Coverage correlation coefficient

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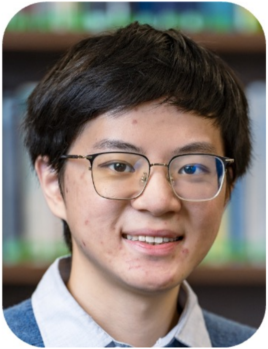
Collaborators



Xuzhi Yang
LSE

Mona Azadkia
LSE

Problem description

- **Setup**: $(X_1, Y_1), \ldots, (X_n, Y_n) \overset{\text{iid}}{\sim} P^{(X,Y)}$ on $\mathbb{R}^{d_X + d_Y}$
- **Goal**: Find a correlation coefficient such that it
  (i) has a simple and interpretable form
  (ii) has distribution-free null asymptotics
  (iii) converges to 0 $\iff X \perp\!\!\!\perp Y$,
  converges to 1 $\iff P^{(X,Y)}$ is singular w.r.t. $P^X \otimes P^Y$.

What are correlations?

- A measure of statistical association that quantifies how two variables tend to vary together.
- Closely related to independence testing, but quantifies the extend of association between 0 (no association) and 1 (maximum association).

Correlation	Dist. free	Relationship
Pearson's r	$\times$	linear
Spearman's ρ	✓	monotone
Kendall's τ	✓	monotone
distance correlation	$\times$	isometric
Chatterjee's ξ	✓	$y = f(x)$

Chatterjee's correlation

- Recent renewed interest in nonparametric correlation statistics (Dette et al., 2013; Chatterjee, 2021; Azadkia and Chatterjee, 2021; Deb et al., 2020; Wiesel, 2022; Azadkia and Roudaki, 2025)
- Among them, Chatterjee's correlation has seen rapid adoption in practice.
- **Definition:** Given $(X_i, Y_i)_{i\in[n]}$, rearrange as $(X_{(1)}, Y_{(1)}), \ldots, (X_{(n)}, Y_{(n)})$ s.t. $X_{(1)} \le \cdots \le X_{(n)}$, then

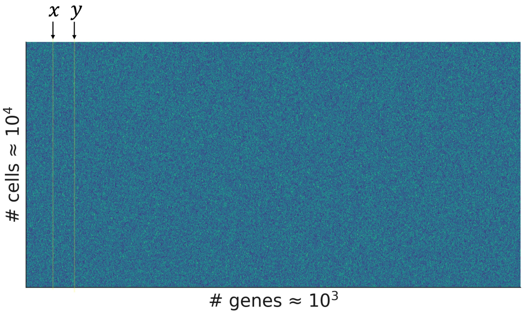
$$\xi_n^{X,Y} := 1 - \frac{\sum_{i=1}^{n-1} |r_{i+1} - r_i|}{(n^2-1)/3},$$

  where $r_i =$ "rank of $Y_{(i)}$" $= \#\{j : Y_{(j)} \le Y_{(i)}\}$.
- $\xi_n^{X,Y}$ can be viewed as a sample version of

$$\xi^{X,Y} := \frac{\int_{\mathbb{R}} \mathrm{Var}(\mathbb{E}[\mathbb{1}\{Y \ge t\} \mid X])\, dP^Y(t)}{\int_{\mathbb{R}} \mathrm{Var}(\mathbb{1}\{Y \ge t\})\, dP^Y(t)}.$$

Why is it successful?

