Tools for Dynamic Optimisation in Discrete Time

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This note discusses the various methods one can use to solve the discrete-time version of the infinitely-lived consumer's consumption-saving choice. At first we assume *no* uncertainty. We will eventually derive equation (1) of RBC lecture 2, slide 2.

The consumer's problem is:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t . u(c_t)$$

subject to *either*

the lifetime budget constraint

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \tag{1}$$

or

the two-period budget constraint

$$c_t + a_{t+1} = (1+r_t)a_t + y_t \tag{2}$$

with a_0 given. c_t is consumption in period t, y_t is income, a_t is assets (or the stock of savings), r_t is the interest rate, $u(.)$ is concave, and $\beta \in (0,1)$ is the discount factor.

Notice that the two formulations of the budget constraint (BC) above are entirely equivalent. Indeed, starting from the two-period BC we can derive the lifetime BC by recursive substitution as follows (let $R_t = (1+r_t)$):

$$
\begin{aligned}
c_t + a_{t+1} &= R_t a_t + y_t \\
a_t &= {R_t}^{-1} a_{t+1} + {R_t}^{-1} c_t - {R_t}^{-1} y_t \\
a_1 &= {R_1}^{-1} a_2 + {R_1}^{-1} c_1 - {R_1}^{-1} y_1 \\
a_1 &= {R_1}^{-1} \left[{R_2}^{-1} a_3 + {R_2}^{-1} c_2 - {R_2}^{-1} y_2 \right] + {R_1}^{-1} c_1 - {R_1}^{-1} y_1 \\
a_1 &= \ldots\ldots\ldots\ldots\ldots\ldots \\
a_1 &= \ldots\ldots\ldots\ldots\ldots\ldots \\
a_1 &= \sum_{t=1}^{\infty} R^{-t} c_t - \sum_{t=1}^{\infty} R^{-t} y_t \\
a_0 &= a_1 + c_0 - y_0 \\
a_0 &= \sum_{t=0}^{\infty} R^{-t} c_t - \sum_{t=0}^{\infty} R^{-t} y_t
\end{aligned}
$$

We will now confront four approaches to deriving the Euler equation for consumption.

1 The Bellman Equation

This approach recognizes that the consumer's complex problem can be written as a two-period problem, where the two-period problem is to maximise the sum of contemporaneous utility plus the consumer's continuation value next period discounted one period. To re-write the consumer's problem in this form, we define the value function $V(a_0)$ as the maximum value the function above can attain under the restrictions, that is:

$$
\begin{aligned}
V(a_0) &= \max_{\{c_t : c_t + a_{t+1} = (1+r_t)a_t + y_t | a_0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t . u(c_t) \\
&= \max_{\{c_t : c_t + a_{t+1} = (1+r_t)a_t + y_t | a_0\}_{t=0}^{\infty}} \left\{ u(c_0) + \sum_{t=1}^{\infty} \beta^t . u(c_t) \right\} \\
&= \max_{\{c_0 : c_0 + a_1 = (1+r_0)a_0 + y_0 | a_0\}} \left\{ u(c_0) + \beta \left[\max_{\{c_t : c_t + a_{t+1} = (1+r_t)a_t + y_t | a_1\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} . u(c_t) \right] \right\} \\
&= \max_{\{c_0 : c_0 + a_1 = (1+r_0)a_0 + y_0 | a_0\}} \left\{ u(c_0) + \beta V(a_1) \right\}
\end{aligned}
$$

The last equation says that the consumer's problem in period 0 is to choose c_0 (and therefore a_1) to maximise utility in period 0 plus his continuation value starting in period 1 discounted back to period 0. This is very intuitive. Note the above two-period formulation of the consumer's problem relates to any point in time, so we obtain the following *Bellman equation*:

$$V(a_t) = \max_{\{c_t : c_t + a_{t+1} = (1+r_t)a_t + y_t | a_t\}} \{u\,(c_t) + \beta V(a_{t+1})\}$$

Substituting the budget constraint, the Bellman equation becomes:

$$V(a_t) = \max_{\{a_{t+1} | a_t\}} \{u\,((1+r_t)a_t + y_t - a_{t+1}) + \beta V(a_{t+1})\}$$

The F.O.C. is:

$$-u'(c_t) + \beta V'(a_{t+1}) = 0 \tag{3}$$

The Envelope Theorem implies:

$$V'(a_t) = u'(c_t)(1 + r_t)$$

Iterating forward one period:

$$V'(a_{t+1}) = u'(c_{t+1})(1 + r_{t+1}) \tag{4}$$

Combining equations (3) and (4) we arrive at the *Euler equation*:

