On-Line Appendix for "On the Fragility of DeFi Lending"

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1 The Model Setup

In this appendix, we model sentiment equilibrium and firesale in the DeFi lending protocol. For expositional clarity, the convenience yields of cryptocurrencies of different qualities are modeled as dividend yields with different statistical properties. The main text provides a micro-foundation for these convenience yields of cryptocurrencies arising from their role as payment instruments.

The economy is set in discrete time and lasts forever.[1] There are many potential borrowers with identical preferences. There is a fixed set of crypto asset. Each borrower can hold at most one unit of the crypto asset.[2] There are also many potential lenders who lend funds to a liquidity pool each period. The lending protocol intermediates DeFi lending via smart contracts. All agents can consume/produce a numeraire good at the end of each period with a constant per unit utility/cost.

Gains from Trade and the Lending Platform A borrower needs funding that can be provided by lenders. There are gains from trade as the value per-unit of funding to a borrower is $z > 1$, while the per-unit cost of providing funding by lenders is normalized to one. In the DeFi setting, borrowers are anonymous and cannot commit to paying their debt. To overcome the commitment problem, loans must be backed by collateral. DeFi lending relies on a smart contract to implement a collateralized loan.

[1] In reality, interest payment on the borrowing in the lending protocols is continuously compounded and can be terminated at any time. Therefore, we can interpret that each time period in our model is relatively short.

[2] One might interpret this asset as a portfolio of crypto currencies used as collateral to borrow from a DeFi liquidity pool.

The DeFi intermediary determines the terms of the smart contract. Collateral is locked into the smart contract and released to the borrower if and only if a repayment is received.[3]

In DeFi lending protocols such as Aave, borrowers predominantly borrow stablecoins such as USDT and USDC using risky crypto assets as collaterals (e.g. ETH, BTC, YFI, YNX). As stablecoins are regarded as medium of exchange and unit of account in DeFi, they are used to fund various transactions or to increase leverage in crypto investment. We can interpret z as the value accrued to borrowers when using stablecoins borrowed from lenders for speculative or productive purposes.[4]

Crypto Asset's Properties and Information Environment We assume that all crypto assets are ex-ante identical and pay random dividend $\widetilde{\delta}$ at each period and survive to the next period with random probability $\widetilde{s}$.[5]

If crypto assets do not pay dividend, crypto assets are purely speculative assets. We do not take such an extreme position, mainly because we think of "dividend" more broadly. For example, crypto asset's dividend value may come from its role in enabling payments. Crypto assets, especially utility tokens of proprietary blockchains, act as mediums of exchanges for protocols developed on respective chains. Since search and matching technologies vary across chains, these crypto assets yield different convenience yields. In an on-line appendix, we micro found random dividend $\widetilde{\delta}$ by showing that crypto assets generate a form of dividend endogenously arising from convenience yields because these crypto assets can be exchanged for consumption goods in future periods in a framework based on Lagos and Wright (2005). A crypto asset with a more efficient matching technology has a larger probability to be exchanged for consumption goods, and hence yields a larger utility gain from this convenience. There are other private benefits that might accrue from holding a crypto asset such as governance rights.

Additionally, crypto assets might generate pecuniary payoffs. For example, some protocols offer staking returns to asset holders. Certain assets are in high demand and able to generate rental income. These non-pecuniary and pecuniary benefits are random for a host of reasons which we capture through the randomly evolving quality of the asset.

We assume that the beginning of a period, each asset receives an iid quality shock. Specifically, with probability $1-\lambda$, the quality of an asset is high (H) and probability λ it is low (L). The distribution

[3]Chiu, Kahn, and Koeppl (2022) study how a smart contract helps mitigate commitment problems in decentralized lending.

[4]It is straight-forward to introduce governance tokens issued by the intermediary - the lending platform. Governance token holders then provide insurance to lenders by acting as residual claimants. Given risk neutrality, the equilibrium outcome remains the same.

[5]We use ˜ to denote random variables.

of $(\widetilde{\delta}, \widetilde{s})$ is F_Q if asset quality is $Q \in \{H, L\}$ .We assume F_H first-order stochastically dominates F_L and denote expectation with respect to F_Q with $\mathbb{E}_Q$.

To simplify the analysis we make further assumptions on the distributions. We assume that a high-quality asset pays dividend $\delta > 0$ at the end of the period and survives to the next period with probability $s = 1$. A low-quality asset does not pay any dividends today ($\delta = 0$) and it survives to the next period with probability $s \in [0, 1]$. Here, $1 - s$ captures whether the quality shock has persistent effects on the dividend flow of the crypto asset, also reflecting the volatility of the survival probability of a crypto asset.

We assume that the crypto asset pays positive dividend in some states (that is, when it is high quality). The main role of this assumption is to eliminate non-monetary equilibrium (the equilibrium where crypto asset is worthless). In our model the asset has collateral service and can have positive price even if it does not pay any dividend. However, there can also be an equilibrium where the asset is worthless because current lenders believe future lenders will not accept the asset the asset as collateral. Positive dividend eliminates the latter equilibrium.

Next, we model asymmetric information between borrowers and lenders. The source of private information could be multitude. As we motivated earlier in the introduction section, the delay of Oracle in updating asset value might give collateral asset holders an information advantage.[6]

Owners might also have a better information about future convenience benefits generated by the crypto assets. Asymmetric information might be about private valuation of the asset rather than the dividend payoff.[7]

Specifically, we assume that at the beginning of each period, the borrower of a crypto asset privately learns the asset's quality (i.e., whether it is high or low). After observing the quality shock, the borrower decides whether and how much to borrow from the platform. The borrower then receives the private return from the loan (which is z times the loan size), and observes the realization of $(\widetilde{\delta}, \widetilde{s})$. Given the information, the borrower decides whether to repay the loan or default. The asset's quality and the state $(\widetilde{\delta}, \widetilde{s})$ are both publicly revealed at the end of each period. In the next period, some low-type assets do not survive and are replaced by new ones that are ex-ante identical. In the main model, we assume that

[6]Instead of selling the overvalued (by the Oracle) asset in the DeFi exchange and incur a price impact, borrowing against it yields a larger return for the asset owners.

[7]Our results do not depend on the asymmetric information on the common value component of the dividends. In Appendix A.8, we explore an alternative setup where there is asymmetric information concerning borrowers' private valuation. The main results hold. In Appendix A.9, we show that our setup can also be extended to time-varying information friction.

borrowers receive private information every period and low quality asset pays lower dividend and may also die. The critical source of asymmetric information in our model, as evidenced by our assumption, is about dividend. In the Appendix, we consider the more general case where private information arrives only infrequently with probability χ, which can capture the degree of information imperfection.

Asset Price At the end of each period, agents meet in an exchange market to trade the assets by transferring the numeraire good. At this point, the private information is revealed publicly. The end-of-period ex-dividend price of a crypto asset that will survive to the next period is denoted as ϕ_t. At the end of the period, a high quality asset receives $\delta + \phi_t$ and a low quality asset receives ϕ_t with survival probability s. In the exchange market, each borrower can acquire at most one unit of crypto asset to the next period.[8]

Smart Contract As discussed in the introduction, DeFi lending is anonymous and collateralized via a smart contract. The smart contract is a debt contract that specifies, at each time t, the haircut and interest rate (h, R_t) set by the lending protocol. The haircut defines the debt limit per unit of collateral according to:

$$D_t \equiv \Phi_t(1 - h) \tag{1}$$

where Φ_t is the contractual price underlying the DeFi debt contract. The borrowing limit is set by applying a pre-specified haircut on Φ_t. In many real world settings (such as Libor contracts and DeFi lending contracts) the contractual price Φ_t is set by traders in a forward looking manner.[9] In DeFi lending the contractual price Φ_t comes from an Oracle that scans price quotes from many (centralized or decentralized) exchanges. All these contractual prices share common characteristics – they are forward looking and reflect both future ex-dividend price and some amount of the asset's promised dividend/convenience yield, δ_t, received at the end of the period.

In our model, these two components of Φ_t are constructed differently. The dividend/convenience yield component is exogenously given, risky and subject to adverse selection, while the ex-dividend price component is determined in equilibrium. This distinction is important because **in equilibrium**, agents anticipate future (ex-dividend) price ϕ_t correctly even though borrowers and lenders are asymmetrically

[8]The dynamic structure of the model is based on Lagos and Wright (2005).

[9]If trading is synchronous, Φ_t should be the "current" traded price. In our model, the current price, at the beginning of the period when the DeFi contract is set, is **not** the past ex-dividend price ϕ_{t-1} nor the future ex-dividend price ϕ_t since neither of these prices capture the value of the asset at the beginning of the period.

informed about the dividend. So it is also **rational** for agents to set the forward looking ex-dividend price component of Φ_t, to be the same as ϕ_t.

We choose the promised dividend/interest/convenience yield of the asset, that is δ, to compute the dividend component of Φ_t. This is to match the current industry practice in setting the haircut and margin loan limit. For example, for repo contracts, the haircut is on the "dirty" price of an asset. This price includes the asset's "clean" price quote (or ex-dividend price quoted on exchanges), and accrued interest (or dividend) even when the promised interest (or dividend) payment is risky and the realized value might be lower. This is especially true for fixed-income securities, such as exchange rate, bills and bonds. In DeFi, one can map prices quoted on Binance (a centralized exchange) or Uniswap/Curve (decentralized exchanges) as the sum of a risky convenience yield component of these crypto assets that is subject to asymmetric information and a clean price component. For these reason we choose to specify the contractual price as $\Phi_t = \delta + \phi_t$. Note that our main results hold as long as the specification of the contractual price has an endogenously determined equilibrium price and a dividend component (which is subject to asymmetric information). [10]

In practice, the DeFi loan interest rate in the smart contract is a function of the utilization ratio i.e., the ratio of demand and supply for funding, and the collateral specific haircut is *infrequently* updated. To capture the economic impact of these features, we assume in our main model that the smart contract specifies a *flexible* market clearing interest rate and a *fixed* haircut. We investigate the flexible haircut case in an extension.

DeFi Lending & Borrowers In each period, if the borrower borrows ℓ_t units of funding, the face value of the debt is $R_t\ell_t$. After observing the asset quality, the borrower raises funding from a DeFi protocol by executing the lending contract. Given (R_t, D_t), a type $Q = H, L$ borrower chooses how much collateral a_t to pledge and how much loan ℓ_t to borrow from the pool:

$$\max_{a_t,\ell_t} z\ell_t - \mathbb{E}_Q \min\{\ell_t R_t, a_t(\delta + \widetilde{s}\phi_t)\}$$

subject to a collateral constraint

$$\ell_t R_t \leq a_t D_t$$

where D_t is the debt limit pinned down by (1). By borrowing ℓ_t and pledging a_t, the borrower obtains $z\ell_t$ from the loan but needs to either repay $\ell_t R_t$ or lose the collateral value $a_t(\widetilde{\delta} + \widetilde{s}\phi_t)$. The collateral

[10] Besides matching industry practice, this specification also leads to clean expositions. We could also use some other function of δ (for example expected dividend) such as $\Phi_t = E[\delta]+\phi_t$. This modification would not affect our main results since this function of dividend is subject to some degree of adverse selection.

value discounted by the haircut needs to be sufficiently high to cover the loan repayment. Note that, without loss of generality, we can assume that the collateral constraint is binding: $\ell_t R_t = a_t D_t$.[11] So the solution for the borrowing decision is given by

$$a_{it} \in \arg\max_{a_t \in [0,1]} a_t[zD_t/R_t - \mathbb{E}_Q \min\{D_t, \widetilde{\delta} + \widetilde{s}\phi_t\}]. \tag{2}$$

Hence, it is optimal to set $a_t \in \{0, 1\}$. When the term inside the square bracket is positive, the borrower pledges $a_t = 1$ to borrow $\ell_t = D_t/R_t$ and promises to repay D_t. Default happens whenever $D_t > \widetilde{\delta} + \widetilde{s}\phi_t$. When the term inside the square bracket is non-positive, the borrower does not borrow: $a_t = \ell_t = 0$. Since $\mathbb{E}_H \min\{D_t, \delta + \phi_t\} = D_t \geq \mathbb{E}_L \min\{D_t, \widetilde{s}\phi_t\}$, we have $a_{Lt} \geq a_{Ht}$ and $\ell_{Lt} \geq \ell_{Ht}$. That is, the low-type borrowers have higher incentives to borrow than the high-type. When both types borrow, we have a *pooling* outcome. When only the low-type borrows, we have a *separating* outcome.

DeFi Lending & Lenders The intermediary has no initial funding. It obtains funding q_t from the lenders to finance loans to borrowers. When the loan matures, the intermediary passes the cash flows – either the repayment of the borrowers or the resale value of the collateral (in case of a default) – to the lenders, after collecting an intermediation fee (discussed below).[12] Note that the borrower's borrowing decision, $a_{i,t}$ where $i \in \{L, H\}$, is quality dependent, meaning that lenders face adverse selection in DeFi lending. Since lenders are not able to distinguish between low and high quality borrowers at the time of lending, the choice of funding size q_t does not depend on the underlying asset quality. Of course, in equilibrium, lenders take into account the expected quality of the collateral mix backing the loan.

