The Golden Rule of Capital Accumulation in the Solow Model

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In the Solow model the steady-state (S.S.) level of consumption per effective worker is given by $\tilde{c}^* = (1-s)f(\tilde{k}^*)$, where s is the constant savings rate and $\tilde{k}$ is capital per effective worker. Recall that in S.S. $sf(\tilde{k}^*) = (\delta+\nu+\xi)\tilde{k}^*$, hence $\tilde{c}^* = f(\tilde{k}^*) - (\delta+\nu+\xi)\tilde{k}^*$. The value of $\tilde{k}^*$ that maximises $\tilde{c}^*$ is denoted by $\tilde{k}_{gold}$ and is the solution to[2]:

$$f'(\tilde{k}_{gold}) = \delta + \nu + \xi \tag{1}$$

The condition above (Eq. (1)) is known as the *golden rule of capital accumulation*. The intuition is clear. To maximise consumption in S.S. the economy must save until the marginal product of capital $f'(\tilde{k})$ is equal to the effective depreciation rate $(\delta+\nu+\xi)$. At which point, any further increase in saving, and hence capital, will push the marginal product of capital below the effective depreciation rate.

A Modified Solow Model

Now assume that instead of a constant exogenous savings rate, households decide to save all capital income and consume all labour income. We will show that (i) this economy converges to a S.S. and that (ii) $\tilde{k}^*$ is equal to the golden rule level $\tilde{k}_{gold}$.

(i)

If all capital income is saved and all labour income is consumed, then investment is equal to rK, and investment per effective worker is $r\tilde{k}$. Under competitive markets, $r = \frac{\partial F(K,AL)}{\partial K} = \frac{\partial (ALf(\tilde{k})}{\partial K} = f'(\tilde{k})$. So investment per effective worker is $f'(\tilde{k})\tilde{k}$. Therefore, the fundamental equation of the Solow model is now:

$$\begin{aligned} \dot{\tilde{k}} &= f'(\tilde{k})\tilde{k} - (\delta+\nu+\xi)\tilde{k} \\ &= \tilde{k}\left[f'(\tilde{k}) - (\delta+\nu+\xi)\right] \end{aligned}$$

or growth rate of capital per effective worker is:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = f'(\tilde{k}) - (\delta+\nu+\xi)$$

[1] Any errors are my own.

[2] It is the first-order condition to $\max_{\tilde{k}^*}\left[f(\tilde{k}^*) - (\delta+\nu+\xi)\tilde{k}^*\right]$.

Plotting these two terms on the right-hand side of the above equation separately on a graph with $\tilde{k}$ on the horizontal axis, the first term $f'(\tilde{k})$ is a downward sloping convex line, and the second term $(\delta+\nu+\xi)$ is a horizontal line. These two must therefore intersect at some $\tilde{k}$, call it $\tilde{k}^*$. For $\tilde{k} < \tilde{k}^*$ the first term exceeds the second and the growth rate of capital is positive. Conversely, if $\tilde{k} > \tilde{k}^*$ the second term exceeds the first and the growth rate of capital is negative. Hence, this economy converges to a S.S., that is, where $\tilde{k}$ is a constant.

(ii)

Consumption per effective worker is $f(\tilde{k})-f'(\tilde{k})\tilde{k}$ (i.e. output minus saving). Steady-state consumption per effective worker is, as before:

$$\tilde{c}^* = f(\tilde{k}^*) - f'(\tilde{k}^*)\tilde{k}^* = f(\tilde{k}^*) - (\delta + \nu + \xi)\tilde{k}^*$$

where the second equality uses the fact that from part (i) $f'(\tilde{k}^*) = (\delta + \nu + \xi)$ in steady-state.

Choosing $\tilde{k}^*$ to maximise consumption in S.S., the first-order condition is, as before:

$$f'(\tilde{k}_{gold}) = \delta + \nu + \xi$$

where $\tilde{k}_{gold}$ is the golden rule level of capital. From our answer in part (i), we know that in steady-state $f'(\tilde{k}^*) = (\delta + \nu + \xi)$ and hence $\tilde{k}^* = \tilde{k}_{gold}$ and this economy DOES achieve the golden rule of capital accumulation. What have we learnt? To maximise concumption for current and all future generations, the economy should re-invest all capital income and consume all labour income. Is this optimal? Well, utility is increasing in consumption, but it depends on how much consumers discount the future. In other words, we need to write down households' intertemporal utility maximisation problem (the Ramsey-Cass-Koopmans Growth Model). If households are impatient to consume then it is optimal to have a level of steady-state capital which is less than the golden rule. Finally, it is never optimal to have a level of capital above the golden rule.