- Can capture functional relationship of Y on X
  - if $Y = f(X)$, then $\xi_n^{X,Y} \xrightarrow{\mathrm{p}} \xi^{X,Y} = 1$
- Distribution free null CLT
  - if $X \perp\!\!\!\perp Y$, then $\sqrt{n}\xi_n^{X,Y} \xrightarrow{\mathrm{d}} N(0, \mathrm{variance})$
  - asymp valid p-values
- Fast to compute: $O(n \log n)$ time
- **A use case**: detecting gene relationship in single-cell RNA-seq experiment



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- Fast to compute: $O(n \log n)$ time
- **A use case**: detecting gene relationship in single-cell RNA-seq experiment
  - heavy multiple testing burden
  - each test needs to be carried out quickly
  - resampling-based tests are computationally prohibitive
- **Issues:**
  - Asymmetry
  - Unable to capture implicit functional relationship

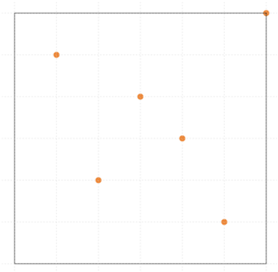
A geometric re-interpretation

- We have

$$\xi_n^{X,Y} := 1 - \frac{\sum_{i=1}^{n-1} |r_{i+1} - r_i|}{(n^2-1)/3} \approx 1 - \frac{\sum_{i=1}^{n-1} \left|\frac{r_{i+1}}{n} - \frac{r_i}{n}\right| \cdot \frac{1}{n}}{1/3}.$$

Example:

$$\begin{array}{c} 3 \\ 6 \\ 4 \\ 2 \\ 5 \\ 1 \end{array} \begin{pmatrix} -1.2 & -0.7 \\ 2.3 & 1.2 \\ -1.1 & -0.8 \\ -1.3 & -1.4 \\ -0.5 & -1.6 \\ -2.2 & 0.3 \end{pmatrix} \begin{array}{c} 4 \\ 6 \\ 3 \\ 2 \\ 1 \\ 5 \end{array}$$



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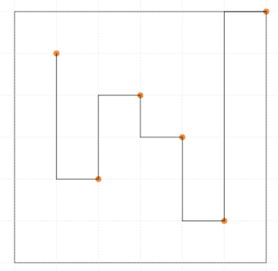

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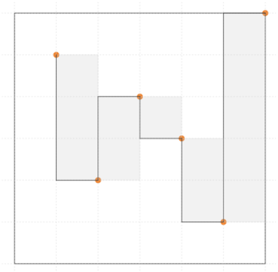

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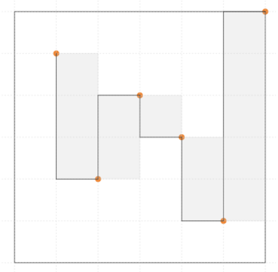
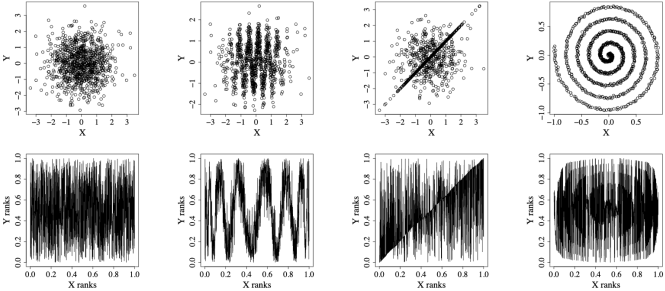


A geometric re-interpretation

- Chatterjee's correlation is related to covered area in $[0,1]^2$ of a 'thickened' line plot of normalised x- and y-ranks.



$\xi_n^{X,Y} = -0.019$ $p_\xi = 0.83$ | $\xi_n^{X,Y} = 0.46$ $p_\xi < 10^{-16}$ | $\xi_n^{X,Y} = 0.23$ $p_\xi < 10^{-16}$ | $\xi_n^{X,Y} = 0.0094$ $p_\xi = 0.32$

Coverage correlation coefficient

Coverage correlation

- Given $(X_i, Y_i)_{i\in[n]}$, let

$$R_i^X := n^{-1}\{j : X_j \le X_i\} \quad \text{and} \quad R_i^Y := n^{-1}\{j : Y_j \le Y_i\}$$

  be normalised x- and y-ranks.
- Draw squares of area $1/n$ centred at points (R_i^X, R_i^Y)
- The area in $[0,1]^2$ uncovered by the squares, $\mathcal{V}_n$, is called the *vacancy*.