We assume that the lending market is competitive. That is, given $\{a_{i,t}\}_{i\in\{L,H\}}$, D_t, and ϕ_t, funding supply q_t satisfies the following zero profit condition:

$$q_t = \frac{1}{1+f}\left\{\frac{1}{a_{L,t}\lambda + a_{H,t}(1-\lambda)}\left[a_{L,t}\lambda\mathbb{E}_L \min\{D_t, \widetilde{s}\phi_t\} + a_{H,t}(1-\lambda)\min\{D_t, \delta+\phi_t\}\right]\right\} \tag{3}$$

where $f < z - 1$ is a fixed fee charged by the intermediary per unit of loan.[13]

[11] To see this, suppose (ℓ^*, a^*) is optimal and $\ell^* R < a^* D$. Since the objective function is (weakly) decreasing in a, lowering a (weakly) increases the objective. The increase is strict if $as\phi < \ell R$ for some realization of s.

[12] In reality, some lending protocols have a backstop provided by "equity holders". For example, Aave's Safety Module incentivizes its governance token holders to lock their AAVE tokens as a mitigation tool in case of a shortfall event. The feature can be incorporated into our model easily by introducing some risk neutral agents who absorb the default risk and promise a constant payoff q to the lenders. Risk neutrality and zero profit condition imply that we will get exactly the same result.

[13] When the loan matures the intermediary takes qf either from the repayment or from the resale value of the collateral. The remaining amount goes to the lender. The assumption of $f < z - 1$ ensures that the net gain from loans is positive.

When $a_{L,t} = a_{H,t} = 1$ (both types are borrowing) or when $a_{L,t} = 1$, $a_{H,t} = 0$ and the realized type is L, the funding supply is fully utilized and the funding market clears. In the separating case, if the realized type is H then there is no demand for funding. In this case, we assume the intermediary returns the funding supply to the lenders without charging a fee.

The intermediary's payoff is given by

$$f[\lambda a_{L,t} + (1-\lambda)\, a_{H,t}]q_t. \tag{4}$$

In section 3.5, we consider the case where the intermediary flexibly chooses the haircut. In that case, the intermediary chooses h_t to maximize (4) taking $(a_{i,t})_{i\in\{L,H\}}$ and ϕ_t as given.

Determination of the Crypto Asset Price At the end of each period, borrowers bid for the crypto asset to use as collateral for future period. Therefore, the price of a crypto asset at the end of period t, $\phi_{t,}$, is given by its continuation value to the borrower:

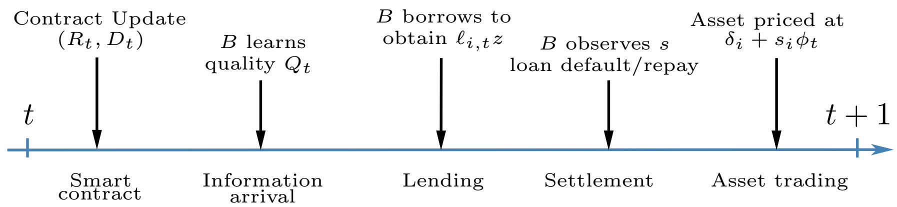
$$\phi_t = \beta \underbrace{\left\{ \begin{array}{c} \lambda \left(a_{L,t+1}\mathbb{E}_L \left(zD_{t+1}/R_{t+1} - \min\{D_{t+1}, \widetilde{s}\phi_{t+1}\}\right)\right) \\ + (1-\lambda)\, a_{H,t+1} \left(zD_{t+1}/R_{t+1} - \min\{D_{t+1}, \delta + \phi_{t+1}\}\right) \end{array} \right\}}_{\text{Collateral Value}} \tag{5}$$
$$+\beta \underbrace{\left\{\lambda \left(\mathbb{E}_L \widetilde{s}\right) \phi_{t+1} + (1-\lambda)\left(\delta + \phi_{t+1}\right)\right\}}_{\text{Fundamental Value}}$$

where β is the discount factor. We assume $0 < \beta < 1/z$.[14] The continuation value of the asset, is simply the sum of two terms: the fundamental value of the asset which is the discounted value of future dividend and asset resale price, and the collateral value. Importantly, the collateral value of the asset depends on endogenous variables, $(a_{i,t+1})_{i\in\{L,H\}}$, D_{t+1}, R_{t+1} and ϕ_{t+1}, which in turn depend on the extent of asymmetric information in future DeFi lending markets.

Timing The time-line is summarized in Figure (1). In the beginning of each period, the smart contract specifies the debt limit D_t (or equivalently the haircut h) and the loan interest rate. Next, borrower receives private information about the quality of the asset and decides whether to borrow from the lending platform by pledging collateral to the smart contract and lenders supply funding subject to zero profit condition. After this stage, the borrower's type is revealed, and the borrower either repays the loan or defaults and loses the collateral. If the asset survives then its price is determined, consumption takes place and the borrower works to acquire assets for the next period.

[14]This assumption is to make sure that the equilibrium asset prices are bounded.

Figure 1: Timeline



Note that in this timeline, the lending platform is exposed to information friction and the asset market is frictionless, and we assume that they do not open simultaneously, which reflects the natural timing of information revelation process. In reality, a privately informed borrower can choose to offload the underlying asset in a lending platform by borrowing a stablecoin loan against it or conduct an outright sale in an exchange (that is, an asset market). However, theoretically, adverse selection problem is more severe in an asset exchange since the borrower is selling an equity contract and less so in a lending platform since the borrower is selling a debt contract.[15]Empirically, there are other technical frictions in selling crypto assets on decentralized and centralized exchanges on blockchains. Transferring crypto assets to an off-chain centralized exchange is often subject to a long time lag before the assets can be traded, while transactions on an on-chain decentralized exchange are often subject to market illiquidity and price slippage. Therefore, for expositional clarity and without loss of generality, we assume that the asset market with frictions does not open simultaneously with the lending platform.

Equilibrium Definition Given haircut h and fee f, an equilibrium consists of asset prices $\{\phi_{t,}\}_{t=0}^{\infty}$, debt thresholds $\{D_t\}_{t=0}^{\infty}$, loan rates $\{R_t\}_{t=0}^{\infty}$, funding size $\{q_t\}_{t=0}^{\infty}$ and collateral quantities $\{a_{Lt}, a_{Ht}\}_{t=0}^{\infty}$ such that

1. borrowers' loan decisions are optimal (condition 2),
2. lenders earn zero profit (condition 3),
3. funding supply equals funding demand, i.e. $q_t = D_t/R_t$, and
4. the asset pricing equation is satisfied (condition 5).

[15]Ozdenoren, Yuan, and Zhang (2021) have shown the optimal security for privately informed borrowers to sell in a similar setting consists of a debt contract (which both high and low quality borrowers sell) and a residual equity contract (which only the low quality borrowers sell).

2 Equilibrium in Lending Market

We begin the analysis by describing the equilibrium in the DeFi lending market for a given asset price ϕ.[16] To study the borrowers' decision, we first define the degree of *information insensitivity* as the ratio of the expected value of the debt contract for types L and H, i.e., $\zeta(\phi; h) = \mathbb{E}_L \min\{D, \widetilde{s}\phi\}/D \in (0,1]$ where $D = ((1-\lambda)\delta + \phi)(1-h)$. As this ratio increases, the expected values of the debt under the low versus high become closer, and the adverse selection problem becomes less severe.

There are two cases depending on whether the high-type borrowers are active. In the pooling case, condition (3) implies that the equilibrium funding supplied by lenders is

$$q^P = \frac{1}{1+f}[\lambda \mathbb{E}_L \min\{D, \widetilde{s}\phi\} + (1-\lambda)D].$$

Interest rate is pinned down by $q^P = D/R^P$, that is,

$$R^P = \frac{D(1+f)}{\lambda \mathbb{E}_L [\min\{D, \widetilde{s}\phi\}] + (1-\lambda)D}.$$

In the separating case, the funding from lenders is given by

$$q^S = \frac{1}{1+f}\mathbb{E}_L \min\{D, \widetilde{s}\phi\}.$$

and the interest rate pinned down by $q^S = D/R^S$, that is,

$$R^S = \frac{D(1+f)}{\mathbb{E}_L [\min\{D, \widetilde{s}\phi\}]}.$$

Define $\overline{\zeta} \equiv 1 - \frac{z-1-f}{z\lambda}$. The next proposition characterizes the equilibrium in the DeFi lending market for a given asset price ϕ.

Proposition 1. *Given asset price ϕ, if the degree of information insensitivity $\zeta(\phi; h) > \overline{\zeta}$, then borrowers' equilibrium funding obtained from DeFi lending is $q = q^P$, interest rate is $R = R^P$ and collateral choices for H type borrower and L type borrower are $a_L = a_H = 1$. If the degree of information insensitivity $\zeta(\phi; h) < \overline{\zeta}$, then borrowers' equilibrium funding from DeFi lending is $q = q^S$, interest rate is $R = R^S$, and collateral choices for H type borrower and L type borrower are $a_L = 1$ and $a_H = 0$. The former condition, for a pooling equilibrium, is easier to satisfy when asset price ϕ, haircut h or productivity from borrowers' private investment z is higher.*

Proposition 1 implies that, given asset price ϕ, there is a unique equilibrium in DeFi lending. It is a pooling (separating) outcome when the debt contract is sufficiently informationally insensitive (sensitive).

[16] In this section we drop the time subscript t from all the variables to ease the notation.

In particular, when the degree of information insensitivity $\zeta(\phi; h)$ is above the threshold ζ, the adverse selection problem is not too severe and both types borrow. In this case, the loan size is the pooling quantity $q = q^P$. When the degree of information insensitivity is below the threshold, the adverse selection problem is severe and only the low type borrows. In this case, the loan size is the separating amount $q = q^S$. Furthermore, the loan rate in a pooling equilibrium is lower than that in a separating equilibrium.

Note that $\zeta(\phi; h) = \mathbb{E}_L \min\{1, \frac{\widetilde{s}\phi}{(\delta+\phi)(1-h)}\}$. As a result, the debt contract becomes informationally less sensitive for a high ϕ and for a high h. The above proposition also indicates that in addition to the parameter λ that characterizes type heterogeneity, the net gains from trade, $z/(1+f)$, is also an important determinant of adverse selection: a lower $z/(1+f)$ leads to a higher $\overline{\zeta}$. In particular, even if there is very little asymmetric information about the quality of the debt contract (i.e., when $\zeta(\phi; h)$ is slightly below 1), as $z/(1+f)$ approaches 1 (so that $\overline{\zeta}$ is close 1), the DeFi lending will be in a separating equilibrium. In other words, when net gains from trade is low, even a slight amount of asymmetric information results in adverse selection problem.

3 Multiple Equilibria in Dynamic DeFi Lending

The analysis in the previous section takes the asset price as given. In this section, we characterize the stationary equilibrium where asset prices are endogenously determined. We demonstrate that DeFi lending is fragile in the sense that it exhibits dynamic multiplicity in prices. Specifically, we show that there might be multiple equilibria in the DeFi lending market justified by different crypto asset prices. The multiple asset prices are in turn justified by the different equilibria in DeFi lending. Since we are focusing on stationary equilibria, we drop the time subscripts.

3.1 Characterization of Stationary Equilibria

3.1.1 Pooling equilibrium

In a stationary pooling equilibrium, all borrowers borrow ($a_L = a_H = 1$). This equilibrium exists when there is an asset price ϕ^P satisfying the equation

$$\phi^P = \beta\left[(z-1-f)q^P\right] + \beta(1-\lambda)\delta + \beta(\lambda\mathbb{E}_L\widetilde{s} + (1-\lambda))\phi^P. \tag{6}$$

The loan size is given by

$$q^P = \frac{1}{1+f}\left(\lambda\mathbb{E}_L\left[\min\{D^P, \widetilde{s}\phi^P\}\right] + (1-\lambda)D^P\right),$$

where $D^P = \left(\delta + \phi^P\right)(1-h)$. In addition, it has to satisfy the high-type borrowers' incentive constraint to pool:

$$\zeta\left(\phi^P;h\right) = \mathbb{E}_L \min\{1, \frac{\widetilde{s}\phi^P}{(\delta+\phi^P)(1-h)}\} \geq \overline{\zeta}. \tag{7}$$

3.1.2 Separating Equilibrium

In a separating equilibrium, only the low-type borrowers borrow (i.e., $a_H = 0$, $a_L = 1$). This equilibrium exists when there is an asset price ϕ^S satisfying the equation

$$\phi^S = \beta\left(\lambda(z-1-f)q^S + (1-\lambda)\delta + (\lambda\mathbb{E}_L\widetilde{s} + (1-\lambda))\phi^S\right). \tag{8}$$

The loan size is given by

$$\frac{D^S}{R} = q^S = \frac{1}{1+f}\mathbb{E}_L\left[\min\{D^S, \widetilde{s}\phi^S\}\right],$$

where $D^S = \left(\delta + \phi^S\right)(1-h)$. In addition, high-type's incentive constraint to pool is violated:

$$\zeta\left(\phi^S;h\right) < \overline{\zeta}. \tag{9}$$

3.2 Existence and Uniqueness

We first focus on the asset pricing equations (6) and (8).

