RBC Problem Set 1: Remarks

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Problem set 1 encourages you to think about certain aspects of the canonical RBC model without actually writing down the full model. Here we briefly discuss some of these.

Question 1

We are asked to identify the successes and failures of the basic RBC model. Recall, RBC theory is judged on whether realistic exogenous shocks to TFP can generate the quantitative characteristics (volatility, persistence, comovement) of business cycles observed in the data. The biggest failure of the canonical RBC model is its inability to generate enough volatility in total hours (employment) to match the data. The reason for this can be seen in Question 2(b) below.

Question 2(a)

The consumer's one-period problem writes as:

$$\max_{\{c_1, l_1\}} \left[\ln c_1 + b \frac{(1-l_1)^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

subject to the budget constraint:

$$c_1 = w_1 l_1 \tag{2}$$

where c is consumption and l is labour supply (the fraction of time spent working). Using (2) to substitute for c_1 in (1) and taking the FOC w.r.t. l_1 yields:

$$\frac{1}{l_1} = b\,(1-l_1)^{-\gamma} \tag{3}$$

Equation (3) implicitly defines l_1. We notice that the choice of l_1 does not depend on the wage. Therefore, it must be that the income and substitution effects of a change in the wage exactly offset eachother.[1] This happens when the utility function takes the log form.

Remark: this is a so-called admissible utility function for a balanced growth path (see King, Plosser and Rebelo, 1988). We know from our earlier work on neoclassical growth theory that in steady state the wage grows at the rate of technological progress. In the data, there is no long-run trend in total hours worked (employment) so we want a utility function that on the non-stochastic

[1] The wage is the price of leisure. A change in the price of leisure has both income and substitution effects on the consumption of leisure.

growth path gives rise to constant total hours (employment). We have shown that the utility function above achieves this.

Question 2(b)

The RBC model - a neoclassical general equilibrium model - forces the economy to be on its labour supply curve (one of the household's FOCs). Question 2(b) asks us to derive this household FOC for the two-period model.[2]

The household's two-period problem is:

$$\begin{aligned}\max_{\{c_1,c_2,l_1,l_2\}} &= \ln(c_1) + b\frac{(1-l_1)^{1-\gamma}}{1-\gamma} + \beta\left[\ln(c_2) + b\frac{(1-l_2)^{1-\gamma}}{1-\gamma}\right] \\ &\quad -\lambda\left[c_1 + \frac{c_2}{1+r} - w_1 l_1 - \frac{w_2 l_2}{1+r}\right] \end{aligned} \tag{4}$$

where β is the discount factor, λ is the Lagrange multiplier on the lifetime budget constraint, and $b, \gamma > 0$.

The FOCs w.r.t. l_1 and l_2 are respectively:

$$b\,(1-l_1)^{-\gamma} = \lambda w_1 \tag{5}$$

$$\beta b\,(1-l_2)^{-\gamma} = \lambda\frac{w_2}{1+r} \tag{6}$$

Combining (5) and (6) to eliminate λ yields the household's intertemporal FOC for labour supply (leisure):[3]

$$\frac{1-l_2}{1-l_1} = \left[\beta\,(1+r)\,\frac{w_1}{w_2}\right]^{1/\gamma} \tag{7}$$

Equation (7) says that a higher relative wage in period 1 induces higher relative labour supply in period 1. This equation is the key to understanding the biggest failure of the RBC model identified in Question 1. In the data, the real wage is much less volatile than output, yet total hours is as volatile as output. To match the volatility in labour supply observed in the data, we need to make γ very small (less than 1). To see this, the elasticity of substitution between relative leisure in the two periods and the relative wage is:

[2] The analogous expression for the infinite horizon case can be found at

[3] The observant reader may find it strange that equation (7) at first sight appears to imply that labour supply would grow at a constant rate on the non-stochastic BGP (since $\frac{w_1}{w_2} = \frac{1}{1+\xi}$ where ξ is the rate of technological progress). In fact labour supply is constant on the BGP because from the household's intertemporal FOC for consumption, $\frac{1}{c_t} = \beta E_t\left[\frac{1+r_{t+1}}{c_{t+1}}\right]$, we have $1+r = \frac{1+\xi}{\beta}$ on the BGP, and so $\frac{1-l_2}{1-l_1} = \left[\beta\,(1+r)\,\frac{w_1}{w_2}\right]^{1/\gamma} = 1$ on the BGP.

$$\frac{\partial\left(\frac{1-l_2}{1-l_1}\right)}{\partial\left(\frac{w_1}{w_2}\right)}\frac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{1-l_2}{1-l_1}\right)}=\frac{1}{\gamma} \tag{8}$$

Therefore, $\frac{1}{\gamma}$ measures the percentage change in relative leisure as a result of a one percent change in the relative wage. In other words, it is the willingness of the household to substitute leisure (and hence labour supply) intertemporally. Micro estimates of $1/\gamma$ lie between 0 and 1, which is far too small to generate a big change in relative labour supply from a small change in the relative wage. In the literature various "fixes" to the problem have been suggested. Examples are indivisible labour (Hansen 1985, Rogerson 1988) and search and matching (Merz, 1995). Indivisible labour (e.g. either work 8 hours plus or nothing) prevents the smooth adjustment in hours worked of an individual. Similarly, search and matching in the labour market generates shifts into and out of unemployment. Both of these frictions in the labour market generate a more elastic labour supply curve, so that large changes in relative total hours can be generated from a small change in the relative wage. Notice that in the basic Walrasian RBC model the variation in total hours comes only from the intensive margin - that is, changes in hours worked per individual - there is no unemployment. Indivisible labour and search both create variation on the extensive margin - that is, movements into and out of unemployment. The extensive margin is empirically very relevant, indeed most of the cyclical variation in total hours worked is due to shifts into and out of unemployment.