Example:

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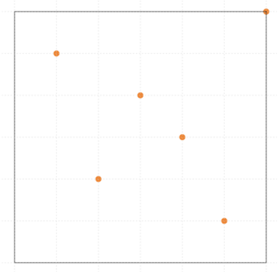

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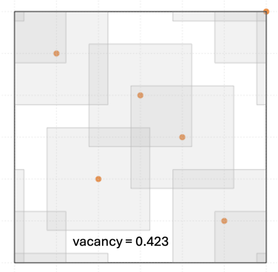


Coverage correlation in general dimensions

- Draw reference points:

$$U_1, \dots, U_n \overset{\text{iid}}{\sim} \mathrm{Unif}([0,1]^{d_X}), \quad V_1, \dots, V_n \overset{\text{iid}}{\sim} \mathrm{Unif}([0,1]^{d_Y}).$$

- Let

$$\pi^X := \operatorname*{arg\,min}_{\pi \in \mathcal{S}_n} \sum_{i=1}^n \|U_{\pi(i)} - X_i\|_2^2, \quad \pi^Y := \operatorname*{arg\,min}_{\pi \in \mathcal{S}_n} \sum_{i=1}^n \|V_{\pi(i)} - Y_i\|_2^2,$$

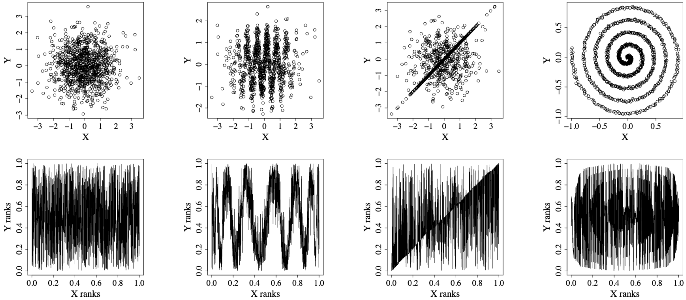
  and write $R_i := (U_{\pi^X(i)}, V_{\pi^Y(i)})$ for the Monge–Kantorovich ranks (Chernozhukov et al., 2017; Hallin et al. 2021) in dimension $d := d_X + d_Y$.
- We define the vacancy as

$$\mathcal{V}_n := 1 - \mathrm{vol}\biggl(\bigcup_{i=1}^n B_\infty\Bigl(R_i, \frac{1}{2n^{1/d}}\Bigr)\biggr).$$

- **Definition**: the coverage correlation coefficient is defined as the normalised excess vacancy

$$\kappa_n^{X,Y} := \frac{\mathcal{V}_n - e^{-1}}{1 - e^{-1}}.$$

Comparison to Chatterjee's correlation



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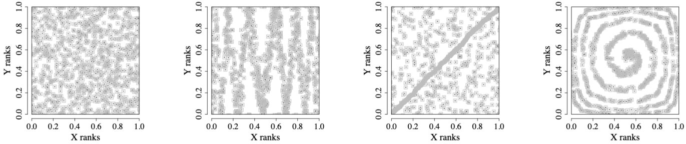


$\kappa_n^{X,Y} = -0.003$ $\quad p_\kappa = 0.69$

$\kappa_n^{X,Y} = 0.12$ $\quad p_\kappa < 10^{-16}$

$\kappa_n^{X,Y} = 0.22$ $\quad p_\kappa < 10^{-16}$

$\kappa_n^{X,Y} = 0.21$ $\quad p_\kappa < 10^{-16}$

Properties of the coverage correlation coefficient

Fast computation

- We need to evaluate volume of union of n axis-aligned hypercubes
- Special case of Klee's problem (Klee, 1977)
- When $d_X = d_Y = 1$, Bentley's algorithm solves this in $O(n \log n)$ time using a segment tree data structure (Ben-Or, 1983).
- In higher dimensions, the best algorithm runs in $O(n^{d/2})$ (Chan, 2013).
- But we can do better through geometric hashing, to get average run time of $O(n \log^{d-1} n)$.
- In multivariate settings, computing Monge–Kantorovich ranks becomes the rate limiting step. The Hungarian algorithm has $O(n^3)$ worst case complexity.