Lemma 1. *Equation (6) has a unique solution ϕ^P and equation (8) has a unique solution ϕ^S. Also, $\phi^P \geq \phi^S$.*

Lemma 1 implies that there exists at most one pooling and one separating stationary equilibrium. If they co-exist, the price in the pooling equilibrium is higher than that in the separating equilibrium. It is also easy to show that both prices are higher than the fundamental price of the asset in autarky, $\underline{\phi} = \frac{\beta(1-\lambda)\delta}{1-\beta(\lambda\mathbb{E}(s_L)+(1-\lambda))}$. This means that the introduction of DeFi lending raises the equilibrium asset price above its fundamental level. Lemma 1 implies that $\zeta(\phi^P;h) \geq \zeta(\phi^S;h)$. Hence, we have the following proposition.

Proposition 2. *There always exists at least one stationary equilibrium:*

- *it is a unique pooling equilibrium when $\overline{\zeta} < \zeta(\phi^S;h)$,*
- *it is a unique separating equilibrium when $\overline{\zeta} > \zeta(\phi^P;h)$,*
- *a pooling equilibrium and a separating equilibrium coexist when $\overline{\zeta} \in [\zeta(\phi^S;h), \zeta(\phi^P;h)]$.*

In the next section, we examine the conditions under which the multiplicity arises.

3.3 Haircut and Multiplicity

In Proposition 2, multiplicity arises due to a dynamic price feedback effect. When the collateral asset price is high, the degree of information insensitivity of the debt contract, $\zeta(\phi^P; h)$, is above the threshold $\overline{\zeta}$. Hence, the adverse selection problem is mild and the high-type borrowers are willing to pool with the low type. In turn, if agents anticipate a pooling equilibrium in future periods, the expected liquidity value of the asset in the next period is large, hence the asset price today is high. Conversely, when the asset price is low, the degree of information insensitivity of the debt contract, $\zeta(\phi^S; h)$, is below the threshold $\overline{\zeta}$. Therefore, the adverse selection problem is severe and the high type retains the asset and chooses not to borrow. In turn, if agents anticipate a separating equilibrium in future periods, the liquidity value of the asset is limited and thus the asset price today is low. As a result, the asset prices are self-fulfilling in this economy.

The haircut is a key parameter controlling the degree of information sensitivity. Setting a lower haircut makes the debt contract informationally more sensitive, magnifying the adverse selection problem. Defining two thresholds

$$\kappa_P \equiv \frac{\zeta}{\beta z[(1-\lambda)+\zeta\lambda]}$$
$$\kappa_S \equiv \frac{\zeta}{\beta[(1-\lambda)+\zeta\lambda z]} < \kappa_P,$$

we have the following result.

Proposition 3. *Suppose the expected survival probability of the crypto asset satisfies* $\mathbb{E}_L\widetilde{s} \in (\kappa_P, \kappa_S)$. *There exists a threshold for haircut such that when the haircut h is below this threshold, there are multiple equilibria.*

3.3.1 Example: Two-point distribution

We now use an example to illustrate the effects of h on the equilibrium outcome. The full analysis is given in the Appendix. Suppose $\widetilde{s}$ is drawn from a two-point distribution such that $s = 1$ with probability π, and $s = 0$ with probability $1-\pi$. Consider the separating equilibrium. When $s = 0$, a low-type borrower always defaults. When $s = 1$, the low-type defaults if $D^S = \left(\delta + \phi^S\right)(1-h) > \phi^S$ and repays if $D^S \leq \phi^S$. We can rewrite this condition to show that there exists a threshold level $\underline{h}^S$ such that when $s = 1$, the low-type defaults if $h < \underline{h}^S$ and repays if $h \geq \underline{h}^S$. In the former case, the low type always defaults so the face value of the loan and consequently the loan size do not depend on the haircut. In the latter case, the low type repays the loan in the good state (i.e., $s = 1$), hence the loan size depends on the face value of the debt. Since the face value of debt declines as the haircut increases, the loan size decreases in h.

We define $\zeta^S(h) \equiv \zeta(\phi^S(h); h)$. That is, we obtain $\zeta^S(h)$ by substituting the price ϕ^S as a function of haircut given fixed values for all other exogenous variables. We define $\zeta^P(h)$ similarly. Using (9), a separating equilibrium exists if $\zeta^S(h) \leq \overline{\zeta}$. The threshold $\zeta^S(h)$ is strictly increasing in h for $h < \underline{h}^S$. The reason is that the high type never defaults, so the expected value of the contract under the high type declines as h increases. The low type, on the other, always defaults and the expected value of the contract under the low type is independent of h. Hence, the information sensitivity of the contract decreases as h increases and it becomes harder to support a separating equilibrium. For $h \geq \underline{h}^S$, $\zeta^S(h) = \pi$ and a separating equilibrium exists whenever $\pi < \overline{\zeta}$. That is, once the haircut is large enough, increasing it further does not affect the information sensitivity of the contract. The reason is that, in this case, the high type always pays the face value and the low type pays the face value only in the good state. As the haircut increases, the face value decreases but the value of the contract declines at the same rate for both types so its information sensitivity remains constant.

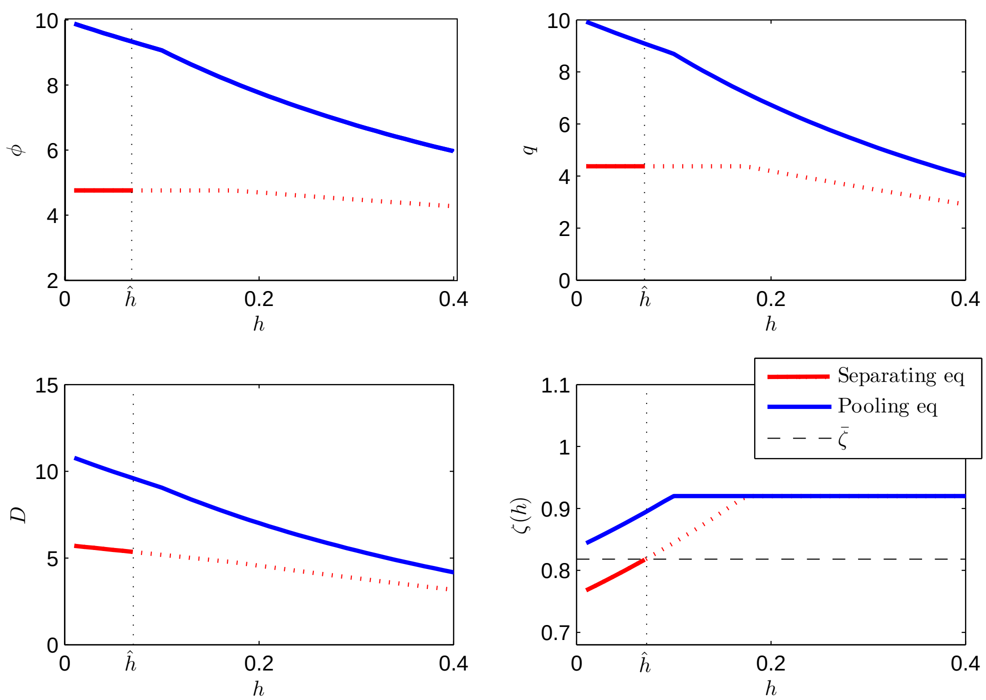
We analyze the pooling equilibrium similarly, and find a threshold $\underline{h}^P < \underline{h}^S$ such that when $s = 1$, the low-type defaults if $h < \underline{h}^P$ and repays if $h \geq \underline{h}^P$. A pooling equilibrium exists if $\zeta^P(h) \geq \overline{\zeta}$. The threshold $\zeta^P(h)$ is strictly increasing in h and $\zeta^P(h) > \zeta^S(h)$ for $h < \underline{h}^P$. For $h \geq \underline{h}^P$, $\zeta^P(h) = \pi$ and a pooling equilibrium exists whenever $\pi > \overline{\zeta}$.

Putting these facts together we see that whenever $h < \underline{h}^S$, we have $\zeta^S(h) < \zeta^P(h)$. Hence when $\overline{\zeta}$ is in this range the two equilibria coexist. When the haircut exceeds $\underline{h}^S$, there can only be a unique equilibrium depending on whether $\overline{\zeta}$ is above or below π.

Figure 2 plots the effects of h on the asset price, the loan size, the debt limit and the degree of information insensitivity of the contract. The red and blue curves indicate respectively the separating and pooling equilibria, assuming their existence. The parameter values used are $z = 1.1$, $\lambda = 0.5$, $\beta = 0.9$, $\delta = 1$, $\pi = 0.92$, $f = 0$, which satisfy the condition $\mathbb{E}_L\widetilde{s} \in (\kappa_P, \kappa_S)$ in Proposition 3. The bottom right plot compares the degrees of information insensitivity to the threshold $\overline{\zeta}$ which is captured by the horizontal dash line. When h is close to zero, the dash line appears above the red curve and below the blue curve, confirming the multiplicity result in Proposition 3. The other three plots also confirm the earlier result that the asset price, loan size and debt limit are higher in a pooling equilibrium. In this example, multiplicity can be ruled out and pooling can be supported by setting $h > \hat{h} = 7.1\%$ where $\overline{\zeta} = \zeta^S(\hat{h})$.[17]

[17]When $h > \hat{h}$, separating equilibrium cannot be sustained and hence in Figure 2 red lines depicting separating equilibriums become red dotted lines in this region.

Figure 2: Effects of Haircut h



3.4 Sentiment Equilibrium

In the middle region where multiple self-fulfilling equilibria coexist, it is possible to construct *sentiment equilibria* where agents' expectations depend on non-fundamental sunspot states. In a static game with multiple equilibria, as in Diamond and Dybvig (1983), an equilibrium is chosen by a sunspot. In a dynamic setting, the economy can switch between equilibria (based on a sunspot) and we refer to different phases as different sentiments. Clearly, in a dynamic setting construction is more delicate because one has to take into account that equilibrium can switch again in the future. We are not the first to use this dynamic notion of sentiment equilibrium. Similar notions have been used by among others including Hassan and Mertens (2011), Benhabib, Wang, and Wen (2015), Asriyan, Fuchs, and Green (2017), etc. By constructing sentiment equilibria, we offer potential empirical testable hypotheses relating measurable sentiment index with equilibrium price and quantities.

Suppose that there are K sentiment states indexed from 1 to K. We let $\sigma_{kk'}$ be the Markov transition probability from sentiment state k to k'. In the presence of sentiments we modify the model as follows.

Let ϕ^k be the price of the asset, R^k be the loan rate, and $D^k = \left(\delta + \phi^k\right)(1-h)$ be the debt limit in sentiment state k. Quantities of collateral a_L^k, a_H^k chosen by each type must be optimal given the price and rate at each sentiment state k. The loan size chosen by the lender in sentiment state k is given by:

$$q^k = \lambda E_L \left[\min\{D^k, s\phi^k\}\right] + (1-\lambda)D^k$$

The price of crypto asset in sentiment state k is given by:

$$\phi^k = \beta \sum_{k=1}^{K} \sigma_{kk'} \left\{ \lambda \int_{\underline{s}}^{\bar{s}} s_L \phi^{k'} dF(s_L) + (1-\lambda)\left(\delta + \phi^{k'}\right) \right.$$
$$\left. + \lambda a_L^{k'} \int_{\underline{s}}^{\bar{s}} \left(zD^{k'}/R^{k'} - \min\{D^{k'}, s_L\phi^{k'})\}\right) dF(s_L) + (1-\lambda)\, a_H^{k'} \left(zD^{k'}/R^{k'} - D^{k'}\right) \right\}.$$

We want to construct a *non-trivial sentiment equilibrium* such that the economy supports a pooling outcome in states $k = 1, ..., \bar{k}$ and a separating outcome in states $k = \bar{k}+1, ..., K$. By continuity, one can obtain the following result.

Proposition 4. *Suppose* $\mathbb{E}(s) \in (\kappa_P, \kappa_S)$ *and haircut is not too big. Then for* σ_{kk} *large enough, there exists a non-trivial sentiment equilibrium.*

To demonstrate non-trivial sentiment equilibrium and examine equilibrium properties, we provide the following two numerical examples. In both examples we assume $\widetilde{s}$ is drawn from a two-point distribution such that $s = 1$ with probability π, and $s = 0$ with probability $1 - \pi$.

