Real Business Cycle Models

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1 Intro

RBC models belong to a class of models called *Dynamic Stochastic General Equilibrium* (DSGE) Models. They are *dynamic* because investment is a key endogenous variable, which facilitates the smoothing of household consumption over time. They are *stochastic* because an exogenous shock process is introduced to trigger fluctuations in the endogenous variables around their balanced growth paths (that is, a business cycle). RBC models consider exogenous shocks to productivity and/ or government expenditure (i.e. real shocks). The model is written in discrete time because business cycles are fluctuations with a periodicity of between 2 and 8 years.

It is a general equilibrium model, that is, households and firms are maximising utility and profits respectively, and markets clear.

2 Baseline RBC Model

Define output Y, capital K, TFP A, labour supply N, consumption C. real interest rate R, and the real wage W. The economy is characterised by the following set of 7 equations:

Production function:

$$Y_t = K_t^{1-\alpha}(A_t N_t)^{\alpha} \tag{1}$$

where $\alpha \in (0,1)$.
Capital accumulation:

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t \tag{2}$$

NB: there is no government so $Y_t = C_t + I_t$.

Technology:

$$\ln A_t = \ln G + \ln A^*_{t+1} + \phi \ln \tilde{A}_{t-1} + \varepsilon_t \tag{3}$$

FOCs of Firms:

$$R_t = (1-\alpha)\left(\frac{A_t N_t}{K_t}\right)^{\alpha} + (1-\delta) \tag{4}$$

$$W_t = \alpha A_t^{\alpha}\left(\frac{K_t}{N_t}\right)^{1-\alpha} \tag{5}$$

where $R_t \, (\equiv (1+r_t))$ is the gross interest rate.

FOCs of the representative household:

$$\frac{1}{C_t} = \beta E_t\left[\frac{R_{t+1}}{C_{t+1}}\right] \tag{6}$$

$$\frac{W_t}{C_t} = \theta\,(1-N_t)^{-\gamma} \tag{7}$$

3 Deriving the Household and Firm FOCs

We will now derive the FOCs of the household: equations (6) and (7). Because this is a Walrasian model, the first and second welfare theorems hold and hence the solution to the decentralised problem is identical to that of a social planner. Nonetheless, I solve both cases here, first the decentralised competitive equilibrium and second the social planner's problem.

3.1 The Decentralised Competitive Equilibrium

The general competitive equilibrium is a stochastic process for the endogenous variables $\{Y_t, C_t, N_t, K_t, R_t, W_t, A_t\}_{t=0}^{\infty}$ such that: (1) households are maximising their utility subject to their budget constraint; (2) firms are maximising profits subject to the production function; and finally (3) markets clear.

3.1.1 Households

The *representative* household (taking $\{R_t, W_t\}_{t=0}^{\infty}$ as given) solves:

$$\max_{\{C_t, N_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, 1-l_t)$$

subject to the period budget constraint:

$$a_{t+1} = R_t a_t + W_t l_t - C_t$$

where $l_t \in (0,1)$ is the proportion of time the representative household spends working; and a_t is the stock of assets (savings) held by the representative household.

State: a_t.
Controls: C_t, l_t.
The Bellman equation is:

$$V(a_t, A_t) = \max_{C_t, l_t} \{U(C_t, 1 - l_t) + \beta E\left[V(a_{t+1}, A_{t+1}) \mid A_t\right]\}$$

subject to:

$$a_{t+1} = R_t a_t + W_t l_t - C_t$$

FOCs:

$$U_1(C_t, 1 - l_t) = \beta.E\left[\frac{\partial V(a_{t+1}, A_{t+1})}{\partial a_{t+1}} \mid A_t\right] \tag{8}$$

$$U_2(C_t, 1 - l_t) = \beta.E\left[\frac{\partial V(a_{t+1}, A_{t+1})}{\partial a_{t+1}}.W_t \mid A_t\right] \tag{9}$$

E.T:

$$\frac{\partial V(a_t, A_t)}{\partial a_t} = U_1(C_t, 1 - l_t).R_t$$

Iterating forward to t+1:

$$\frac{\partial V(a_{t+1}, A_{t+1})}{\partial a_{t+1}} = U_1(C_{t+1}, 1 - l_{t+1}).R_{t+1} \tag{10}$$

Substituting eq. (10) into the F.O.C. in eq. (8) we get the INTERTEMPORAL FOC (or Euler equation):

$$U_1(C_t, 1 - l_t) = \beta.E\left[U_1(C_{t+1}, 1 - l_{t+1}).R_{t+1} \mid A_t\right] \tag{11}$$

Substituting eq. (10) into the F.O.C. in eq. (9) we get the INTRATEMPORAL FOC:

$$U_2(C_t, 1 - l_t) = U_1(C_t, 1 - l_t).W_t \tag{12}$$

where we have substituted the Euler equation (11).