Timing comparison

- Average running time in seconds, for $n \in \{125, 250, \ldots, 8000\}$ and $d_X = d_Y \in \{1, 2\}$
- Time larger than 1 minute is not shown

n	d_X	$\kappa_n^{X,Y}$	$\xi_n^{X,Y}$	dCor	HSIC	KMAc	USP
125	1	0.001	0.001	0.008	0.014	0.553	0.010
250	1	0.001	0.001	0.010	0.047	1.06	0.043
500	1	0.002	0.001	0.037	0.192	2.50	0.182
1000	1	0.003	0.001	0.130	1.01	7.35	0.781
2000	1	0.005	0.001	0.498	4.23	26.3	2.98
4000	1	0.010	0.002	2.01	21.6	-	10.8
8000	1	0.019	0.003	7.95	-	-	-
125	2	0.034	-	0.004	0.011	0.514	0.042
250	2	0.076	-	0.014	0.042	1.05	0.164
500	2	0.177	-	0.052	0.186	2.52	0.720
1000	2	0.567	-	0.176	0.975	7.50	3.17
2000	2	1.93	-	0.694	4.45	27.5	10.8
4000	2	6.16	-	2.77	21.5	-	43.9
8000	2	24.4	-	11.4	-	-	-

Interpretable population quantity

- **Definition**: $f : \mathbb{R} \to \mathbb{R}$ is convex with $f(1) = 0$. For any two p.m. μ, ν on space $\mathcal{S}$, let $\mathrm{d}\mu = h\mathrm{d}\nu + \mathrm{d}\nu^{\perp}$ be the Lebesgue–Radon–Nikodym decomposition. The f-divergence between μ and ν is

$$D_f(\mu \,\|\, \nu) := \int_{\mathcal{S}} f \circ h \,\mathrm{d}\nu + \nu^{\perp}(\mathcal{S}) \lim_{t\to\infty} \frac{f(t)}{t}$$

Theorem. Let $P^{(X,Y)}$ be a Borel probability measure on $\mathbb{R}^2$ with marginals P^X and P^Y. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = (e^{-x} - e^{-1})/(1 - e^{-1})$. Given $(X_1, Y_1), \ldots, (X_n, Y_n) \overset{\mathrm{iid}}{\sim} P^{(X,Y)}$, we have

$$\kappa_n^{X,Y} \xrightarrow{\mathrm{p}} \kappa^{X,Y} := D_f(P^{(X,Y)} \,\|\, P^X \otimes P^Y), \quad \text{as } n \to \infty.$$

Interpretable population quantity

- The f-divergence interpretation immediately implies the following desirable properties of the coverage correlation
  - $\kappa^{X,Y} = \kappa^{Y,X}$. [symmetry]
  - $\kappa^{X,Y} = 0$ iff $X \perp\!\!\!\perp Y$. [zero-independence]
  - $\kappa^{X,Y} = 1$ iff $P^{(X,Y)}$ is singular w.r.t. $P^X \otimes P^Y$. [max-functionality+]
  - If $X \perp\!\!\!\perp Y \mid Z$, then $\kappa^{X,Z} \geq \kappa^{X,Y}$. [information-monotonicity]
  - If $P^{(X^{(n)},Y^{(n)})} \xrightarrow{\mathrm{d}} P^{(X,Y)}$, then $\liminf_{n\to\infty} \kappa^{X^{(n)},Y^{(n)}} \geq \kappa^{X,Y}$. [lower-semicontinuity]