Example 1. Suppose $K = 3$ and $\bar{k} = 1$. The economy stays in the same state with probability σ and moves to the next state with probability $1 - \sigma$ where the next state from 1 is 2, from 2 is 3 and from 3 is 1. We can interpret the three states as follows:

- $k = 1$: Boom state
  - $a_L^1 = a_H^1 = 1$, $q^1 = \lambda\pi \min\{(\delta + \phi^1)(1-h), \phi^1\} + (1-\lambda)(\delta + \phi^1)(1-h)$
- $k = 2$: Crash state
  - $a_L^2 = 1, a_H^2 = 0$, $q^2 = \pi \min\{(\delta + \phi^2)(1-h), \phi^2\}$
- $k = 3$: Recovery state
  - $a_L^3 = 1, a_H^3 = 0$, $q^3 = \pi \min\{(\delta + \phi^3)(1-h), \phi^3\}$

The asset prices are then given by

$$\begin{aligned}\phi^k =& \beta\sigma_{k1}\left[(z-1)q^1 + (1-\lambda)\delta + (\lambda\pi + (1-\lambda))\phi^1\right] \\ &+ \beta\sigma_{k2}\left[\lambda(z-1)q^2 + (1-\lambda)\delta + (\lambda\pi + (1-\lambda))\phi^2\right] \\ &+ \beta\sigma_{k3}\left[\lambda(z-1)q^3 + (1-\lambda)\delta + (\lambda\pi + (1-\lambda))\phi^3\right]\end{aligned}$$

Figure 3 below plots the effects of sentiment states on asset prices and total lending. When $\sigma = 0.95$, the sentiment state is sufficiently persistent so that the above sentiment equilibrium exists. As shown, the sentiment dynamics drive the endogenous asset price cycle: The asset price declines when the economy enters the crash state, jumps up when the economy moves from the crash state to the recovery state, and jumps up further when the economy returns to the boom state. Note that the total lending, $\left(\lambda a_L^k + (1-\lambda)a_H^k\right) q^k$ is "pro-cyclical" in the sense that it is positively correlated with the asset price.

Next, we show a similar pro-cyclical pattern of lending and asset prices in an example where there are more (than three) states and a state moves to an up or a down state with an equal probability. In this example, equilibrium lending and asset prices are more volatile.

Example 2. Let $K = 10$. If the economy is in state k in a given period, in the next period sentiment stays the same with probability σ. From states $k \in \{2, \ldots, K-1\}$ economy moves to state $k-1$ with probability $(1-\sigma)/2$ and to state $k+1$ with probability $(1-\sigma)/2$. From state 1 economy moves to state 2 with probability $1-\sigma$. From state K economy moves to state $K-1$ with probability $1-\sigma$. Figure 4 plots a simulation for 5000 periods when $\sigma = 0.95$ and $\bar{k} = 6$.

3.5 Uniqueness under Flexible Design of Debt limit

We have shown that DeFi lending subject to a rigid haircut can lead to multiplicity when the debt contract is too informationally sensitive. In this extension, we show that a flexible contract design supports a unique equilibrium in the case and generates higher social surplus from lending compared to the case with a rigid haircut.

Under flexible design, the smart contract is no longer subject to constraint (1). Instead, in each period, the intermediary, in this case, the DeFi protocol, can choose any feasible debt contract, $y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t) = \min(D_t, \widetilde{\delta} + \widetilde{s}\phi_t)$ for $0 \le D_t \le \delta + \phi_t$. Let $\widehat{z}$ denote the marginal value of obtaining funding from lenders deducting the intermediation fee f to the intermediary,

$$\widehat{z} = \frac{z}{1+f}.$$

Figure 3: Sentiment Equilibrium Example 1

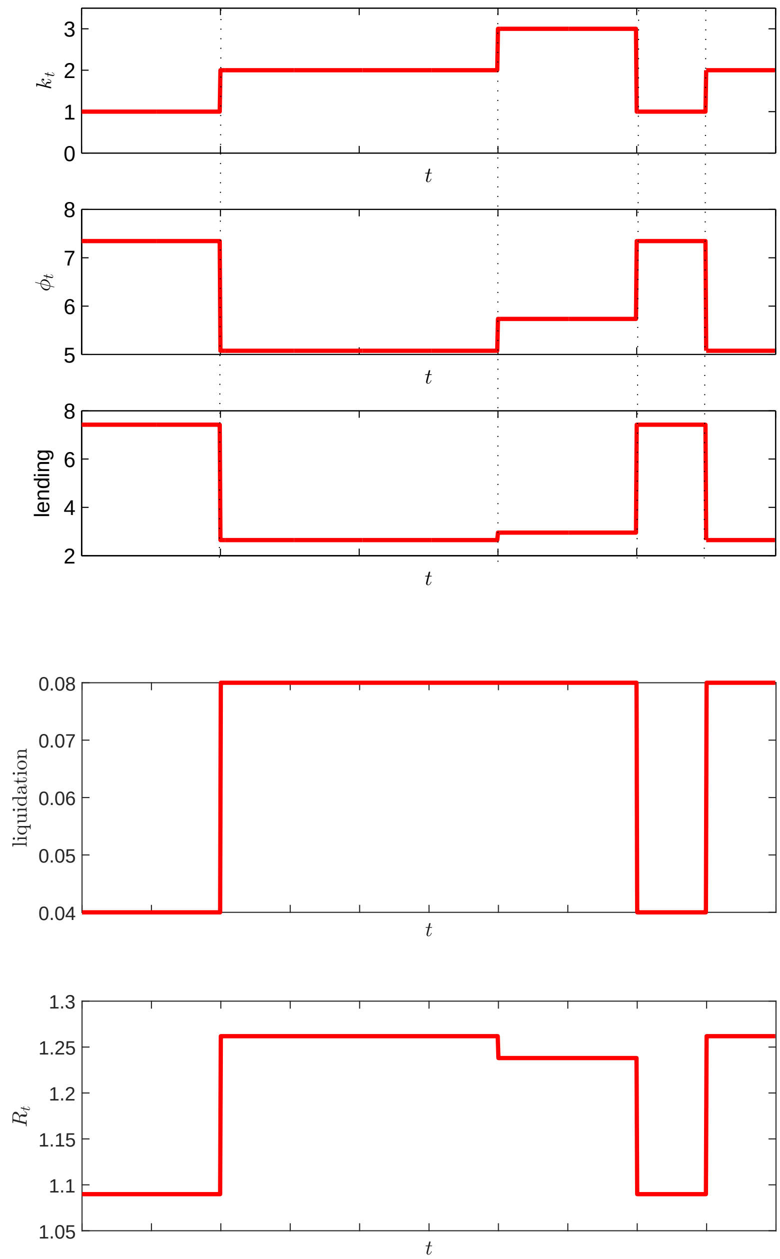
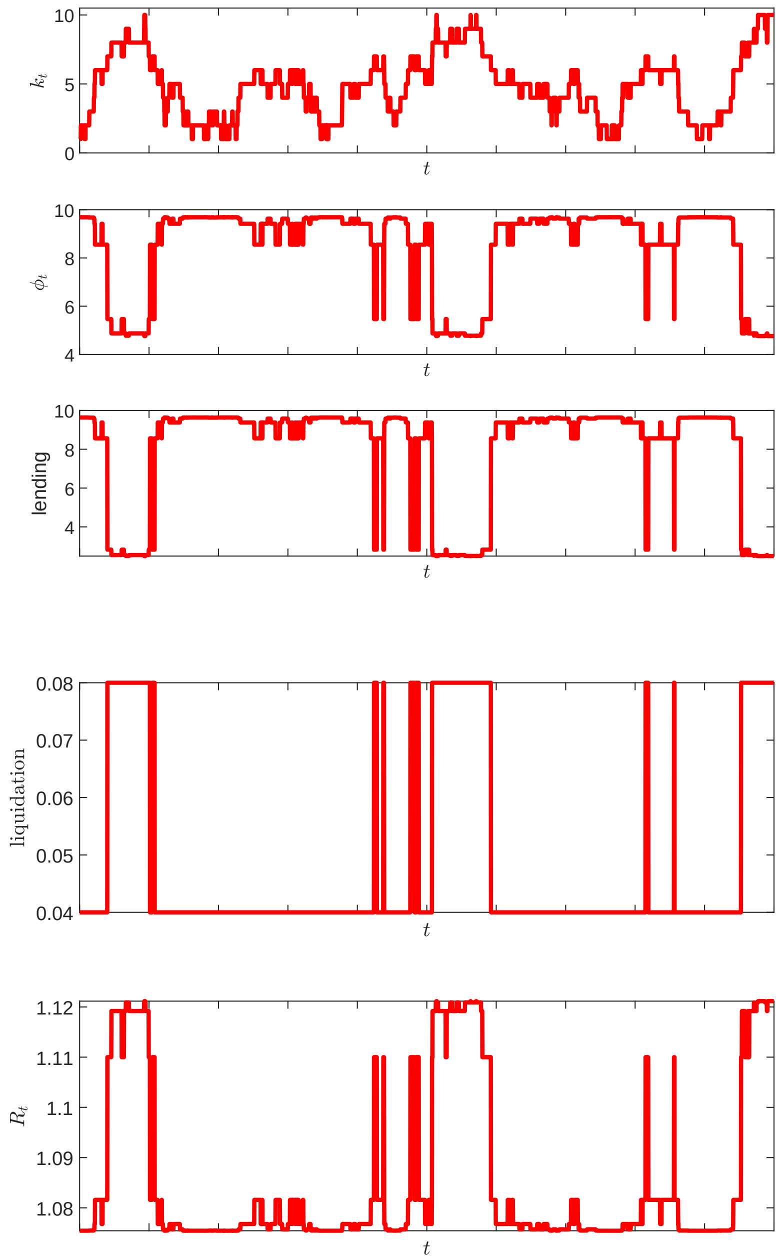


Figure 4: Sentiment Equilibrium Example 2



Recall from (4) that intermediary maximizes the expected loan size times the intermediation fee:

$$f[\lambda + (1-\lambda)\, a_{H,t}]q_t\left(y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t)\right)$$

The loan size is:

$$q_t\left(y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t)\right) = \frac{1}{1+f}\frac{[\lambda \mathbb{E}_L + a_{H,t}\,(1-\lambda)\,\mathbb{E}_H]\, y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t)}{\lambda + a_{H,t}\,(1-\lambda)} \tag{10}$$

where

$$a_{H,t} = \begin{cases} 1 & \text{if } \widehat{z}\,[\lambda \mathbb{E}_L + (1-\lambda)\mathbb{E}_H]\, y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t) \geq \mathbb{E}_H y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t) \\ 0 & \text{otherwise} \end{cases}. \tag{11}$$

Equivalently the intermediary maximizes

$$[\lambda \mathbb{E}_L + a_{H,t}\,(1-\lambda)\,\mathbb{E}_H]\, y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t) \tag{12}$$

subject to (11). In words, the intermediary takes the price ϕ_t as given and sets the debt threshold D to maximize the expected loan size taking into account the impact of the contract on the funding that the lenders are willing to supply. The value of the asset to the borrower is:

$$\begin{aligned} V_t = \max_{0 \leq D \leq \delta + \phi_t} \lambda & \left[\widehat{z} q_t\left(y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t)\right) - \mathbb{E}_L y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t) + \mathbb{E}_L\left(\widetilde{\delta} + \widetilde{s}\phi_t\right)\right] \\ & + (1-\lambda)\left[a_{H,t}\left\{\widehat{z} q_t\left(y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t)\right) - \mathbb{E}_H y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t)\right\} + \mathbb{E}_H\left(\widetilde{\delta} + \widetilde{s}\phi_t\right)\right] \end{aligned} \tag{13}$$

Given the optimal design, the asset price at the end of the previous period equals

$$\phi_{t-1} = \beta V_t. \tag{14}$$

An equilibrium under flexible design of smart contracts is debt face value D_t, the borrower's value for the asset at the beginning of period t V_t, and the resale price of the asset at the end of period t ϕ_t such that (i) D_t maximizes (12) taking ϕ_t as given and, (ii) V_t, and ϕ_t satisfy (13) and (14).

We also make the same simplifying assumptions on the distribution of $\left(\widetilde{\delta}, \widetilde{s}\right)$ that we make in the rigid haircut case. That is, we assume that a high-quality asset pays dividend $\delta > 0$ at the end of the period and survives to the next period with certainty which implies:

$$\mathbb{E}_H y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t) = y(D_t, \delta + \phi_t);$$

and the low type asset does not pay any dividends and it survives to the next period with probability $s \in [0,1]$ which is drawn from a distribution F which implies:

$$\mathbb{E}_L y(D_t, \widetilde{\delta} + \widetilde{s}\phi_t) = \int_{\underline{s}}^{\bar{s}} y(D_t, s_L\phi_t) dF\left(s_L\right).$$

The following proposition describes the optimal debt threshold and the implied haircut as a function of the asset price ϕ_t.

