

Equivalence between intersection cuts and the corner polyhedron

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Abstract

Intersection cuts were introduced by Balas and the corner polyhedron by Gomory. Balas showed that intersection cuts are valid for the corner polyhedron. In this paper we show that, conversely, every nontrivial facet-defining inequality for the corner polyhedron is an intersection cut.

Keywords: integer programming, cutting plane, corner polyhedron, intersection cut

We consider a mixed integer linear set

$$\begin{aligned} Ax &= b \\ x_j &\in \mathbb{Z} \quad \text{for } j = 1, \dots, p \\ x_j &\geq 0 \quad \text{for } j = 1, \dots, n \end{aligned} \tag{1}$$

where $p \leq n$, A is a matrix in $\mathbb{Q}^{m \times n}$, and b is a column vector in \mathbb{Q}^m . We assume that A has full row rank m . Given a feasible basis B , let $N = \{1, \dots, n\} \setminus B$ index the nonbasic variables. We rewrite the system $Ax = b$ as

$$x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad \text{for } i \in B. \tag{2}$$

where $\bar{b}_i \geq 0$, $i \in B$.

The corner polyhedron introduced by Gomory [5] is obtained from (1) by dropping the nonnegativity restriction on all the basic variables x_i , $i \in B$, in (2). Note that in this

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relaxation we can drop the constraints $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for all $i \in B \cap \{p+1, \dots, n\}$ since these variables x_i are continuous and only appear in one equation and no other constraint. Therefore from now on we assume that all basic variables in (2) are integer variables, i.e. $B \subseteq \{1, \dots, p\}$.

Therefore the relaxation of (1) introduced by Gomory is the mixed integer set

$$\begin{aligned} x_i &= \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j && \text{for } i \in B \\ x_i &\in \mathbb{Z} && \text{for } i = 1, \dots, p \\ x_j &\geq 0 && \text{for } j \in N. \end{aligned} \quad (3)$$

The convex hull of the feasible solutions to (3) is called the *corner polyhedron* relative to the basis B and it is denoted by $\text{corner}(B)$.

Let $P(B)$ be the linear relaxation of (3). $P(B)$ is an $|N|$ -dimensional polyhedron with a unique vertex, namely $\bar{x}_i = \bar{b}_i$, for $i \in B$, $\bar{x}_j = 0$, for $j \in N$, and with $|N|$ extreme rays, \bar{r}^j for $j \in N$, defined by

$$\bar{r}_h^j = \begin{cases} -\bar{a}_{hj} & \text{if } h \in B, \\ 1 & \text{if } j = h, \\ 0 & \text{if } h \in N \setminus \{j\}. \end{cases} \quad (4)$$

The corner polyhedron has been investigated over the last few decades. It is well known, and easy to prove, that $\text{corner}(B)$ is nonempty if and only if there exists a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ satisfying $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for all $i \in B$. Furthermore, if $\text{corner}(B)$ is nonempty, then it is $|N|$ -dimensional, and its extreme rays are \bar{r}^j for $j \in N$. We will assume throughout this note that $\text{corner}(B)$ is nonempty.

Clearly, $P(B)$ coincides with $\text{corner}(B)$ when $\bar{x}_i \in \mathbb{Z}$ for $i = 1, \dots, p$. If this is not the case, \bar{x} does not belong to $\text{corner}(B)$ and we address the problem of finding valid inequalities for the set (1) that are violated by the point \bar{x} . Balas [1] proposed the following construction to generate valid inequalities for the corner polyhedron that cut off the basic solution \bar{x} .

Consider a closed convex set $C \subseteq \mathbb{R}^n$ such that the interior of C contains the point \bar{x} . Assume that the interior of C contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. For $j \in N$, define

$$\alpha_j = \max\{\alpha \geq 0 : \bar{x} + \alpha \bar{r}^j \in C\}.$$

Since \bar{x} is in the interior of C , $\alpha_j > 0$. When \bar{r}^j belongs the recession cone of C , we have $\alpha_j = +\infty$. Define $\frac{1}{+\infty} = 0$. The inequality

$$\sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1 \quad (5)$$

is the intersection cut defined by C .

Balas [1] showed that intersection cuts are valid for $\text{corner}(B)$. The following theorem provides a converse statement, namely that $\text{corner}(B)$ is defined by the intersection cuts.