  (Rényi, 1959; Móri and Székely, 2019; Borgonovo et al., 2025)

Proof sketch of the theorem

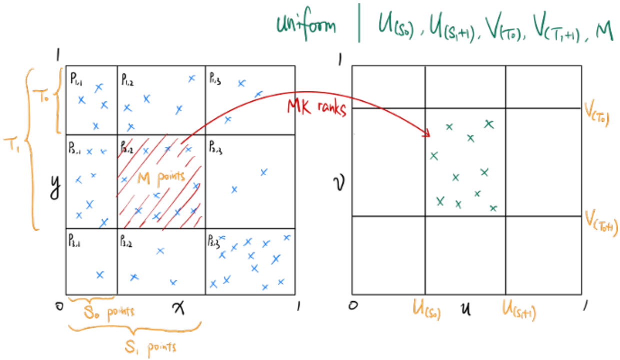
- We first reduce to the case where marginals are $\mathrm{Unif}[0,1]$, so $P^{(X,Y)}$ is simply the copula
- If $X \perp\!\!\!\perp Y$, the proof is easy: for an independent point $W \sim \mathrm{Unif}([0,1]^2)$ we have

$$\begin{aligned}\mathbb{E}(\mathcal{V}_n) &= \mathbb{E}\biggl[\mathbb{P}\biggl\{W \notin \bigcup_{i=1}^n B\Bigl(R_i, \frac{1}{2\sqrt{n}}\Bigr) \biggm| R_1, \ldots, R_n\biggr\}\biggr] \\ &= \mathbb{E}\biggl[\mathbb{P}\biggl\{R_i \notin B\Bigl(W, \frac{1}{2\sqrt{n}}\Bigr)\ \forall\, i \in [n] \biggm| W\biggr\}\biggr] = (1-1/n)^n \to e^{-1},\end{aligned}$$

  and similarly $\mathrm{Var}(\mathcal{V}_n) \to 0$.
- In general this argument does not work since R_i's are dependent.

Proof sketch

- We approximate the absolutely continuous part of $P^{(X,Y)}$ by blockwise constant density



- Use Poissonisation to show the approximation has negligible effect on vacancy
- Show the singular part of $P^{(X,Y)}$ has negligible contribution.

Distribution free null CLT

Theorem. We have

$$\sqrt{n}\kappa_n^{X,Y} \xrightarrow{\mathrm{d}} N(0,\sigma^2),$$

where

$$\sigma^2 := \frac{1}{(e-1)^2}\sum_{k=2}^{\infty}\frac{1}{k!}\left(\frac{2}{k+1}\right)^d.$$

- The statistic is distribution free under the null
- When $d_X = d_Y = 1$, $\sigma^2 = (e-1)^{-2}(4\mathrm{Ei}(1) - 4\gamma_0 - 5) = 0.091992\ldots$
- We have an asymptotically valid p-value that can be used to test independence:

$$p_\kappa := 1 - \Phi(\sqrt{n}\kappa_n^{X,Y}/\sigma).$$

- Proof uses ideas from the area of coverage processes (Hall, 1988)

Numerical studies

Size control

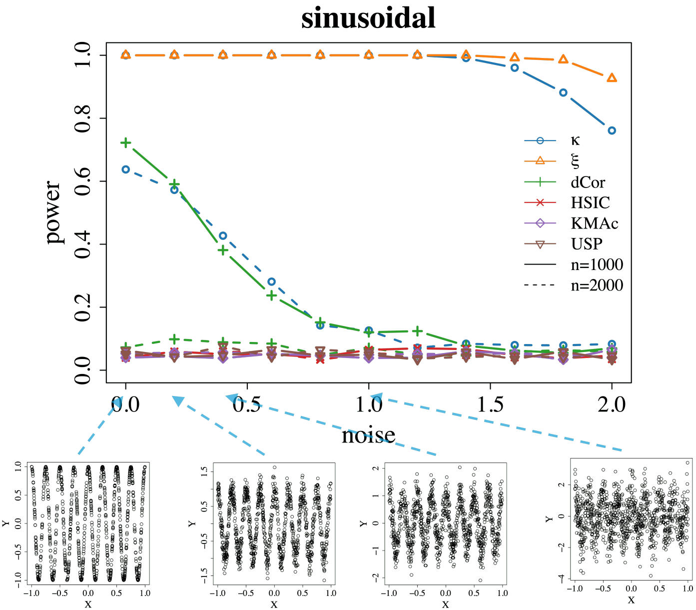
- Valid finite sample size control
- Slightly conservative when n is small, but asymptotically well-calibrated