Proposition 5. *If* $\mathbb{E}_L s < 1 + \frac{1}{\lambda \widehat{z}} - \frac{1}{\lambda}$ *then let* s^* *be the unique solution to:*

$$\widehat{z}\left[\lambda \mathbb{E}_L \min(s^*, s) + (1-\lambda)s^*\right] = s^*.$$

In this case, the equilibrium contract is a pooling one ($a_{H,t} = 1$) *with face value* $D_t = s^* \phi_t$ *when*

$$\lambda \mathbb{E}_L \min(s^*, s) + (1-\lambda)\, s^* - \lambda \mathbb{E}_L s \geq 0.$$

Otherwise, the equilibrium contract is a separating one ($a_{H,t} = 0$) *with face value* $D_t = \delta + \phi_t$*. The implied haircut is:*

$$h_t = \begin{cases} 0 & \text{if } \lambda \mathbb{E}_L \min(s^*, s) + (1-\lambda)\, s^* - \lambda \mathbb{E}_L s < 0, \\ 1 - \frac{s^* \phi_t}{(1-\lambda)\delta + \phi_t} & \text{if } \lambda \mathbb{E}_L \min(s^*, s) + (1-\lambda)\, s^* - \lambda \mathbb{E}_L s \geq 0. \end{cases}$$

If $\mathbb{E}_L s > 1 + \frac{1}{\lambda \widehat{z}} - \frac{1}{\lambda}$*, the equilibrium contract is a pooling one with face value* $D = d^* + \phi$ *where*

$$d^* = \min\left\{\delta, \frac{\widehat{z}\left[\lambda E_L s + (1-\lambda)\right] - 1}{1 - \widehat{z}(1-\lambda)}\phi\right\}.$$

The implied haircut is

$$h_t = \max\left\{0, 1 - \frac{\widehat{z}\lambda E_L s}{1 - \widehat{z}(1-\lambda)} \frac{\phi_t}{\delta + \phi_t}\right\}.$$

Moreover, given any end-of-period price ϕ_t*, the asset price in the previous period and the lending volume are higher than those under the rigid DeFi contract.*

Note that the optimal haircut rule is not a fixed number or a simple linear rule but non-linear in price ϕ_t. The proposition shows that the flexible contract generates more social surplus. For example, when ϕ_t is high (which makes the debt contract informationally less sensitive), the intermediary can increase D_t to induce a higher lending volume which raises the surplus from lending. In contrast, when ϕ_t is low (which makes the contract informationally more sensitive), the intermediary may choose to lower D_t to maintain a pooling outcome. Depending on the parameter values, the intermediary may also choose to raise D_t to induce a separating equilibrium. This flexibility in adjusting D_t implies that, given any end-of-period price ϕ_t, the price of asset in the previous period and the loan size are weakly greater than those under the rigid DeFi contract.

The following proposition shows that the flexibility in setting the haircut optimally in response to changes in the asset price leads to a unique stationary equilibrium with a fixed realized equilibrium haircut.

Proposition 6. *Under flexible optimal debt limit there exists a unique stationary equilibrium that Pareto dominates the one under DeFi.*

The above result suggests that the rigid haircut rule (1) imposed by the DeFi smart contract generates financial instability in the form of multiple equilibria, and potential sentiment driven equilibria (e.g. Asriyan, Fuchs, and Green (2017)), and lowers welfare. Can a DeFi smart contract be pre-programmed to replicate the flexible contract design? This can be challenging in practice. First, flexible contract cannot be implemented using simple linear hair-cut rules that are typically en-coded in DeFi contracts. Second, the optimal debt threshold depends on information that may not be readily available on-chain (e.g., z, λ). Alternatively, the lending protocol can replace the algorithm by a human risk manager who can adjust risk parameters in real time according to the latest information. Relying fully on a trusted third party, however, can be controversial for a DeFi protocol. Our results highlight the difficulty in achieving stability and efficiency in a decentralized environment subject to informational frictions.

3.6 Liquidation and Fire Sale

In this section, we incorporate the price impact of liquidation sale of defaulted contracts from DeFi lending protocol that is empirically investigated in Lehar and Parlour (2022). To incorporate the possibility of fire-sales we modify the model as follows. There are two states: normal and fire-sale with probabilities σ and $1-\sigma$, respectively, known to all agents at the beginning of the period. In the fire-sale state, DeFi capital is not fast moving enough within the period to correct the temporary price impact due to the collateral liquidation from the defaulted DeFi debt contracts and fire-sale happens. These probabilities are iid across periods. We denote by ϕ and ϕ^f the price of the collateral asset in the normal state and the fire-sale state, respectively. Since the collateral price is lower in the fire-sale state and the states are known to all participants at the beginning of the period, the amount of collateralizable lending is different between the normal and the fire-sale state. We denote R^f the fire-sale loan rate, $D^f = \left(\delta + \phi^f\right)(1-h)$ the debt limit, a_L^f and a_H^f the amount of collateral pledged by types L and H in the fire-sale state. We denote by $\mathcal{M}$ is the amount of collateral being liquidated, and Λ the price impact of the liquidation.

As in the main model, the loan size chosen by the lender in the normal state based on the haircut rule $D = (\delta + \phi)(1-h)$ is given by the lender's break-even zero profit condition as follows:

$$q_t = \frac{1}{1+f}\left\{\frac{1}{a_{L,t}\lambda + a_{H,t}(1-\lambda)}\left[a_{L,t}\lambda \mathbb{E}_L \min\{D_t, s\phi_t\} + a_{H,t}(1-\lambda)\min\{D_t, \delta + \phi_t\}\right]\right\},$$

and the price of crypto asset in the normal state is given by:

$$
\begin{aligned}
\phi_t =& \beta\sigma\lambda \left(a_{L,t+1}\mathbb{E}_L \left(zD_{t+1}/R_{t+1} - \min\{D_{t+1}, s\phi_{t+1}\}\right)\right) \\
&+\beta\sigma\left(1-\lambda\right) a_{H,t+1}\left(zD_{t+1}/R_{t+1} - D_{t+1}\right) \\
&+\beta\sigma\left\{\lambda\left(\mathbb{E}_L s\right)\phi_{t+1} + \left(1-\lambda\right)\left(\delta+\phi_{t+1}\right)\right\} \\
&+\beta\left(1-\sigma\right)\lambda\left(a^f_{L,t+1}\mathbb{E}_L\left(zD^f_{t+1}/R^f_{t+1} - \min\{D^f_{t+1}, s\phi^f_{t+1}\}\right)\right) \\
&+\beta\left(1-\sigma\right)\left(1-\lambda\right)\left(a^f_{L,t+1}\mathbb{E}_L\left(zD^f_{t+1}/R^f_{t+1} - \min\{D^f_{t+1}, \delta+\phi^f_{t+1}\}\right)\right) \\
&+\beta\left(1-\sigma\right)\left\{\lambda\left(\mathbb{E}_L s\right)\phi^f_{t+1} + \left(1-\lambda\right)\left(\delta+\phi^f_{t+1}\right)\right\}.
\end{aligned}
$$

Similarly, the loan size chosen by the lender in the distressed state is given by:

$$
q^f = \lambda\frac{1}{1+f}\left\{\frac{1}{a_{L,t}\lambda + a_{H,t}\left(1-\lambda\right)}\left[a_{L,t}\lambda\mathbb{E}_L\min\left\{D^f_t, s\phi^f_t\right\} + a_{H,t}\left(1-\lambda\right)\min\left\{D^f_t, \delta+\phi^f_t\right\}\right]\right\},
$$

but the price of crypto asset in the distressed state with fire sale has a price impact discount which is specified as follows:

$$
\begin{aligned}
\phi^f_t =& \beta\sigma\lambda \left(a_{L,t+1}\mathbb{E}_L \left(zD_{t+1}/R_{t+1} - \min\{D_{t+1}, s\phi_{t+1}\}\right)\right) \\
&+\beta\sigma\left(1-\lambda\right) a_{H,t+1}\left(zD_{t+1}/R_{t+1} - D_{t+1}\right) \\
&+\beta\sigma\left\{\lambda\left(\mathbb{E}_L s\right)\phi_{t+1} + \left(1-\lambda\right)\left(\delta+\phi_{t+1}\right)\right\} \\
&+\beta\left(1-\sigma\right)\lambda\left(a^f_{L,t+1}\mathbb{E}_L\left(zD^f_{t+1}/R^f_{t+1} - \min\{D^f_{t+1}, s\phi^f_{t+1}\}\right)\right) \\
&+\beta\left(1-\sigma\right)\left(1-\lambda\right)\left(a^f_{L,t+1}\mathbb{E}_L\left(zD^f_{t+1}/R^f_{t+1} - \min\{D^f_{t+1}, \delta+\phi^f_{t+1}\}\right)\right) \\
&+\beta\left(1-\sigma\right)\left\{\lambda\left(\mathbb{E}_L s\right)\phi^f_{t+1} + \left(1-\lambda\right)\left(\delta+\phi^f_{t+1}\right)\right\} \\
&- \underbrace{\Lambda\mathcal{M}_{t+1}}_{\text{Fire-sale Discount}}
\end{aligned}
$$

The key difference between the normal and fire-sale prices is the last term in the fire-sale price which captures the fire-sale price impact of collateral liquidations. This price impact is temporary and within the period. We assume that $\Lambda \leq \beta(1-\lambda)\delta$. This assumption is to make sure that the price impact does not dominate asset fundamentals completely. That is, the fire sale does not lead to a negative asset price but results in a significant discount.

Next, we first solve for equilibrium outcomes in the special case where fire sale is certain every period ($\sigma = 0$) to demonstrate the direct impact of a fire sale. In this case, the stationary pooling equilibrium asset price ϕ^{fP} satisfies the equation:

$$\phi^{fP} = \beta\left[(z-1-f)q^P\right] + \beta(1-\lambda)\delta + \beta(\lambda\mathbb{E}_L\widetilde{s} + (1-\lambda))\phi^{fP} - \Lambda\mathcal{M}, \tag{15}$$

where

$$\mathcal{M}^P = \lambda \Pr\left(\widetilde{s}\phi^{fP} < D^{fP}\right) = \lambda \Pr\left(\widetilde{s} < \frac{D^{fP}}{\phi^{fP}}\right) = \lambda F_L\left(\frac{D^{fP}}{\phi^{fP}}\right).$$

and the stationary separating equilibrium asset price ϕ^{fS} satisfies the equation:

$$\phi^{fS} = \beta\left(\lambda(z-1-f)q^S + (1-\lambda)\delta + (\lambda\mathbb{E}_L\widetilde{s} + (1-\lambda))\phi^{fS}\right) - \Lambda\mathcal{M}, \tag{16}$$

$$\mathcal{M}^S = \lambda \Pr\left(\widetilde{s}\phi^{fS} < D^{fS}\right) = \lambda \Pr\left(\widetilde{s} < \frac{D^{fS}}{\phi^{fS}}\right) = \lambda F_L\left(\frac{D^{fS}}{\phi^{fS}}\right).$$

It is obvious that the size of fire-sale discount is related to the magnitude of the price impact Λ and the amount of default.

Lemma 2. *Equations (15) and (16) each have at least one solution. The largest solution of (15) is larger than all solutions of (16) and the smallest solution (16) is less than all solutions of (15). Consequently, when these equations both have a unique solution,* $\phi^{fP} \geq \phi^{fS}$.

Let $\underline{\phi}^{fS}$ be the smallest separating equilibrium price and $\overline{\phi}^{fP}$ be the largest pooling equilibrium price. Following the same steps as in the proofs of the main model, we establish the following existence result.

Proposition 7. *In the case of fire sales every period ($\sigma = 0$), there always exists at least one stationary equilibrium. Moreover,*

- *only pooling equilibria exist when* $\overline{\zeta} < \zeta(\underline{\phi}^{fS}; h)$,
- *only separating equilibria exist when* $\overline{\zeta} > \zeta(\overline{\phi}^{fP}; h)$,
- *there is at least one pooling equilibrium and one separating equilibrium when* $\overline{\zeta} \in [\zeta(\underline{\phi}^{fS}; h), \zeta(\overline{\phi}^{fP}; h)]$.

However, the conditions under which the multiplicity arises would be different from the case without any fire sale presented in the main model. This is because lower collateral asset prices make DeFi debts more information sensitive, resulting in more adverse selection. The two thresholds for multiplicity are now:

$$\check{\kappa}_S = \frac{\overline{\zeta}\left(\delta - \Lambda\lambda\right)}{\beta\delta\left((1-\lambda) + \overline{\zeta}\lambda z\right) - \Lambda\lambda}$$

and

$$\check{\kappa}_P = \frac{\overline{\zeta}\,(\delta - \Lambda\lambda)}{\beta z \delta \left((1-\lambda) + \overline{\zeta}\lambda\right) - \Lambda\lambda} < \check{\kappa}_S.$$

Now, we can prove the counterpart of Proposition 3 in the setting with fire sales.