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$$

2 Lagrange with period multipliers

Using the infinite sequence of period budget constraints given in equation (2) the Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t.u\,(c_t) + \sum_{t=0}^{\infty} \lambda_t\,[(1+r_t)a_t + y_t - c_t - a_{t+1}]$$

F.O.C.s:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t.u'\,(c_t) - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \beta^{t+1}.u'\,(c_{t+1}) - \lambda_{t+1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0$$

Eliminating the Lagrange multipliers gives us:

$$\beta^t.u'(c_t) = \beta^{t+1}.(1 + r_{t+1}).u'(c_{t+1})$$

Simplifying gives us the Euler equation:

$$u'(c_t) = \beta(1 + r_{t+1}).u'(c_{t+1})$$

3 Lagrange with one multiplier

Using the lifetime budget constraint given in equation (1) the Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t.u(c_t) + \lambda \left[a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \right]$$

F.O.C.'s:

$$\beta^t.u'(c_t) = \frac{\lambda}{(1+r)^t}$$

$$\beta^{t+1}.u'(c_{t+1}) = \frac{\lambda}{(1+r)^{t+1}}$$

Combining the F.O.C.'s we get:

$$\beta^t.u'(c_t) = \beta^{t+1}.(1 + r).u'(c_{t+1})$$

Simplifying gives us the Euler equation:

$$u'(c_t) = \beta(1 + r).u'(c_{t+1})$$

4 Perturbation

The consumer's utility is given by $u(c_t)+ \ \beta u(c_{t+1}) +$ .... Consider the following perturbation: decrease consumption in period t by ε and increase consumption in period t+1 by $\varepsilon(1 + r_{t+1})$. Notice that such a perturbation satisfies the budget constraint in equation (1). Under the specified perturbation the consumer's utility is given by $u(c_t - \varepsilon)+ \ \beta u(c_{t+1} + \varepsilon(1 + r_{t+1})) +$ .... The consumer's problem is to choose ε to maximise his utility.

Taking a 1st order Taylor expansion of the perturbed utility about $\varepsilon = 0$ we get: $u(c_t) - u'(c_t).\varepsilon+ \ \beta \left[u(c_{t+1}) + u'(c_{t+1}).\varepsilon(1 + r_{t+1})\right] +$ ....

The F.O.C. w.r.t. ε is:

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1}) \tag{5}$$

which is the Euler equation.

The intuition for the Euler equation comes from using the perturbation method. Consider a deviation whereby the consumer reduces his consumption today by one unit, which he saves and earns a return $(1 + r)$ and hence can consume an additional $(1 + r)$ units of consumption next period. The Euler equation tells us that since the consumer was originally maximising his utility, this *marginal* deviation will leave him indifferent. Indeed, the LHS of the Euler equation (eq. 5) tells us the loss of utility from decreasing consumption by one unit in period t (minus one times by the marginal utility). The RHS of the Euler equation tells us the gain in utility from saving the one unit, earning a return $(1 + r)$ and consuming $(1 + r)$ more units in period t+1 ($(1 + r)$ times by the marginal utility discounted back to period t).

The Euler Equation under Uncertainty

In the consumption-saving problem considered above, the consumer's infinite stream of future income beyond period t was assumed to be known with certainty in period t. In RBC models, real shocks introduce uncertainty about future rates of return and income. The consumer has an expectation of what his future income stream will be and uses this to make decisions. In period t, y_{t+1} and therefore c_{t+1} is unknown. Therefore, we must carry the expectation operator through the workings. This is easy to do. Consider the consumption-saving problem but with stochastic income $\{y_t\}$, which is Markov. The consumer's problem is:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t.u\left(c_t\right)$$

subject to

$$c_t + a_{t+1} = (1 + r_t)a_t + y_t$$

where $y_{t+1} = f(y_t, \upsilon_{t+1})$; υ_{t+1} is i.i.d., and E_0 is the mathematical expectation conditional on information known at time 0.

The Bellman equation is:

$$V(a_t, y_t) = \max_{\{a_{t+1}|a_t\}} E_t \left\{u\left((1 + r_t)a_t + y_t - a_{t+1}\right) + \beta V(a_{t+1}, y_{t+1})\right\}$$

The F.O.C. is:

$$-u'(c_t) + \beta E_t \left[\frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}} \right] = 0$$

The Envelope Theorem implies:

$$\frac{\partial V(a_t, y_t)}{\partial a_t} = u'(c_t)(1 + r_t)$$

Iterating forward one period:

$$\frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}} = u'(c_{t+1})(1 + r_{t+1})$$

Combining equations (3) and (4) we arrive at the Euler equation:

$$u'(c_t) = \beta E_t \left[(1 + r_{t+1}) u'(c_{t+1}) \right]$$

This is equation (1) of RBC lecture 2, slide 2.