n	d_X	$\alpha = 1\%$	$\alpha = 2.5\%$	$\alpha = 5\%$	$\alpha = 10\%$
10	1	$0.69_{(0.03)}$	$1.54_{(0.04)}$	$3.03_{(0.05)}$	$6.02_{(0.08)}$
100	1	$0.93_{(0.03)}$	$2.27_{(0.05)}$	$4.34_{(0.06)}$	$8.78_{(0.09)}$
1000	1	$0.96_{(0.03)}$	$2.34_{(0.05)}$	$4.76_{(0.07)}$	$9.50_{(0.09)}$
10	2	$0.55_{(0.02)}$	$1.18_{(0.03)}$	$2.10_{(0.05)}$	$4.08_{(0.06)}$
100	2	$0.94_{(0.03)}$	$2.11_{(0.05)}$	$4.12_{(0.06)}$	$8.03_{(0.09)}$
1000	2	$0.97_{(0.03)}$	$2.36_{(0.05)}$	$4.66_{(0.07)}$	$9.30_{(0.09)}$

Power comparison

- We compare power of independent testing based on coverage correlation coefficient at 5% nominal level against competitors
- Competitors:
  - Chatterjee's correlation ($\xi_n^{X,Y}$) (Chatterjee, 2021)
  - distance correlation (dCor) (Székely et al., 2007)
  - Hilbert–Schmidt Independence Criterion (HSIC) (Gretton et al., 2008)
  - kernel measure of association (KMAc) (Deb et al., 2020)
  - U-statistics permutation test (USP) (Berrett et al., 2021)
- Five data generating mechanisms with $(n, d) \in \{(1000, 1), (2000, 2)\}$ at different noise levels γ.

Power comparison



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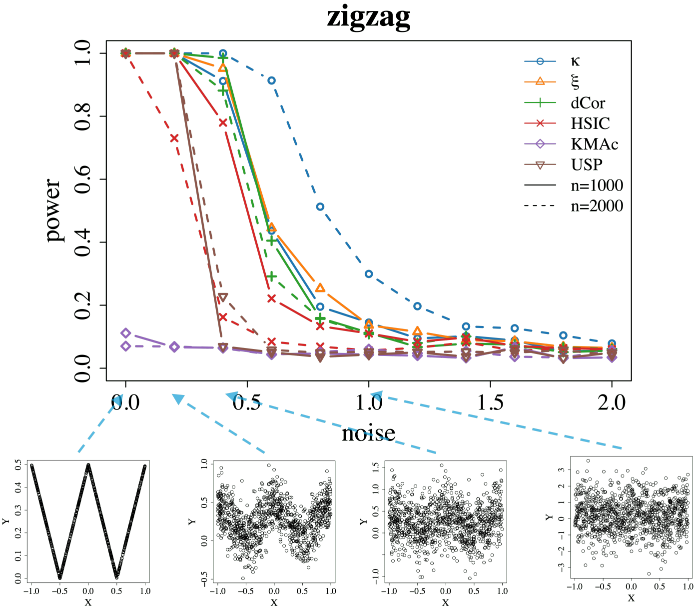