Proposition 8. *Suppose fire sales occur every period ($\sigma = 0$), and the expected survival probability of the crypto asset satisfies $\mathbb{E}_L \widetilde{s} \in (\check{\kappa}_P, \check{\kappa}_S)$. There exists a threshold for haircut such that when the haircut h is below this threshold, there are multiple equilibria.*

It is easy to see that $\kappa_S < \check{\kappa}_S$ and $\kappa_P < \check{\kappa}_P$ which implies together with Proposition 8 that the region for multiplicity shifts up in the presence of fire sale. That is, multiple equilibria might occur for crypto assets with better fundamentals (that is, a higher survival probability), indicating that adverse selection is more severe with fire sale. Besides a direct impact on crypto asset price (temporary or not), fire sale in our model also triggers the feedback loop identified earlier: a lower asset price makes DeFi debt more information sensitive, results in more adverse selection in the DeFi lending market, leading to the withdrawal of the funding from the lenders, which in turn justifies even a lower asset price, and so on. This feedback loop might result in equilibrium multiplicity and hence amplifies the downward price impact from fire sale leading to more volatile equilibrium outcomes.

Finally, we study the case when fire sales might be temporary ($1 > \sigma > 0$). In this case, the impact of liquidation is only within the period and with probability σ, the state next period will return to normal. Following proposition 8, it is straightforward to show that the result of multiple equilibria is robust. Furthermore, we show that the anticipation of multiplicity in the fire-sale state might lead to multiplicity in the normal state, which is another channel that information friction amplifies the fire sale which making the price impact of the sale permanent even in the case when the fire sales are temporary. Hence the following theoretical result of price multiplicity offers a potential explanation of the empirical finding of permanent price impact of fire sale in Lehar and Parlour (2022).

Proposition 9. *When the probability of fire sale at each period is ($0 < \sigma < 1$), suppose the expected survival probability of the crypto asset satisfies $\mathbb{E}_L \widetilde{s} \in (\check{\kappa}_P, \check{\kappa}_S)$. There exists a threshold for haircut such that when the haircut h is below this threshold, there are multiple equilibrium outcomes in both normal and distressed states.*

References

Asriyan, Vladimir, William Fuchs, and Brett Green (2017). "Liquidity sentiments". *Working paper.*

Benhabib, Jess, Pengfei Wang, and Yi Wen (2015). "Sentiments and Aggregate Demand Fluctuations". *Econometrica* 83.2, pp. 549–585. DOI: https://doi.org/10.3982/ECTA11085. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.3982/ECTA11085. URL: https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA11085.

Chiu, J., C. Kahn, and T. Koeppl (2022). *Grasping De(centralized) Fi(nance) through the Lens of Economic Theory*. URL: https://ssrn.com/abstract=4221027.

Diamond, Douglas W. and Philip H. Dybvig (1983). "Bank runs, deposit insurance, and liquidity". *Journal of Political Economy* 91.3, pp. 401–419.

Hassan, Tarek A. and Thomas M. Mertens (2011). "Market Sentiment: A Tragedy of the Commons". *American Economic Review* 101.3, pp. 402–05. DOI: 10.1257/aer.101.3.402. URL: https://www.aeaweb.org/articles?id=10.1257/aer.101.3.402.

Lagos, Ricardo and Randall Wright (2005). "A unified framework for monetary theory and policy analysis". *Journal of Political Economy* 113.3, pp. 463–484.

Lehar, Alfred and Christine A Parlour (2022). *Systemic Fragility in Decentralized Markets*. Tech. rep. BIS.

Ozdenoren, Emre, Kathy Yuan, and Shengxing Zhang (2021). *Dynamic Asset-Backed Security Design*. London School of Economics.

A Appendix

A.1 Proof of Proposition 1

Condition (2) implies that, in a pooling equilibrium, the high-type borrower is willing to borrow if and only if

$$zq^P \geq \mathbb{E}\min\{D, \delta + \phi\},$$

which is equivalent to

$$\mathbb{E}y_L(s_L, \phi)/\mathbb{E}y_H(\phi) \geq \overline{\zeta}.$$

If $\mathbb{E}y_L(s_L, \phi)/\mathbb{E}y_H(\phi) > \overline{\zeta}$ then it is optimal for the intermediary to set $R = R^P$. To see this, note that at this rate lenders provide loan q^P and, by assumption, the high type borrower indeed chooses to borrow. This is clearly optimal because setting a higher rate lowers total lending and at a lower rate lenders do not break even. If $\mathbb{E}y_L(s_L, \phi)/\mathbb{E}y_H(\phi) < \overline{\zeta}$ then the intermediary's problem is solved by setting $R = R^S$. In this case, if the intermediary lowers the rate sufficiently below R^P then the high type would borrow. However, at that rate lenders would make negative profit.

Since $\mathbb{E}y_L(s_L,\phi)/\mathbb{E}y_H(\phi) = \mathbb{E}\min\{1, \frac{s_L\phi}{(\delta+\phi)(1-h)}\}$, a higher ϕ or h make the condition for the pooling outcome easier to satisfy.

A.2 Proof of Lemma 1

First, we define functions

$$\hat{q}^S(\phi) = \frac{1}{1+f}\mathbb{E}_L\left[\min\{(1-h)(\phi+\delta), s\phi\}\right],$$
$$\hat{q}^P(\phi) = \frac{1}{1+f}\left[\lambda\mathbb{E}_L\left[\min\{(1-h)(\phi+\delta), s\phi\}\right] + (1-\lambda)(1-h)(\phi+\delta)\right].$$

$$\mathbb{E}_L\left[\min\{(1-h)(\phi+\delta), s\phi\}\right] = \left(\int_0^{\hat{s}(\phi)} s dF_L\left(s\right)\right)\phi + \left(1 - F_L\left(\hat{s}\left(\phi\right)\right)\right)(1-h)(\phi+\delta)$$

where $\hat{s}\left(\phi\right) = \frac{(1-h)(\phi+\delta)}{\phi}$ if $(1-h)\delta < h\phi$ and $\hat{s}\left(\phi\right) = 1$ otherwise. Note $\hat{s}'\left(\phi\right) = -\frac{(1-h)\delta}{\phi^2}$ if $(1-h)\delta < h\phi$ and $\hat{s}'\left(\phi\right) = 0$ otherwise.

$$\begin{aligned}(1+f)\,\hat{q}^{P\prime}(\phi) = \lambda&\left[\left(\int_0^{\hat{s}(\phi)} s dF_L\left(s\right)\right) + \left(1 - F_L\left(\hat{s}\left(\phi\right)\right)\right)(1-h)\right]\\ &+ \lambda f_L\left(\hat{s}\left(\phi\right)\right)\hat{s}'\left(\phi\right)\left(\hat{s}\left(\phi\right)\phi - (1-h)(\phi+\delta)\right) + (1-\lambda)(1-h)\end{aligned}$$

If $(1-h)\delta < h\phi$ we have $\hat{s}\left(\phi\right)\phi - (1-h)(\phi+\delta) = 0$. If $(1-h)\delta \geq h\phi$ we have $\hat{s}'\left(\phi\right) = 0$. Hence

$$\begin{aligned}(1+f)\,\hat{q}^{P\prime}(\phi) = \lambda&\left[\left(\int_0^{\hat{s}(\phi)} s dF_L\left(s\right)\right) + \left(1 - F_L\left(\hat{s}\left(\phi\right)\right)\right)(1-h)\right]\\ &+ (1-\lambda)(1-h) < 1.\end{aligned}$$

Similarly

$$(1+f)\,\hat{q}^{S\prime}(\phi) = \left[\left(\int_0^{\hat{s}(\phi)} s dF_L\left(s\right)\right) + \left(1 - F_L\left(\hat{s}\left(\phi\right)\right)\right)(1-h)\right] < 1.$$

Note that their difference is

$$\begin{aligned}&\hat{q}^P(\phi) - \hat{q}^S(\phi)\\ =&\frac{1-\lambda}{1+f}[(1-\lambda)(1-h)(\phi+\delta) - \mathbb{E}\min\{(1-\lambda)(1-h)(\phi+\delta), s_L\phi\}]\\ \geq&0,\end{aligned}$$

Similarly, we define functions

$$\hat{\phi}^P(\phi) = \beta\left[(z-1-f)\hat{q}^P(\phi)\right] + \beta(1-\lambda)\delta + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda))\phi,$$

$$\hat{\phi}^S(\phi) = \beta\lambda(z-1-f)\hat{q}^S(\phi) + \beta(1-\lambda)\delta + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda))\phi.$$

Note:

$$\hat{\phi}^{P\prime}(\phi) = \beta\left[(z-1-f)\hat{q}^{P\prime}(\phi)\right] + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda)) < 1,$$
$$\hat{\phi}^{S\prime}(\phi) = \beta\lambda(z-1-f)\hat{q}^{S\prime}(\phi) + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda)) < 1,$$

$$\hat{\phi}^P(0) = \beta(1-\lambda)\delta + \beta\frac{(z-1-f)(1-\lambda)(1-h)\delta}{1+f} > \beta(1-\lambda)\delta = \hat{\phi}^S(0),$$
$$\hat{\phi}^{P\prime}(\phi) > 0 \text{ and } \hat{\phi}^{S\prime}(\phi) > 0.$$

Furthermore, the difference between the two functions is

$$\begin{aligned}&\hat{\phi}^P(\phi) - \hat{\phi}^S(\phi)\\ =&\beta(1-\lambda)(z-1-f)\hat{q}^P(\phi) + \beta\lambda(z-1-f)(\hat{q}^P(\phi) - \hat{q}^S(\phi)) > 0.\end{aligned}$$

The above properties imply that both functions have a unique fixed point and that $\phi^P > \phi^S$.

A.3 Proof of Proposition 3

For simplicity we set $f = 0$ but the result also holds for $f > 0$. In a separating equilibrium debt limit, loan size and asset price when $h = 0$ are given by:

$$D^S = \left(\delta + \phi^S\right)$$

$$q^S = \mathbb{E}_L(s)\phi^S$$

$$\phi^S = \frac{\beta(1-\lambda)\delta}{1 - \beta[\lambda z\mathbb{E}_L(s) + (1-\lambda)]}$$

Plugging the asset price into the condition for the existence of a separating equilibrium we obtain:

$$\zeta\left(\phi^S; 0\right) = \frac{\mathbb{E}_L(s)\phi^S}{(\delta + \phi^S)} < \overline{\zeta}$$

Rearranging we find that a separating equilibrium exists at $h = 0$ when

$$\mathbb{E}_L(s) < \frac{\bar{\zeta}}{\beta[(1-\lambda) + \bar{\zeta}\lambda z]} \equiv \kappa_S.$$

Furthermore, if the above condition holds, a separating also exists in a neighborhood of $h = 0$.

Similarly, in a pooling equilibrium debt limit, loan size and asset price when $h = 0$ are given by:

$$D^P = \left(\delta + \phi^P\right)$$

$$q^P = \lambda \mathbb{E}_L(s)\phi^P + (1-\lambda)(\delta + \phi^P)$$

$$\phi^P = \frac{\beta z(1-\lambda)\delta}{1 - \beta z[\lambda \mathbb{E}(s) + (1-\lambda)]}$$

Plugging the asset price into the condition for the existence of a pooling equilibrium we obtain:

$$\zeta\left(\phi^P; 0\right) = \frac{\mathbb{E}_L(s)\phi^P}{(\delta + \phi^P)} > \bar{\zeta}$$

Rearranging we find that a pooling equilibrium exists at $h = 0$ when

$$\mathbb{E}_L(s) > \frac{\bar{\zeta}}{\beta z[(1-\lambda) + \bar{\zeta}\lambda]} \equiv \kappa_P < \kappa_S$$

Furthermore, if the above condition holds, a pooling also exists in a neighborhood of $h = 0$.

Therefore, when $\mathbb{E}(s) \in (\kappa_P, \kappa_S)$, there are multiple equilibria in a neighborhood of $h = 0$ which implies that there is a threshold for haircut below which multiple equilibria exist.

A.4 Two-point Distribution Example

A.4.1 Separating Equilibrium

Suppose $s_L = 1$ w.p. π, and $s_L = 0$ w.p. $1 - \pi$.

In a separating equilibrium:

Debt limit:

$$D^S = \left(\delta + \phi^S\right)(1-h)$$

Loan size:

$$\ell_L = q^S = \mathbb{E}\left[\min\{D^S, s\phi^S\}\right] = \pi \min\{D^S, \phi^S\}$$

There are two cases.

Case (i) $D^S > \phi^S$

This is true when

$$\delta\frac{1-h}{h} > \phi^S.$$

We then have

$$q^S = \pi\phi^S,$$
$$\phi^S = \frac{\beta(1-\lambda)\delta}{1-\beta[\lambda z\pi + (1-\lambda)]}.$$

The existence of separating equilibrium requires

$$\zeta^S(h) = \frac{\pi\phi^S}{(\delta+\phi^S)(1-h)} < \zeta.$$

We define a threshold

$$\underline{h}^S \equiv \frac{\delta}{\phi^S+\delta} = \frac{1-\beta[\lambda z\pi + (1-\lambda)]}{1-\beta\lambda z\pi}.$$

When the haircut is lower than the threshold $\underline{h}$, the low type borrowers default even when $s_L = 1$. In this case, the loan size is equal to the expected value of the asset, $\pi\phi^S$, which does not depend on the haircut. Hence, the asset price is also independent of h. An increase in h, however, makes it harder to support a separating equilibrium as the contract becomes less information sensitive.