A valid inequality for $\text{corner}(B)$ is *trivial* if it is implied by the nonnegativity constraints $x_j \geq 0$, $j \in N$. Every nontrivial valid inequality for $\text{corner}(B)$ can be expressed in terms of the nonbasic variables only, and can be written in the form $\sum_{j \in N} \gamma_j x_j \geq 1$, where $\gamma_j \geq 0$ for all $j \in N$. We say that such an inequality is *minimal* if there is no other valid inequality

$\sum_{j \in N} \gamma'_j x_j \geq 1$ for $\text{corner}(B)$ such that $\gamma'_j \leq \gamma_j$ for all $j \in N$, and the inequality is strict for at least one index $j \in N$. We say that it is *rational* if $\gamma_j \in \mathbb{Q}$ for all $j \in N$. Since A and b are rational, every nontrivial facet-defining inequality for $\text{corner}(B)$ is rational and minimal.

Theorem 1. *If $\text{corner}(B)$ is nonempty, every nontrivial rational minimal valid inequality for $\text{corner}(B)$ is an intersection cut.*

Proof. Assume that $\text{corner}(B)$ is nonempty, and let $\sum_{j \in N} \gamma_j x_j \geq 1$ be a rational minimal valid inequality for $\text{corner}(B)$. Consider the simplex

$$S = \{x \in \mathbb{R}^n \mid \sum_{j \in N} \gamma_j x_j \leq 1, x_j \geq 0 \text{ for } j \in N, x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \text{ for } i \in B\}.$$

We will enlarge S into a convex set C containing \bar{x} in its interior but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, and observe that $\sum_{j \in N} \gamma_j x_j \leq 1$ is the intersection cut defined by C . This is straightforward in the pure integer case, since we only need to relax the inequalities $x_j \geq 0$, $j \in N$. The crux of the proof is how to do this in the mixed integer case. We first project S onto the space of integer variables (note that any polytope may arise as the projection of a simplex), then we enlarge it to a set K and construct a cylinder over K .

(1) *No face F of S containing \bar{x} has a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its relative interior (including the improper face $F = S$).*

Indeed, let \tilde{x} be a point of S in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. Since $S \subseteq P(B)$, \tilde{x} belongs to $\text{corner}(B)$ and since $\sum_{j \in N} \gamma_j x_j \geq 1$ is a valid inequality for $\text{corner}(B)$, then $\sum_{j \in N} \gamma_j \tilde{x}_j = 1$.

Let F be a face of S containing \bar{x} , and suppose that \tilde{x} is in the relative interior of F . Then there exists a scalar $\lambda > 1$ such that $z = \bar{x} + \lambda(\tilde{x} - \bar{x})$ is in F . Since $\bar{x}_j = 0$, $j \in N$ and $\sum_{j \in N} \gamma_j \tilde{x}_j = 1$, then $\sum_{j \in N} \gamma_j z_j > 1$. This contradicts the fact that $z \in S$ and (1) is proven.

Let $\tilde{S} = S + L$ where $L = \{0\}^p \times \mathbb{R}^{n-p}$. Since S is a rational polyhedron and the lineality space of \tilde{S} contains L , the polyhedron \tilde{S} can be expressed as $\tilde{S} = \{x \in \mathbb{R}^n \mid \sum_{j=1}^p c_j^i x_j \leq d_i, i = 1, \dots, t\}$ for some $c^1, \dots, c^t \in \mathbb{Z}^p$ and $d_1, \dots, d_t \in \mathbb{Z}$. (Indeed $\{x \in \mathbb{R}^p \mid \sum_{j=1}^p c_j^i x_j \leq d_i, i = 1, \dots, t\}$ is the projection of S onto \mathbb{R}^p .)

(2) *No face of \tilde{S} containing \bar{x} has a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its relative interior.*

Let \tilde{F} be a face of \tilde{S} and let $F = S \cap \tilde{F}$. Then F is a face of S and $\tilde{F} = F + L$. Therefore, since L is in the lineality space of \tilde{F} , we have $\text{relint}(\tilde{F}) = \text{relint}(F) + L$, where $\text{relint}(\cdot)$ denotes the relative interior of a set. Assume \tilde{F} contains \bar{x} . Since \bar{x} belongs to S and $F = S \cap \tilde{F}$, we have $\bar{x} \in F$. Assume $\text{relint}(\tilde{F})$ contains \tilde{x} in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. Then $\tilde{x} + L$ is contained in $\text{relint}(\tilde{F}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Since $\text{relint}(\tilde{F}) = \text{relint}(F) + L$, we have $\tilde{x} + L$ contains a point in $\text{relint}(F) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, a contradiction to (1). This proves (2).