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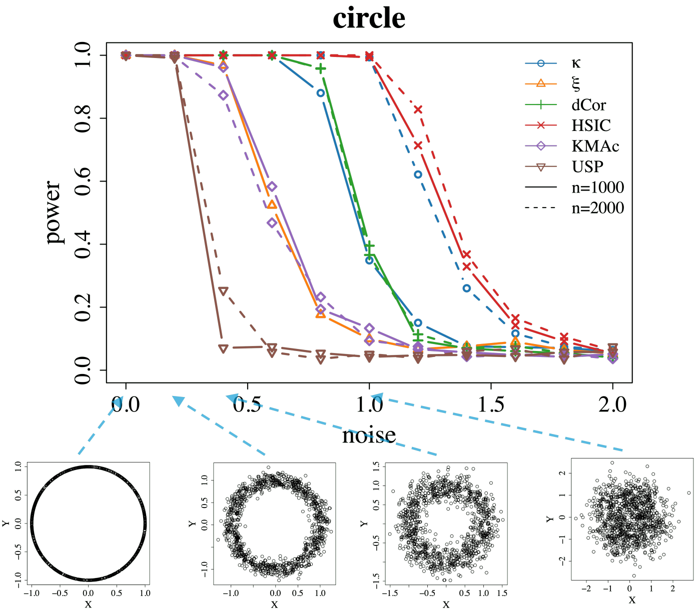

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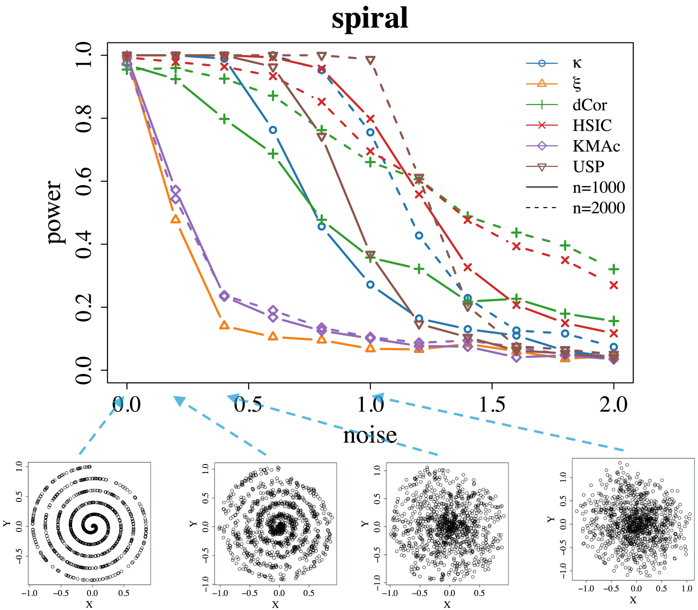

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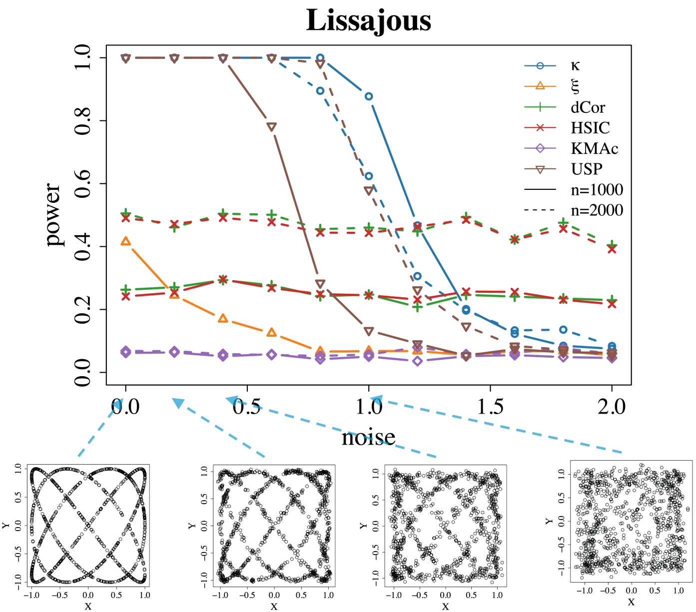
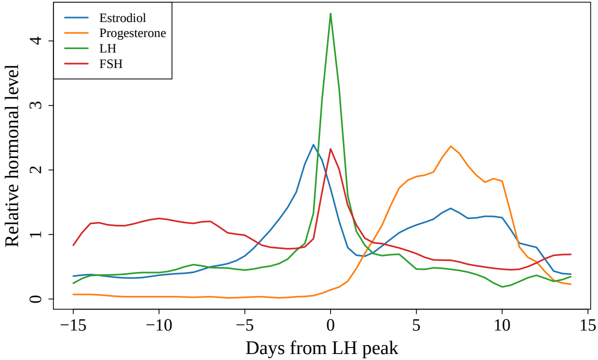