Case (ii) $D^S < \phi^S$

This is true when

$$\delta\frac{1-h}{h} < \phi^S.$$

We then have

$$q^S = \pi(\delta+\phi^S)(1-h)$$
$$\phi^S = \frac{\beta(\lambda(z-1)\pi(1-h) + (1-\lambda))\delta}{1-\beta[\lambda(z-1)\pi(1-h) + (1-\lambda) + \lambda\pi]}.$$

The existence of separating equilibrium requires

$$\zeta^S(h) = \pi < \zeta.$$

When the haircut is higher than the threshold $\underline{h}$, the low type pays back the loan to retain the collateral when $s_L = 1$. In this case, the loan size is equal to the πD. Hence, the asset price is decreasing in h. A separating equilibrium exists whenever $\pi < \zeta$ as h does not affect the information sensitivity of the contract.

A.4.2 Pooling Equilibrium

In a pooling equilibrium:

Debt limit:

$$D^P = \left(\delta + \phi^P\right)(1-h)$$

Loan size:

$$q^P = \lambda \mathbb{E}\left[\min\{D^P, s\phi^P\}\right] + (1-\lambda)D^P = \lambda\pi \min\{D^P, \phi^P\} + (1-\lambda)D^P$$

There are two cases.

Case (i) $D^P > \phi^P$

This is true when

$$\delta\frac{1-h}{h} > \phi^P.$$

We then have

$$\begin{aligned} q^P &= \lambda\pi\phi^P + (1-\lambda)D^P \\ \phi^P &= \frac{\beta(1-\lambda)\delta[(z-1)(1-h)+1]}{1-\beta[\lambda(z-1)\pi + (z-1)(1-\lambda)(1-h) + \lambda\pi + 1 - \lambda]} \end{aligned}$$

The existence of separating equilibrium requires

$$\zeta^P(h) = \frac{\pi\phi^P}{(\delta + \phi^P)(1-h)} > \zeta.$$

We can again define a threshold

$$\underline{h}^P \equiv \frac{1-\beta[\lambda(z-1)\pi + (z-1)(1-\lambda) + \lambda\pi + 1 - \lambda]}{1 - z\beta\lambda\pi - \beta(z-1)(1-\lambda)} < \underline{h}^S$$

such that this case holds when $h < \underline{h}^P$.

Case (ii) $D^P < \phi^P$

This is true when

$$\delta\frac{1-h}{h} < \phi^P.$$

We then have

$$q^P = \lambda\pi D^P + (1-\lambda)D^P$$
$$\phi^P = \beta\delta\frac{(z-1)(\lambda\pi+1-\lambda)(1-h)+(1-\lambda)}{1-\beta[(z-1)(\lambda\pi+1-\lambda)(1-h)+\lambda\pi+1-\lambda]}$$

The existence of pooling equilibrium requires

$$\zeta^P(h) = \pi > \zeta.$$

A.5 Proof of Uniqueness Under a Flexible Smart Contract

Denote the debt contract $y(D, \widetilde{\delta}+\widetilde{s}\phi) = \min(D, \widetilde{\delta}+\widetilde{s}\phi)$. We prove the result for the main model where

$$\mathbb{E}_H y(D, \widetilde{\delta}+\widetilde{s}\phi) = y(D, \delta+\phi);$$

and

$$\mathbb{E}_L y(D, \widetilde{\delta}+\widetilde{s}\phi) = \int_{\underline{s}}^{\bar{s}} y(D, s\phi)dF(s).$$

The arguments, however, generalize to the more general case with some modifications.

Denote $D^* \leq \delta+\phi$ the maximum face value so that the incentive constraint of the high type borrower is satisfied

$$\widehat{z}\left[\lambda\mathbb{E}_L y(D, \widetilde{\delta}+\widetilde{s}\phi) + (1-\lambda)\mathbb{E}_H y(D, \widetilde{\delta}+\widetilde{s}\phi)\right] \geq \mathbb{E}_H y(D, \widetilde{\delta}+\widetilde{s}\phi)$$

in which case there is a pooling equilibrium.

When the intermediary designs the smart deposit contract flexibly, it aims to maximize the expected trading volume. Specifically, the intermediary chooses D, or equivalently haircut, to maximize expected trade volume $[\lambda\mathbb{E}_L + a_{H,t}(1-\lambda)\mathbb{E}_H]\min(D, \widetilde{\delta}+\widetilde{s}\phi)$ taking ϕ as given. Note that the intermediary's payoff is increasing in D as long as the equilibrium does not switch from pooling to separating. Hence, if the intermediary chooses a contract that leads to a pooling outcome, then $D = D^*$, and if the intermediary chooses a contract that leads to a separating outcome, then $D = \delta+\phi$.

Next we look at the two cases:

Pooling case:

If $D < \phi$, we can denote $\hat{s} = D/\phi$. In this case, all terms in the incentive constraint for the high type are proportional to the asset price ϕ, which drops out of the constraint. So, the high type's incentive constraint is satisfied iff

$$\widehat{z}[\lambda\mathbb{E}_L\min(\hat{s}, s) + (1-\lambda)\hat{s}] \geq \hat{s}$$

Let $\mathcal{F}(\hat{s}) \equiv \widehat{z}\left[\lambda \mathbb{E}_L \min(\hat{s}, s) + (1-\lambda)\hat{s}\right] - \hat{s}$ and note the high type's incentive constraint is satisfied iff $\mathcal{F}(\hat{s}) \geq 0$. $\mathcal{F}(\hat{s})$ has the following properties:

$$\begin{aligned} \mathcal{F}(0) &\geq 0 \\ \mathcal{F}'(0) &= \widehat{z} - 1 > 0 \\ \mathcal{F}''(\hat{s}) &= -\widehat{z}\lambda f(\hat{s}) < 0 \end{aligned}$$

So $\mathcal{F}(\hat{s})$ is concave and strictly positive when $\hat{s}$ is close to 0. Suppose the information friction is severe enough so that $\mathcal{F}(1) = \widehat{z}(\lambda \mathbb{E}_L s + (1-\lambda)) - 1 < 0$, or equivalently $\mathbb{E}_L s < \frac{1-(1-\lambda)\widehat{z}}{\lambda \widehat{z}} = 1 + \frac{1}{\lambda \widehat{z}} - \frac{1}{\lambda} < 1$. In this case, there exists a unique threshold $0 < s^* < 1$ such that $\mathcal{F}(s^*) = 0$. Since the asset price ϕ drops out, threshold s^* does not depend on ϕ.

Taking next period asset price ϕ as given, the asset price in the current period under pooling equilibrium is

$$\phi^P(\phi) = \beta\left[\left(\widehat{z} - 1\right)\left(\lambda \mathbb{E}_L \min(s^*, s) + (1-\lambda)\, s^*\right)\phi + \lambda \phi \mathbb{E}_L s + (1-\lambda)(\delta + \phi)\right] \tag{A.1}$$

which has the following property

$$\begin{aligned} \frac{\partial \phi^P(\phi)}{\partial \phi} &= \beta\left[\left(\widehat{z} - 1\right)\left(\lambda \mathbb{E}_L \min(s^*, s) + (1-\lambda)\, s^*\right) + \lambda \mathbb{E}_L s + (1-\lambda)\right] < 1 \\ \phi^P(0) &= \beta(1-\lambda)\delta. \end{aligned}$$

So, $\phi^P(\phi)$ is a straight line with slope $\frac{\partial \phi^P(\phi)}{\partial \phi}$ and intercept $\phi^P(0) = \beta(1-\lambda)\delta$. Hence there is a unique steady state price satisfying $\phi^P(\phi) = \phi$.

Suppose information friction is not so severe so that $\mathcal{F}(1) > 0$, or equivalently, $1 > \mathbb{E}_L s > 1 + \frac{1}{\lambda \widehat{z}} - \frac{1}{\lambda}$. In this case, the face value of the debt is $D^* \geq \phi$. Let $d^*(\phi) = D^* - \phi$. There are two possibilities: either high type's incentive constraint is binding and there is $d^*(\phi) \leq \delta$ that satisfies:

$$\widehat{z}\left[\lambda \phi \mathbb{E}_L s + (1-\lambda)(d^*(\phi) + \phi)\right] = d^*(\phi) + \phi$$

or the high-type's incentive constraint is slack for all D. In the former case

$$d^*(\phi) = \frac{\widehat{z}\left[\lambda \mathbb{E}_L s + (1-\lambda)\right] - 1}{1 - \widehat{z}(1-\lambda)} \phi.$$

In the latter case $d^*(\phi) = \delta$. If $\frac{\widehat{z}[\lambda \mathbb{E}_L s + (1-\lambda)] - 1}{1 - \widehat{z}(1-\lambda)} \phi < \delta$,

$$\phi^P(\phi) = \beta\left[\frac{\lambda \widehat{z}}{1 - \widehat{z}(1-\lambda)} \lambda \mathbb{E}_L s \phi + (1-\lambda)(\delta + \phi)\right]. \tag{A.2}$$

Note,

$$\phi^P(0) = \beta(1-\lambda)\delta,$$

$$\frac{\partial \phi^P(\phi)}{\partial \phi} = \beta\left(\frac{\lambda\widehat{z}}{1-\widehat{z}(1-\lambda)}\lambda\mathbb{E}_L s + 1 - \lambda\right).$$

Hence $\phi^P(\phi)$ is a straight line with slope $\frac{\partial \phi^P(\phi)}{\partial \phi}$ and intercept $\phi^P(0)$.

If $\frac{\widehat{z}[\lambda\mathbb{E}_L s+(1-\lambda)]}{1-\widehat{z}(1-\lambda)}\phi > \delta$,

$$\begin{aligned}\phi^P(\phi) &= \beta\widehat{z}\left[\lambda\mathbb{E}_L s\phi + (1-\lambda)(\delta+\phi)\right] \\ &= \beta\widehat{z}\left[(1-\lambda)\delta + (\lambda\mathbb{E}_L s + 1 - \lambda)\phi\right].\end{aligned}$$

Note,

$$\phi^P(0) = \beta\widehat{z}(1-\lambda)\delta,$$

$$\frac{\partial \phi^P(\phi)}{\partial \phi} = \beta\widehat{z}(\lambda\mathbb{E}_L s + 1 - \lambda) < 1$$

By comparing the slopes of $\phi^P(\phi)$ when $\frac{\widehat{z}[\lambda\mathbb{E}_L s+(1-\lambda)]}{1-\widehat{z}(1-\lambda)}\phi$ is below and above δ, we can see that $\phi^P(\phi)$ is concave with slope less than 1 when $\frac{\widehat{z}[\lambda\mathbb{E}_L s+(1-\lambda)]}{1-\widehat{z}(1-\lambda)}\phi > \delta$.

Note that when $D^* \geq \phi$ in a pooling equilibrium or $\mathbb{E}_L s > 1+\frac{1}{\lambda\widehat{z}}-\frac{1}{\lambda}$, the value of a pooling contract is always greater than that of a separating contract. This is because the intermediary designs the contract optimally to maximize the expected trade volume. The expected value of a loan to a low type is the same in a separating equilibrium and a pooling equilibrium when $D^* \geq \phi$. So the intermediary strictly prefers designing a pooling contract as the revenue from the pooling contract strictly dominates that of a separating contract.

Hence when $\mathbb{E}_L s > 1+\frac{1}{\lambda\widehat{z}}-\frac{1}{\lambda}$, we can focus on the pooling equilibrium. From the analysis above, $\phi^P(\phi)$ is concave with slope less than 1 when $\frac{\widehat{z}[\lambda\mathbb{E}_L s+(1-\lambda)]}{1-\widehat{z}(1-\lambda)}\phi > \delta$. Hence, in this part of the parameter space there exists a unique equilibrium where the loan is traded in a pooling equilibrium.

Separating case:

As argued above, when analyzing the optimal contract in a separating equilibrium, we can focus on the parameter space where

$$E_L s < 1 + \frac{1}{\lambda\widehat{z}} - \frac{1}{\lambda}. \tag{A.3}$$

If the optimal contract supports a separating equilibrium, the intermediary would set $D = \delta + \phi$ to maximize the loan size to the low type. In the special parametrization of the model, any face value between ϕ and $\delta + \phi$ generates the same revenue from borrowing because a low quality asset does not pay any dividend. More generally, low quality assets could pay positive dividend. So the maximum face value $D = \delta + \phi$ is a more robust form of debt design in the separating case.

Given the face value $D = \delta + \phi$, the incentive constraint for the high type not to borrow is

$$\delta + \phi \geq \widehat{z}\mathbb{E}_L s\phi \tag{A.4}$$

Note that condition (A.3) implies that

$$\widehat{z}\mathbb{E}_L s < 1 + (\widehat{z} - 1)\,(1 - \frac{1}{\lambda}) < 1.$$

The condition for the existence of a separating equilibrium,(A.4), always holds.