3.1.2 Firms

Taking the stochastc prices $\{R_t, W_t\}_{t=0}^{\infty}$ as given, the *representative* firm solves $\forall t$:

$$\max_{\{K_t, N_t\}_{t=0}^{\infty}} \pi = Y_t - W_t N_t - (R_t - 1 + \delta)K_t$$

subject to the production function:

$$Y_t = K_t^{1-\alpha}(A_t N_t)^{\alpha}$$

Note: there are no intertemporal elements here so the representative firm's problem is a simple static one.

FOCs (although should use perturbation argument b/c hessian is not negative definite w/ Cobb Douglas):

$$R_t = (1-\alpha)\left(\frac{A_t N_t}{K_t}\right)^{\alpha} + (1-\delta)$$

$$W_t = \alpha A_t^{\alpha}\left(\frac{K_t}{N_t}\right)^{1-\alpha}$$

These are equations (4) and (5) above.

3.1.3 Markets Clear

For all t:
w_t clears the labour market: $l_t = N_t$
r_t clears the capital market: $a_t = K_t$

Equations (6) and (7) are the result of evaluating equations (11) and (12) for the admissible utility function:

$$U(C_t, 1-l_t) = \log(C_t) + \theta\frac{(1-l_t)^{1-\gamma}}{1-\gamma}$$

The above utility function is one of a limited set of functional forms for which the model exhibits a *balanced growth path* (see King, Plosser and Rebelo (1988)).[1] A non-stochastic BGP must feature constant N^*. On the BGP we know from the Ramsey (and Solow) model that the wage W_t grows at rate G. However, because of our choice of utility function the household FOC governing the choice of N_t (equation (7)) depends only on the ratio $\left(\frac{W_t}{C_t}\right)$ which is constant on the BGP because both W_t and C_t grow at rate G on the BGP.

3.2 The Social Planner's Problem

The social planner's problem is:

$$\max_{\{C_t, N_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, 1-N_t)$$

[1] Alternatively one could use $U(C_t, 1-N_t) = \frac{C_t^{1-\sigma}}{1-\sigma}\upsilon(1-N_t)$. In which case the inter-temporal FOC becomes $C_t^{-\sigma}\upsilon(1-N_t) = \beta E_t\left[R_{t+1}C_{t+1}^{-\sigma}\upsilon(1-N_{t+1})\right]$ and the intra-temporal FOC is $(1-\sigma)\frac{W_t}{C_t} = \frac{\upsilon'(1-N_t)}{\upsilon(1-N_t)}$. This latter FOC exhibits a constant N^* on the BGP.

subject to equations (1), (2), (3), (4) and (5).
State: K_t.
Controls: C_t, N_t.
The Bellman equation is:

$$V(K_t, A_t) = \max_{C_t, N_t} \{U(C_t, 1 - N_t) + \beta E[V(K_{t+1}, A_{t+1}) \mid A_t]\}$$

subject to:

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t$$

$$Y_t = K_t^{1-\alpha}(A_t N_t)^{\alpha}$$

$$\ln A_t = \ln G + \ln A^*_{t+1} + \phi \ln \tilde{A}_{t-1} + \varepsilon_t$$

FOCs:[2]

$$U_1(C_t, 1 - N_t) = \beta E\left[\frac{\partial V(K_{t+1}, A_{t+1})}{\partial K_{t+1}} \mid A_t\right] \tag{13}$$

$$U_2(C_t, 1 - N_t) = \beta E\left[\frac{\partial V(K_{t+1}, A_{t+1})}{\partial K_{t+1}} \left(\alpha K_t^{1-\alpha} A_t^{\alpha} {N_t}^{\alpha-1}\right) \mid A_t\right] \tag{14}$$

