Solving PS5 using Bellman

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The flow budget constraint for the *generation* born at time t is:

$$c_{1,t} + \frac{c_{2,t+1}}{1+r} + \frac{B_{t+1}}{1+r} = y + B_t$$

or:

$$B_{t+1} = \left(y + B_t - c_{1,t}\right)(1+r) - c_{2,t+1}$$

The Bellman equation for this problem is:

$$V\left(B_t\right) = \max_{\{c_{1,t}, c_{2,t+1} : B_{t+1} = (y+B_t-c_{1,t})(1+r)-c_{2,t+1} | B_t\}} \left[u\left(c_{1,t}, c_{2,t+1}\right) + \theta V\left(B_{t+1}\right)\right]$$

where $u\left(c_{1,t}, c_{2,t+1}\right) = \frac{c_{1,t}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{2,t+1}^{1-\sigma}}{1-\sigma}$.

FOCs:

$$c_{1,t}^{-\sigma} = \theta V'\left(B_{t+1}\right)(1+r) \tag{1}$$

$$\beta c_{2,t+1}^{-\sigma} = \theta V'\left(B_{t+1}\right) \tag{2}$$

By the envelope theorem:

$$V'\left(B_{t+1}\right) = \theta V'\left(B_{t+2}\right)(1+r) \tag{3}$$

Combining equations (1) and (2) yields the *within-generation* Euler equation for consumption, requested in part (c)(i):

$$c_{1,t}^{-\sigma} = \beta(1+r) c_{2,t+1}^{-\sigma} \quad , \forall t \tag{4}$$

Iterating forward one-period on (1) and using (3) yields:

$$c_{1,t+1}^{-\sigma} = \theta V'\left(B_{t+2}\right)(1+r) = V'\left(B_{t+1}\right) \tag{5}$$

Combining (1) and (5) yields the *between-generation* Euler equation for consumption of the young, requested in part (c)(iii):

$$c_{1,t}^{-\sigma} = \theta(1+r) c_{1,t+1}^{-\sigma} \quad , \forall t \tag{6}$$

Finally, combining (4) and (6) yields the *intra-temporal* FOC, requested in part (c)(ii):

$$\beta c_{2,t+1}^{-\sigma} = \theta c_{1,t+1}^{-\sigma} \quad , \forall t$$

or:

$$\beta c_{2,t}^{-\sigma} = \theta c_{1,t}^{-\sigma} \quad , \forall t$$

[1] Any errors are my own.