In a separating equilibrium, the asset price is

$$\phi^S(\phi) = \beta\left[(\widehat{z} - 1)\lambda\mathbb{E}_L s\phi + \lambda\mathbb{E}_L s\phi + (1-\lambda)(\delta + \phi)\right] \tag{A.5}$$

which has the following property

$$\phi^S(0) = \beta(1-\lambda)\delta$$
$$\frac{\partial\phi^S(\phi)}{\partial\phi} = \beta\left(\widehat{z}\lambda\mathbb{E}_L s + 1 - \lambda\right)$$

So in this case, $\phi^S(\phi)$ is a straight line with slope $\frac{\partial\phi^S(\phi)}{\partial\phi}$ and intercept $\phi^S(0) = \beta(1-\lambda)\delta$.

The intermediary chooses the pooling contract if and only if

$$\left[\lambda\mathbb{E}_L + (1-\lambda)\,\mathbb{E}_H\right] y(D, \widetilde{\delta} + \widetilde{s}\phi^P) \geq \lambda\mathbb{E}_L y(D, \widetilde{\delta} + \widetilde{s}\phi^S)$$

or

$$\left[\lambda\mathbb{E}_L \min(s^*, s) + (1-\lambda)\,s^*\right]\phi^P \geq \phi^S\lambda\mathbb{E}_L s$$

where s^* is the unique solution to

$$\widehat{z}\left[\lambda\mathbb{E}_L \min(s^*, s) + (1-\lambda)s^*\right] = s^*.$$

Plugging in for ϕ^P and ϕ^S we can rewrite the inequality as

$$\frac{\left[\lambda\mathbb{E}_L \min(s^*, s) + (1-\lambda)\,s^*\right]}{1 - \beta\left[(\widehat{z}-1)\left(\lambda\mathbb{E}_L \min(s^*, s) + (1-\lambda)\,s^*\right) + \lambda\mathbb{E}_L s + (1-\lambda)\right]} \geq \frac{\lambda\mathbb{E}_L s}{1 - \beta\left[(\widehat{z}-1)\lambda\mathbb{E}_L s + \lambda\mathbb{E}_L s + (1-\lambda)\right]}$$

which holds iff

$$\lambda\mathbb{E}_L \min(s^*, s) + (1-\lambda)\,s^* - \lambda\mathbb{E}_L s \geq 0. \tag{A.6}$$

In either case, the equilibrium is unique.

To summarize the equilibrium characterization, when $\mathbb{E}_L s < 1 + \frac{1}{\lambda \hat{z}} - \frac{1}{\lambda}$, the equilibrium contract is a pooling one with face value $D = s^* \phi < \phi$ when condition (A.6) holds. Otherwise, the equilibrium contract is a separating one with face value $D = \delta + \phi$.

When $\mathbb{E}_L s > 1 + \frac{1}{\lambda \hat{z}} - \frac{1}{\lambda}$, the equilibrium contract is a pooling one with face value $D = d^* + \phi$ where

$$d^* = \min \left\{ \delta, \frac{\widehat{z}\left[\lambda E_L s + (1-\lambda)\right] - 1}{1 - \widehat{z}(1-\lambda)} \phi \right\}.$$

A.6 Proof of Lemma 2

We define $\hat{q}^S(\phi)$ and $\hat{q}^P(\phi)$ exactly as in the proof of Proposition 1 and following similar steps obtain $0 < \hat{q}^{S\prime}(\phi) < 1$, $0 < \hat{q}^{P\prime}(\phi) < 1$, $\hat{q}^P(\phi) \geq \hat{q}^S(\phi)$.

Next, we define functions

$$\hat{\phi}^P(\phi) = \beta\left[(z-1-f)\hat{q}^P(\phi)\right] + \beta(1-\lambda)\delta + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda))\phi - \Lambda\lambda F_L\left(\frac{(1-h)(\phi+\delta)}{\phi}\right),$$

$$\hat{\phi}^S(\phi) = \beta\lambda(z-1-f)\hat{q}^S(\phi) + \beta(1-\lambda)\delta + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda))\phi - \Lambda F_L\left(\frac{(1-h)(\phi+\delta)}{\phi}\right),$$

which have the following properties:

$$\hat{\phi}^P(0) = \beta(1-\lambda)\delta + \beta\frac{(z-1-f)(1-\lambda)(1-h)\delta}{1+f} - \Lambda\lambda > \beta(1-\lambda)\delta - \Lambda = \hat{\phi}^S(0) > 0$$

where the last inequality follows because $\Lambda < \beta(1-\lambda)\delta$ by assumption. Note:

$$\hat{\phi}^{P\prime}(\phi) = \beta\left[(z-1-f)\hat{q}^{P\prime}(\phi)\right] + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda)) + \lambda\Lambda f_L\left(\frac{(1-h)(\phi+\delta)}{\phi}\right)\left(\frac{(1-h)\delta}{\phi^2}\right),$$
$$\hat{\phi}^{S\prime}(\phi) = \beta\lambda(z-1-f)\hat{q}^{S\prime}(\phi) + \beta(\lambda\mathbb{E}(s_L) + (1-\lambda)) + \Lambda f_L\left(\frac{(1-h)(\phi+\delta)}{\phi}\right)\left(\frac{(1-h)\delta}{\phi^2}\right),$$

and

$$\hat{\phi}^{P\prime}(\phi) > 0 \text{ and } \hat{\phi}^{S\prime}(\phi) > 0.$$

Note that both $\hat{\phi}^{P\prime}(\phi)$ and $\hat{\phi}^{S\prime}(\phi)$ are strictly less than 1 for large ϕ. To see this note that the first two terms are strictly less than 1 and the last term becomes small as ϕ increases. Finally, the difference between the two functions is

$$\begin{aligned}
&\hat{\phi}^P(\phi) - \hat{\phi}^S(\phi) \\
=&\beta(1-\lambda)(z-1-f)\hat{q}^P(\phi) + \beta\lambda(z-1-f)(\hat{q}^P(\phi) - \hat{q}^S(\phi)) \\
&+ (1-\lambda)\,\Lambda F_L\left(\frac{(1-h)(\phi+\delta)}{\phi}\right) > 0.
\end{aligned}$$

The above properties imply that both functions have at least one fixed point and the largest fixed point is larger for $\hat{\phi}^P(\phi)$ than $\hat{\phi}^S(\phi)$. So when fixed points are unique pooling price exceeds separating price and otherwise the largest pooling equilibrium price exceeds all separating prices.

A.7 Proof of Proposition 8

For simplicity we set $f = 0$ but the result also holds for $f > 0$. In a separating equilibrium with fire sale every period, debt limit, loan size and asset price when $h = 0$ are given by:

$$D^{fS} = \left(\delta + \phi^{fS}\right)$$

$$q^{fS} = \mathbb{E}_L(s)\phi^{fS}$$

$$\phi^{fS} = \frac{\beta(1-\lambda)\delta - \Lambda\lambda}{1 - \beta[\lambda z \mathbb{E}_L(s) + (1-\lambda)]}$$

Plugging the asset price into the condition for the existence of a separating equilibrium we obtain:

$$\zeta\left(\phi^{fS}; 0\right) = \frac{\mathbb{E}_L(s)\phi^{fS}}{\left(\delta + \phi^{fS}\right)} < \overline{\zeta}$$

Rearranging we find that a separating equilibrium exists at $h = 0$ when

$$\mathbb{E}_L(s) < \frac{\overline{\zeta}\left(\delta - \Lambda\lambda\right)}{\beta\delta\left((1-\lambda) + \overline{\zeta}\lambda z\right) - \Lambda\lambda} \equiv \check{\kappa}_S$$

Similarly, in a pooling equilibrium with fire sale every period, debt limit, loan size and asset price when $h = 0$ are given by:

$$D^{fP} = \left(\delta + \phi^{fP}\right)$$

$$q^{fP} = \lambda\mathbb{E}_L(s)\phi^{fP} + (1-\lambda)(\delta + \phi^{fP})$$

$$\phi^{fP} = \frac{\beta z(1-\lambda)\delta - \Lambda\lambda}{1 - \beta z\left[\lambda\mathbb{E}_L(s) + (1-\lambda)\right]}$$

Rearranging we find that a separating equilibrium exists at $h = 0$ when

$$\mathbb{E}_L(s) > \frac{\overline{\zeta}\left(\delta - \Lambda\lambda\right)}{\beta z\delta\left((1-\lambda) + \overline{\zeta}\lambda\right) - \Lambda\lambda} \equiv \check{\kappa}_P.$$

Clearly $\check{\kappa}_P < \check{\kappa}_S$. Therefore, when $\mathbb{E}(s) \in (\check{\kappa}_P, \check{\kappa}_S)$, there are multiple equilibria in a neighborhood of $h = 0$.

A.8 An Alternative Setup with Unobservable Private Valuation

We briefly consider an alternative setup where the private information is related to borrowers' private valuation of the asset, instead of the asset's common value. We show that the main results hold.

Suppose with probability $1 - \varepsilon$, the state is good ($s = 1$) and the asset pays dividend δ. With probability ε, the state is bad ($s = 0$), it does not pay any dividends. In addition, the borrower has unobservable private valuation. A type $i = H, L$ borrower, if holding an asset, receives a private value $v_i(s)$ before the asset market opens and after the loan is settled. The type i is determined before the loan is made and the information is private. With probability λ, the borrower is of type $i = L$, and the private valuation is $v_L(1) = v$ in the good state and $v_L(0) = 0$ in the bad state. With probability $1 - \lambda$, the borrower's type is $i = H$ and the private valuation is $v_H(1) = v_H(0) = v$. After observing the private information, the borrower borrows from the platform. After observing the realization of δ, the borrower decides whether to repay or to default. After the loan is settled, the borrower, if holding the asset, receives the private valuation. At the end of the period, the asset is traded at $\delta + \phi$ in the good state and at ϕ in the bad state.

The debt limit is given by $D = (\delta + \phi)(1 - h)$. We assume that $v > \delta$. As a result, all borrowers repay in the good state. A low type borrower defaults in the bad state when $D > \phi$. Our analysis will focus on the case of $D \geq \phi$ as it is suboptimal to set $D < \phi$.

In the separating equilibrium, the loan size is

$$q^S = D^S - \varepsilon(D^S - \phi^S)$$

and the asset price is

$$\phi^S = \beta \frac{\lambda(z-1)(1-h)(1-\varepsilon)\delta + (1-\varepsilon)\delta + (1-\varepsilon\lambda)v}{1 - \beta - \beta\lambda(z-1)(1-h(1-\varepsilon))}.$$

The separating equilibrium exists when

$$\frac{(1-\varepsilon)D^S + \varepsilon\phi^S}{D^S} < \zeta.$$

In the pooling equilibrium, the loan size is

$$q^P = D^P + \lambda\varepsilon(\phi^P - D^P)$$

and the asset price is

$$\phi^P = \beta\frac{(z-1)\delta(1-h)(1-\varepsilon\lambda) + \beta(1-\varepsilon)\delta + \beta(1-\varepsilon\lambda)v}{1-\beta-\beta(z-1)(1-h(1-\varepsilon\lambda))}.$$

The pooling equilibrium exists when

$$\frac{(1-\varepsilon)D^P + \varepsilon\phi^P}{D^P} > \zeta.$$

Hence we can reproduce the main multiplicity result.

Proposition 10. *For h not too large, $\phi^P > \phi^S$ and multiplicity exists when*

$$1 - \frac{\varepsilon\delta}{\delta+\phi^P} > \zeta > 1 - \frac{\varepsilon\delta}{\delta+\phi^S}.$$

A.9 Private Information Parameter $\chi < 1$

We have considered the case where there is private information in each period. We now introduce a parameter, χ, to control the degree of information imperfection. With probability $1-\chi$, there is no private information in the sense that there are no low-quality assets (denoted by state 0). All the equilibrium conditions remain the same except that the asset prices satisfy

$$\begin{aligned}\phi_t = \beta\chi \Big\{ \lambda \Big[\int_{\underline{s}}^{\bar{s}} (z\ell_{L,t+1} - \min\{\ell_{L,t+1}R_{t+1}, a_{L,t+1}s_L\phi_{t+1})\} + s_L\phi_{t+1})\, dF(s_L) \Big] \\ +\chi(1-\lambda)\left[z\ell_{H,t+1} - \min\{\ell_{H,t+1}R_{t+1}, a_{H,t+1}(\delta+\phi_{t+1})\} + \delta + \phi_{t+1}\right]\Big\} \\ + \beta(1-\chi)\left[z\ell^0_{t+1} - \min\{\ell^0_{t+1}R^0_{t+1}, a^0_{t+1}(\delta+\phi_{t+1})\} + \delta + \phi_{t+1}\right].\end{aligned}$$

where $a^0 = 1$, $\ell^0_t = q^0_t = \frac{1}{1+f}(\delta+\phi_t)(1-h)$ and $R^0_t = (\delta+\phi_t)(1-h)/q^0_t$. By continuity, all results hold when χ is sufficiently close to 1.