E.T:

$$\frac{\partial V(K_t, A_t)}{\partial K_t} = U_1(C_t, 1 - N_t)\left[(1 - \delta) + (1 - \alpha)K_t^{-\alpha}(A_t N_t)^{\alpha}\right]$$

Iterating forward to t+1:

$$\frac{\partial V(K_{t+1}, A_{t+1})}{\partial K_{t+1}} = U_1(C_{t+1}, 1 - N_{t+1})\left[(1 - \delta) + (1 - \alpha)K_{t+1}^{-\alpha}(A_{t+1} N_{t+1})^{\alpha}\right] \tag{15}$$

Substituting eq. (15) into the F.O.C. in eq. (13) we get the INTERTEMPORAL FOC (or Euler equation):

$$U_1(C_t, 1 - N_t) = \beta E[U_1(C_{t+1}, 1 - N_{t+1}).R_{t+1} \mid A_t] \tag{16}$$

where we have substituted the gross interest rate $R_t = (1 - \alpha)\left(\frac{A_t N_t}{K_t}\right)^{\alpha} + (1 - \delta)$.

[2] It is important that you choose (C_t, N_t) as opposed to (K_{t+1}, N_t) as controls. The reason is that there are two FOCs - one for each control - and each FOC is derived as the partial derivative, that is, *holding the other control constant.* The transition law tells us the dependence between K_{t+1}, C_t, and N_t. Equation (14) is only true if we hold C_t constant as opposed to K_{t+1}.

Substituting eq. (15) into the F.O.C. in eq. (14) we get the INTRATEMPORAL FOC:

$$U_2(C_t, 1 - N_t) = U_1(C_t, 1 - N_t).W_t \tag{17}$$

where we have substituted the Euler equation (16) and the wage rate $W_t = \alpha A_t^{\alpha} \left(\frac{K_t}{N_t}\right)^{1-\alpha}$.

Equations (6) and (7) are the result of evaluating equations (16) and (17) for the admissible utility function:

$$U(C_t, 1 - N_t) = \log(C_t) + \theta \frac{(1 - N_t)^{1-\gamma}}{1 - \gamma}$$

4 Solving the Model

The goal: solve for the evolution of the endogenous variables $\{Y_t, C_t, N_t, K_t, R_t, W_t, A_t\}_{t=0}^{\infty}$ as a function of any realization of the exogenous technology shocks $\{\varepsilon_t\}_{t=0}^{\infty}$ and the parameters of the model $(\alpha, \beta, \delta, \theta, \gamma, \phi, G)$. The impulse response function for an endogenous variable is defined as the evolution of this variable following a one-time-only 1% shock to the productivity parameter ($\varepsilon_t = 1$, $\varepsilon_{t+s} = 0$ $\forall s \geq 1$).

This is not easy because the 7 equations that characterise the model are a mixture of linear and log-linear equations so no closed-form solution exists. There are various methods that can be used. One is to take a first-order Taylor approximation about the BGP for each one of the 7 equations and then use the method of undetermined coefficients.

5 Evaluating the Model

The model does a good job if it can match the stylised facts of business cycles observed in postwar U.S. quarterly data, both qualitative (co-movement) and quantitative (volatility, persistence). Typically this is done by eyeballing the impulse responses. To derive the impulse responses, you will have to pick numbers for the parameters of the model $(\alpha, \beta, \delta, \theta, \gamma, \phi, G)$, this is called *calibration*.

The big failure of the RBC model is matching the fluctuations in employment under reasonable parameter values. To see why, look at the household intratemporal FOC (equation (12)):

$$U_2(C_t, 1 - l_t) = U_1(C_t, 1 - l_t).W_t$$

The above equation tells us that a necessary condition for both C_t and l_t to be procyclical is that W_t is procyclical. Why? Consider a boom: $C \uparrow$ and $l \uparrow$. $C \uparrow \implies U_1 \downarrow$ and $l \uparrow \implies U_2 \uparrow$. These two effects imply $W \uparrow$ since $U_2 = U_1.W$. In the data, C and l are strongly procyclical but W is only mildly procyclical. So we need to assume a very strong substitution away from leisure and into consumption from a change in the wage. That is, we need a very low γ, because $1/\gamma$ is the intertemporal elasticity of substitution for leisure.