Real data

Menstrual cycle hormones

- Data digitised from Stricker et al. (2006).
- Estradiol and progesterone are key reproductive hormones, while LH and FSH regulate their production.
- They form a feedback-controlled system, driving the menstrual cycle.
- All four are correlated, though the dependence may be complex and implicit.



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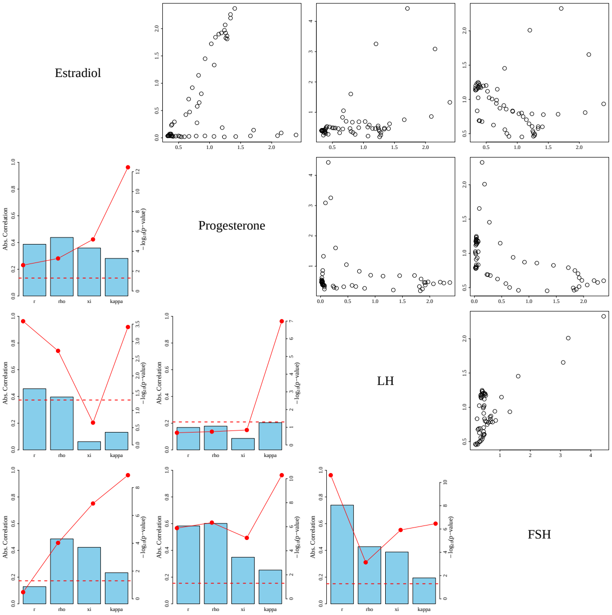
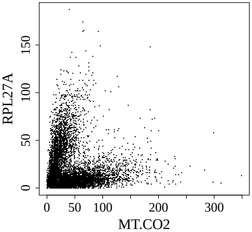
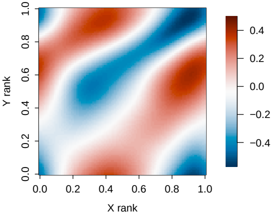


Gene covariation in CD8T cells

- Subset of single-cell RNA sequencing dataset from Suo et al. (2022)
- Gene expression levels of top $p = 1000$ highly variable genes measured in $n = 9369$ CD8⁺ T cells
- We compute all $\binom{p}{2}$ pairwise correlations using Pearson's correlation, Spearman's correlation, Chatterjee's correlation and the coverage correlation and adjust the corresponding p-values via Bonferroni correction.
- 54 gene pairs as significant by coverage correlation but not by any of the other methods





$$p_r = 0.11, \quad p_\rho = 0.10, \quad p_\xi = 0.002, \quad p_\kappa < 10^{-16}$$

Summary

- We develop a new correlation statistic that captures the extent to which the joint distribution is singular w.r.t. product of marginals.
- It is effective in detecting implicit functional dependence.
- Distribution-free null CLT helps generate asymptotically valid p-values.
- **R** package available on CRAN, and **Python** package on GitHub `https://github.com/wangtengyao/covercorr`.

Main reference

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Thank you!

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