(3) *There exists a convex set $K \subset \mathbb{R}^p$ with no point of \mathbb{Z}^p in its interior such that the set $C := K \times \mathbb{R}^{n-p}$ contains \bar{x} in its interior and $\tilde{S} \subseteq C$.*

Assume, without loss of generality, that \bar{x} satisfies at equality the first h constraints defining \tilde{S} (possibly $h = 0$), and none of the other constraints. That is

$$\begin{aligned} \sum_{j=1}^p c_j^i \bar{x}_j &= d_i & i = 1, \dots, h; \\ \sum_{j=1}^p c_j^i \bar{x}_j &< d_i & i = h+1, \dots, t. \end{aligned}$$

Define $d'_i = d_i + 1$ for $i = 1, \dots, h$ and $d'_i = d_i$ for $i = h + 1, \dots, t$, and let $K = \{x \in \mathbb{R}^p \mid \sum_{j=1}^p c_j^i x_j \leq d'_i, \dots, i = 1, \dots, t\}$ and $C = K \times \mathbb{R}^{n-p}$. Note that $C = \{x \in \mathbb{R}^n \mid \sum_{j=1}^p c_j^i x_j \leq d'_i, \dots, i = 1, \dots, t\}$. By construction, \bar{x} is in the interior of C and $\tilde{S} \subseteq C$.

We only need to show that K contains no point of \mathbb{Z}^p in its interior. Suppose not. Then there exists $\tilde{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$ such that \tilde{x} is in the interior of C . Hence $\sum_{j=1}^p c_j^i \tilde{x}_j < d'_i$ for $i = 1, \dots, t$. By definition of d' and since $\tilde{x}_j \in \mathbb{Z}$ for $j = 1, \dots, p$, we have $\sum_{j=1}^p c_j^i \tilde{x}_j \leq d_i$ for $i = 1, \dots, h$, and $\sum_{j=1}^p c_j^i \tilde{x}_j < d_i$ for $i = h + 1, \dots, t$. Let J be the set of indices i such that $\sum_{j=1}^p c_j^i \tilde{x}_j = d_i$, and $\tilde{F} = \{x \in \tilde{S} \mid \sum_{j=1}^p c_j^i \tilde{x}_j = d_i, i \in J\}$. By construction, \tilde{F} is a face of \tilde{S} containing \bar{x} , and \tilde{x} is a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in the relative interior of \tilde{F} , a contradiction to (2) and this proves (3).

Let $\sum_{j \in N} \alpha_j^{-1} x_j \geq 1$ be the intersection cut defined by C , where α_j is defined by (5) for all $j \in N$. Since $\bar{x} + \gamma_j^{-1} \bar{r}^j \in S$ and $S \subseteq C$, the definition of α_j implies $\gamma_j^{-1} \leq \alpha_j$, $j \in N$. By minimality of $\sum_{j \in N} \gamma_j x_j \geq 1$, we have $\gamma_j \leq \alpha_j^{-1}$, thus $\gamma_j = \alpha_j$ for all $j \in N$. \square

We conclude this note with a few consequences. The convex sets needed to generate minimal intersection cuts are of the form $K \times \mathbb{R}^{n-p}$, where K is a maximal lattice-free convex set in \mathbb{R}^p (*lattice-free* means that K does not contain any point of \mathbb{Z}^p in its interior). A theorem of Lovász [7] states that maximal lattice-free convex sets are polyhedra.

An attractive feature of minimal intersection cuts is that they are given by “formulas” depending only on the number p of integer variables and the vector $\bar{b} \in \mathbb{R}^{|B|}$. Indeed, let $\tilde{x} \in \mathbb{R}^p$ be defined by $\tilde{x}_i = \bar{x}_i$, $i = 1, \dots, p$. Given a lattice-free convex set $K \subset \mathbb{R}^p$ containing \tilde{x} in its interior, let $\gamma_K : \mathbb{R}^p \rightarrow \mathbb{R}$ be the gauge of $K - \tilde{x}$, which is defined by

$$\gamma_K(r) = \inf\{t \mid \tilde{x} + t^{-1}r \in K\}, \quad r \in \mathbb{R}^p.$$

Then the intersection cut defined by $K \times \mathbb{R}^{n-p}$ is $\sum_{j \in N} \gamma_K(\tilde{r}^j) s_j \geq 1$, where $\tilde{r}^j \in \mathbb{R}^p$ is defined by $\tilde{r}_i^j = \bar{r}_i^j$, $i = 1, \dots, p$. Note that the above function γ_K gives a valid inequality for any choice of the coefficients \bar{a}_{ij} and for any number of continuous variables.

A function $\psi : \mathbb{R}^p \rightarrow \mathbb{R}$ is a *valid function* if $\sum_{j \in N} \psi(\tilde{r}^j) x_j \geq 1$ is valid for $\text{corner}(B)$ for any choice of the \bar{a}_{ij} s and for any number of continuous variables. