Convergence Dynamics in the Solow Growth Model

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In the steady state (S.S.) of the Solow model, output per capita (y) grows at the rate of technological progress (ξ). Using a '*' to denote S.S. values, $y^*(t) = y^*(0)e^{\xi t}$, or after taking the (natural) log we have $\log y^*(t) = \log y^*(0) + \xi t$. This steady state growth path is the straight line plotted in **slide 13** of the lecture note, it has slope ξ. Here, we are interested in deriving the dynamics of $\log y$ out of steady state. Although we are interested in $\frac{d}{dt}(\log(y))$, it is easier to work with $\frac{d}{dt}(\log(\tilde{y}))$, where $\tilde{y} \equiv \frac{y}{A}$ is output per effective worker. To derive the convergence formula on **slide 12** of the lecture note, we proceed as follows.

Take a 1st order Taylor approx. of the function $\frac{d}{dt}(\log(\tilde{y}))$ about the S.S. $\log(\tilde{y}^*)$:

$$\frac{d}{dt}(\log(\tilde{y})) \approx \frac{d}{dt}(\log(\tilde{y}^*)) + \left.\frac{\partial\left[\frac{d}{dt}(\log(\tilde{y}))\right]}{\partial \log \tilde{y}}\right|_{\tilde{y}=\tilde{y}*} [\log(\tilde{y}) - \log(\tilde{y}^*)]$$

The first term on the right-hand side is zero. Define $\lambda \equiv \left.\frac{\partial\left[\frac{d}{dt}(\log(\tilde{y}))\right]}{\partial \log \tilde{y}}\right|_{\tilde{y}=\tilde{y}*}$ to be the convergence rate, then:

$$\frac{d}{dt}(\log(\tilde{y})) = \lambda [\log(\tilde{y}) - \log(\tilde{y}^*)] \tag{1}$$

Equ. (1) is a FODE in $\log(\tilde{y})$. Solve this FODE to get $\log(\tilde{y}(t))$ as a function of initial output per effective worker, time and S.S. variables. To do this rearrange the FODE to get:

$$\frac{d}{dt}(\log(\tilde{y})) - \lambda \log(\tilde{y}) = -\lambda \log(\tilde{y}^*)$$

The Integrating Factor (I.F.) is $e^{-\lambda t}$. Premultiply by I.F. on both sides to get:

$$\frac{d}{dt}\left[e^{-\lambda t}\log(\tilde{y})\right] = -\lambda e^{-\lambda t}\log(\tilde{y}^*)$$

Integrate w.r.t. time from time 0 to time t:

$$e^{-\lambda t}\log(\tilde{y}(t)) - \log(\tilde{y}(0)) = -\lambda \log(\tilde{y}^*)\int_0^t e^{-\lambda z} dz$$

N.B. $\log(\tilde{y}^*)$ is a constant so we factored it out of the right-hand side integral in the above expression.

$$e^{-\lambda t}\log(\tilde{y}(t)) = \log(\tilde{y}(0)) - \lambda \log(\tilde{y}^*)\left[\frac{e^{-\lambda t} - 1}{-\lambda}\right]$$

$$\log(\tilde{y}(t)) = e^{\lambda t}\log(\tilde{y}(0)) + e^{\lambda t}\log(\tilde{y}^*)\left[e^{-\lambda t} - 1\right]$$

$$\log(\tilde{y}(t)) = \log(\tilde{y}^*) + e^{\lambda t}[\log(\tilde{y}(0)) - \log(\tilde{y}^*)]$$

Recall: $\log(\tilde{y}(t)) = \log(\frac{y(t)}{A(t)}) = \log(y(t)) - \log(A(t))$. So, we can re-write the above equation in terms of $\log(y(t))$:

$$\log(y(t)) - \log(A(t)) = \log(y^*(t)) - \log(A(t)) + e^{\lambda t}[\log(y(0)) - \log(A(0)) - \log(y^*(t)) + \log(A(t))]$$

Recall: $\log(A(t)) = \log(A(0)e^{\xi t}) = \log(A(0)) + \xi t$. Plugging this in we get:

[1] All errors are my own.

$$\log(y(t)) = \log(y^*(t)) + e^{\lambda t}\left[\log(y(0)) - \log(A(0)) - \log(y^*(t)) + \log(A(0)) + \xi t\right]$$