6 A Special Case: Brock-Mirman model

There is a special case of the above model which displays a closed-form solution. The strong assumptions are:

- log utility: $U(C_t, 1-l_t) = \ln(C_t) + b\ln(1-l_t)$
- full depreciation: $\delta = 1$
- (no government) -> which I left out of the above baseline model.

Applying these changes to the 7 equations we have:

Production function:

$$Y_t = K_t^{1-\alpha}(A_t N_t)^\alpha$$

Capital accumulation:

$$K_{t+1} = Y_t - C_t$$

Technology:

$$\ln A_t = \ln G + \ln A^*_{t+1} + \phi \ln \tilde{A}_{t-1} + \varepsilon_t$$

FOCs of Firms:

$$R_t = (1-\alpha)\left(\frac{A_t N_t}{K_t}\right)^\alpha \tag{18}$$

where R_t ($\equiv (1+r_t)$) is the gross interest rate.

$$W_t = \alpha A_t^\alpha \left(\frac{K_t}{N_t}\right)^{1-\alpha} \tag{19}$$

FOCs of the representative household:

$$\frac{1}{C_t} = \beta E_t\left[\frac{R_{t+1}}{C_{t+1}}\right] \tag{20}$$

$$\frac{W_t}{C_t} = \frac{b}{1-N_t} \tag{21}$$

Using $C_t = (1 - s_t)Y_t$, $K_{t+1} = s_t Y_t$, and equation (18), equation (20) becomes:

$$\frac{1}{C_t} = \beta E_t \left[\frac{(1-\alpha)\left(\frac{A_{t+1}N_{t+1}}{K_{t+1}}\right)^{\alpha}}{(1-s_{t+1})Y_{t+1}} \right]$$

$$\frac{1}{C_t} = \beta E_t \left[\frac{(1-\alpha)}{(1-s_{t+1})K_{t+1}^{1-\alpha}K_{t+1}^{\alpha}} \right]$$

$$\frac{1}{C_t} = \beta E_t \left[\frac{(1-\alpha)}{(1-s_{t+1})s_t Y_t} \right]$$

$$\frac{1}{(1-s_t)Y_t} = \frac{(1-\alpha)\beta}{s_t Y_t} E_t \left[\frac{1}{(1-s_{t+1})} \right]$$

$$\frac{1}{(1-s_t)} = \frac{(1-\alpha)\beta}{s_t} E_t \left[\frac{1}{(1-s_{t+1})} \right]$$

Assume $s_t = s_{t+1} \equiv s$. Then $s = (1-\alpha)\beta$. Why? If $s_t < (1-\alpha)\beta$ then $\lim_{t\to\infty} E_t [s_t] = -\infty$. If $s_t > (1-\alpha)\beta$ then $\lim_{t\to\infty} E_t [s_t] = 1$. None of these cases can be optimal.

Substituting the constant saving rate $s = (1-\alpha)\beta$ and equation (19) into equation (21) yields:

$$\frac{W_t}{C_t} = \frac{b}{1-N_t}$$

$$\frac{\frac{\alpha Y_t}{N_t}}{(1-s)Y_t} = \frac{b}{1-N_t}$$

$$\frac{\alpha}{1-s} = \frac{b.N_t}{1-N_t}$$

$$\frac{1-N_t}{N_t} = \frac{b\,(1-s)}{\alpha}$$

$$N_t = \frac{\alpha}{b\,(1-s)+\alpha}$$

$$N = \frac{\alpha}{b\,(1-(1-\alpha)\beta)+\alpha}$$

So N_t is also constant. The saving and labour supply decisions are unaffected by changes in the interest rate and wage rate. This is the result of log utility and full depreciation, that is, the income and substitution effects exactly cancel each other.

6.1 Evaluating the Special Case

- The wage is too procyclical: $W_t = \frac{\alpha Y_t}{N}$.
  - No fluctuations in employment: N constant.
  - Consumption is too procyclical: $C_t = (1 - s_t)Y_t$.
  - Investment is not procyclical enough: $I_t = Y_t - C_t = sY_t = (1 - \alpha)\beta Y_t$.

So the model gets most of the comovements WRONG!.