^s a_1^s)^2 D_t^s + \sum_{m=1}^N (\eta_m')^2(\sigma^i a_{m1}^i)^2 D_{mt}^i}{[\sum_{m=1}^N \eta_m'[\bar{D}_m + b_m a_1^s(\kappa^s + rD_t^s) + a_{m1}^i(\kappa^i\bar{D}^i + rD_{mt}^i)]^2}\right\}. \tag{C.7}$$

We replace $\{D_{mt}^i\}_{m=1,..,N}$ by their expectations $\bar{D}^i$, and calculate the expectation over D_t^s by writing the integral as a sum of two terms, with integration domains $[0, \epsilon]$ and $[\epsilon, M]$. We do not distinguish between stock n and stocks $m \neq n$ because all stocks are symmetric in (C.7).

CAPM R-squared is

$$R^{2,\mathrm{CAPM}} = \frac{\left[\frac{\mathbb{C}\mathrm{ov}(dR_{nt}, dR_{Mt})}{dt}\right]^2}{\frac{\mathbb{V}\mathrm{ar}(dR_{nt})}{dt}\frac{\mathbb{V}\mathrm{ar}(dR_{Mt})}{dt}} = \left(\beta_{nt}^{\mathrm{CAPM}}\right)^2 \frac{\frac{\mathbb{V}\mathrm{ar}(dR_{Mt})}{dt}}{\frac{\mathbb{V}\mathrm{ar}(dR_{nt})}{dt}},$$

and can be computed from the previous moments.