Finally using the S.S. value: $\log(y^*(t)) = \Gamma_0 + \xi t$ we arrive at the convergence expression in **slide 12** of the lecture note:

$$\log(y(t)) = \Gamma_0 + \xi t + e^{\lambda t}\left[\log(y(0)) - \Gamma_0\right]$$

Question 3 of Exercise 1:

Assume a Cobb Douglas aggregate production function $Y = F(K, NA) = K^{\alpha}\,(NA)^{1-\alpha}$ with $\alpha \in (0,1)$. Find Γ_0 and λ.

$$y \equiv \frac{Y}{N} = AF(\frac{k}{A}, 1) \equiv Af\left(\frac{k}{A}\right)$$

where the first equality is due to CRS of F. Take logs of the above and evaluate at S.S. to get:

$$\begin{aligned}
\log(y^*(t)) &= \log\left[f\left(\left[\frac{k}{A}\right]^*\right)\right] + \log A(t)\\
&= \log\left[f\left(\left[\frac{k}{A}\right]^*\right)\right] + \log A(0) + \xi t\\
&= \Gamma_0 + \xi t
\end{aligned}$$

where $\Gamma_0 = \log\left[f\left(\left[\frac{k}{A}\right]^*\right)\right] + \log A(0)$.

In S.S. we know:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \tau\frac{f(\tilde{k})}{\tilde{k}} - (\delta + \nu + \xi) = 0$$

For our Cobb Douglas production function, $f\left(\frac{k}{A}\right) = \left(\frac{k}{A}\right)^{\alpha}$. So:

$$\tilde{k}^{*(\alpha-1)} = \frac{\delta + \nu + \xi}{\tau}$$

$$\tilde{k}^* = \left(\frac{\tau}{\delta + \nu + \xi}\right)^{\frac{1}{1-\alpha}}$$

$\therefore \Gamma_0 = \frac{\alpha}{1-\alpha}\log\left(\frac{\tau}{\delta+\nu+\xi}\right) + \log A(0)$.

Finally, we have to find λ. Recall $\lambda \equiv \left.\frac{\partial\left[\frac{d}{dt}(\log(\tilde{y}))\right]}{\partial \log \tilde{y}}\right|_{\tilde{y}=\tilde{y}*} = \left.\frac{\partial\left[\frac{\dot{\tilde{y}}}{\tilde{y}}\right]}{\partial \log \tilde{y}}\right|_{\tilde{y}=\tilde{y}*} = \left.\frac{\partial\left[\alpha\frac{\dot{\tilde{k}}}{\tilde{k}}\right]}{\partial \log \tilde{y}}\right|_{\tilde{y}=\tilde{y}*}$, where the second equality is due to the Cobb Douglas production function, $\tilde{y} = \tilde{k}^{\alpha}$.

Substitute the fundamental equ. of the Solow model $\frac{\dot{\tilde{k}}}{\tilde{k}} = \tau\frac{f(\tilde{k})}{\tilde{k}} - (\delta + \nu + \xi)$:

$$\begin{aligned}
\lambda &= \frac{\partial\left(\alpha\left[\tau\frac{f(\tilde{k})}{\tilde{k}}-(\delta+\nu+\xi)\right]\right)}{\partial\log\tilde{y}}|_{\tilde{y}=\tilde{y}*} \\
&= \frac{\partial\left(\alpha\left[\tau\frac{\tilde{y}}{\tilde{y}^{1/\alpha}}-(\delta+\nu+\xi)\right]\right)}{\partial\log\tilde{y}}|_{\tilde{y}=\tilde{y}*} \\
&= \frac{\partial\left(\alpha\left[\tau e^{\frac{(\alpha-1)}{\alpha}\log\tilde{y}}-(\delta+\nu+\xi)\right]\right)}{\partial\log\tilde{y}}|_{\tilde{y}=\tilde{y}*} \\
&= (\alpha-1)\tau\tilde{y}^{\frac{(\alpha-1)}{\alpha}}|_{\tilde{y}=\tilde{y}*} \\
&= (\alpha-1)(\delta+\nu+\xi) \\
&< 0
\end{aligned}$$

where the last equality is from substituting $\tau=\frac{\delta+\nu+\xi}{\tilde{y}^{\frac{(\alpha-1)}{\alpha}}}$ in S.S..