Ramsey-Cass-Koopmans Growth Model

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This note discusses some of the aspects of the Ramsey model mentioned in Prof. Danny Quah's handout entitled "Technical note Ramsey-Cass-Koopmans growth" (http://moodle.lse.ac.uk/mod/resource/view.php?id=123481). Refer to page 3. The Inada conditions (part of the neoclassical assumptions) are typically written in terms of the marginal product of capital as:

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) = +\infty$$

and

$$\lim_{\tilde{k}\to \infty} f'(\tilde{k}) = 0.$$

By L'Hôpital's rule, we have $\lim_{\tilde{k}\to 0} \frac{f(\tilde{k})}{\tilde{k}} = \frac{0}{0} = \lim_{\tilde{k}\to 0} f'(\tilde{k})$ and $\lim_{\tilde{k}\to \infty} \frac{f(\tilde{k})}{\tilde{k}} = \frac{\infty}{\infty} = \lim_{\tilde{k}\to \infty} f'(\tilde{k})$ and so in the limit the average product and marginal product are equivalent.[2]

Social Planner's Problem

The following discussion follows pages 4-7. The social planner's problem writes as:

$$\max_{\{\tilde{c},\tilde{k}\}} \int_0^\infty e^{-(\rho-\upsilon)t} U(\tilde{c}(t)A(t))dt$$

s.t.

$$\dot{\tilde{k}}(t) = f(\tilde{k}(t)) - \tilde{c}(t) - \zeta\tilde{k}(t)$$

where $\zeta = \delta + \nu + \xi$ and $\tilde{k}(0)$ is given.

The present value Hamiltonian for this problem is:

$$H = U(\tilde{c}(t)A(t))e^{-(\rho-\upsilon)t} + \left[f(\tilde{k}(t)) - \tilde{c}(t) - \zeta\tilde{k}(t)\right]\lambda(t)e^{-(\rho-\upsilon)t}$$

The F.O.C.s (dropping the time subscripts where no confusion occurs) are:

[1] Any errors are my own.

[2] Away from the limit, the average product exceeds the marginal product when there is diminishing marginal returns. Consider the Cobb Douglas production function: $F(K,AN) = K^\alpha (AN)^{1-\alpha}$, $\alpha \in (0,1)$; $f(\tilde{k}) = \tilde{k}^\alpha$ and $f'(\tilde{k}) = \alpha\tilde{k}^{\alpha-1} \neq \tilde{k}^{\alpha-1} = \frac{f(\tilde{k})}{\tilde{k}}$.

$$\frac{\partial H}{\partial \tilde{c}} = 0 \iff U'(\tilde{c}A)Ae^{-(\rho-\upsilon)t} = \lambda e^{-(\rho-\upsilon)t} \tag{2.7}$$
$$\iff AU'(\tilde{c}A) = \lambda$$

$$\frac{\partial H}{\partial \tilde{k}} = -\frac{d}{dt}\left[\lambda e^{-(\rho-\upsilon)t}\right] \tag{2.8}$$
$$\iff \left(f'(\tilde{k}) - \zeta\right)\lambda e^{-(\rho-\upsilon)t} = -\frac{d}{dt}\left[\lambda e^{-(\rho-\upsilon)t}\right]$$
$$\iff f'(\tilde{k}) - \zeta = -\frac{d}{dt}\left[\ln\left(\lambda e^{-(\rho-\upsilon)t}\right)\right]$$
$$\iff f'(\tilde{k}) - \zeta = -\left[\frac{\dot{\lambda}}{\lambda} - (\rho - \upsilon)\right]$$

Now we derive the Euler equation from equations (2.7) and (2.8):
Take logs of (2.7):

$$\log A + \log U'(\tilde{c}A) = \log \lambda$$

Differentiate w.r.t. time:

$$\frac{\dot{A}}{A} + \frac{U''(\tilde{c}A)}{U'(\tilde{c}A)}\left[\dot{\tilde{c}}A + \dot{A}\tilde{c}\right] = \frac{\dot{\lambda}}{\lambda}$$

$$\frac{\dot{A}}{A} + \left\{\frac{(\tilde{c}A)\,U''(\tilde{c}A)}{U'(\tilde{c}A)}\right\}\frac{\left[\dot{\tilde{c}}A + \dot{A}\tilde{c}\right]}{\tilde{c}A} = \frac{\dot{\lambda}}{\lambda}$$

where the term in curly brackets $\frac{\tilde{c}AU''(\tilde{c}A)}{U'(\tilde{c}A)}$ is "minus" the coefficient of relative risk aversion. Define $R(\tilde{c}A) \equiv -\frac{(\tilde{c}A)U''(\tilde{c}A)}{U'(\tilde{c}A)}$. Then, replacing $\frac{\dot{A}}{A} = \xi$ we have:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = R(\tilde{c}A)^{-1}\left(\xi - \frac{\dot{\lambda}}{\lambda}\right) \tag{2.10}$$

Assume the utility function exhibits CRRA, that is, $R(\tilde{c}A)$ is a constant R. Use (2.8) to eliminate λ from (2.10) and this gives us the following Euler equation:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = R^{-1}\left[\xi + f'(\tilde{k}) - \zeta - (\rho - \upsilon)\right]$$

NB: $\zeta = \delta + \nu + \xi$, so this simplifies to:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} + \xi = R^{-1}\left[f'(\tilde{k}) - \delta - \rho\right]$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \left[f'(\tilde{k}) - (\delta + \rho + R\xi)\right] R^{-1} \tag{2.11}$$

The endogenous variables in the model are $\left\{\tilde{c}(t), \tilde{k}(t), \lambda(t) : t \in [0, \infty)\right\}$. The social planner's solution is governed by 3 equations: the Euler equation, the transition law and the transversality condition. The Euler equation (2.11) is a FODE for consumption, that is, it tells us how consumption in the next instant depends on consumption in the last instant for optimality. The transition law $\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + \nu + \xi)\,\tilde{k}$ is a FODE for the capital stock. Notice that these two FODEs are not independent, the Euler equation depends on $\tilde{k}$ and the transition law depends on $\tilde{c}$. So we have a (non-linear) SYSTEM of differential equations that must be solved simultaneously to find the optimal paths of $\tilde{c}$, $\tilde{k}$ and λ (NB: equation (2.7) gives us λ once we have $\tilde{c}$). Since we have two FODEs, to pin down the levels of $\left\{\tilde{c}(t), \tilde{k}(t), \lambda(t) : t \in [0, \infty)\right\}$ we need two boundary conditions. The capital stock at time zero $\tilde{k}(0)$ is one and the other is the so-called transversality (or terminal) condition $\lim_{t \to \infty} \left\{\tilde{k}(t)\lambda(t)e^{-(\rho - \upsilon)t}\right\} = 0$. Most of the time we are just interested in whether the system tends to a steady state or BGP. Graphically we can do this using a phase diagram or, more formally, by calculating the eigenvalues of the two-variable system of FODEs.