An alternative solution is to argue that wages are in fact more volatile than the aggregate data suggest. For example, the composition of the workforce changes over the business cycle. During a boom, at the margin low skilled are hired (fishing from a pond) and during a downturn these workers are fired first. Consequently the average wage doesn't vary a great deal.

Question 3

This question abstracts from the labour supply decision, the household only chooses consumption. The interest rate is potentially uncertain, $r=\bar{r}+\varepsilon$ and $E(\varepsilon)=0$. Consequently, even though the household's income stream is certain, in effect it is uncertain because of the uncertainty in r. Here we see the effects of future uncertainty on consumption and savings behaviour in a two-period model. We will find that the effect depends on the timing of income.

(a) The period budget constraints are:

$$\begin{aligned} C_1+S_1 &= Y_1 \\ C_2 &= (1+r)S_1 \end{aligned}$$

Substituting out for S_1 we obtain the lifetime budget constraint:

$$C_1 + \frac{C_2}{(1+r)} = Y_1 \tag{9}$$

Using (9) to substitute for C_2, the consumer's problem writes as:

$$\max_{C_1} E_1 \left[\ln C_1 + (1+\theta)^{-1} \ln \left[(1+r)(Y_1 - C_1) \right] \right] \tag{10}$$

The FOC is:

$$E_1 \left[\frac{1}{C_1} - (1+\theta)^{-1} \frac{(1+r)}{(1+r)(Y_1 - C_1)} \right] = 0$$

$$C_1 = \frac{(1+\theta)}{(2+\theta)} Y_1 \tag{11}$$

From equation (11) we see that the choice of C_1 does not depend on the interest rate r. Under log utility the income and substitution effects of a change in the interest rate on present consumption exactly offset eachother. Consequently, since only r is uncertain, uncertainty does not affect consumption in this model.

(b) The period budget constraints are:

$$\begin{aligned} C_1 + S_1 &= 0 \\ C_2 &= (1+r)S_1 + Y_2 \end{aligned}$$

Substituting out for S_1 we obtain the lifetime budget constraint:

$$C_1 + \frac{C_2}{(1+r)} = \frac{Y_2}{(1+r)} \tag{12}$$

Using (9) to substitute for C_2, the consumer's problem writes as:

$$\max_{C_1} E_1 \left[\ln C_1 + (1+\theta)^{-1} \ln \left[Y_2 - C_1 (1+r) \right] \right] \tag{13}$$

The FOC is:

$$E_1 \left[\frac{1}{C_1} - (1+\theta)^{-1} \frac{(1+r)}{Y_2 - C_1 (1+r)} \right] = 0 \tag{14}$$

In this case, the potentially uncertain interest rate does not drop out of the FOC in equation (14).

To see the impact of uncertainty in r on the choice of C_1, we first compute the choice of C_1 when there is NO uncertainty, that is $r = \bar{r}$:

$$\begin{aligned} \frac{1}{C_1} &= (1+\theta)^{-1} \frac{(1+\bar{r})}{Y_2 - C_1(1+\bar{r})} \\ C_1 &= \frac{(1+\theta)}{(2+\theta)} \frac{Y_2}{(1+\bar{r})} \end{aligned} \tag{15}$$

Now lets see what happens when r is uncertain, $r = \bar{r} + \varepsilon$ and $E(\varepsilon) = 0$. From the FOC in (14) and using the fact that the expectation is a linear operator and that C_1 is known:

$$\frac{1}{C_1} = (1+\theta)^{-1} E_1 \left[\frac{(1+r)}{C_2} \right] \tag{16}$$

where we have substituted C_2 for $Y_2 - C_1(1+r)$ using the budget constraint (12).

Using the property of covariance $Cov(A,B) = E(AB) - E(A).E(B)$ we have:

$$\frac{(1+\theta)}{C_1} = E_1 \left[\frac{1}{C_2} \right] E_1 [1+r] + Cov\left(\frac{1}{C_2}, (1+r) \right)$$

Since $\frac{1}{C_2}$ is a convex function of C_2, Jensen's inequality states $E_1 \left[\frac{1}{C_2} \right] > \frac{1}{E_1(C_2)}$, and noting that the covariance term is positive (higher r means that the consumer pays more interest on his first period borrowing, lowering period 2 consumption) then:

$$\frac{(1+\theta)}{C_1} > \frac{1}{E_1(C_2)} E_1 [1+r] \tag{17}$$

$$\frac{(1+\theta)}{C_1} > \frac{1}{E_1(Y_2 - C_1(1+r))} E_1 [1+r] \tag{18}$$

$$\frac{(1+\theta)}{C_1} > \frac{1}{Y_2 - C_1(1+\bar{r})} [1+\bar{r}] \tag{19}$$

$$C_1 < \frac{(1+\theta)}{(2+\theta)} \frac{Y_2}{(1+\bar{r})} \tag{20}$$

Comparing equation (20) with equation (15) we see that consumption under uncertainty is less than consumption under certainty, in the first period. There is precautionary saving. This occurs because the marginal utility $U' = 1/C$ is convex; that is, the third derivative of the utility function is positive.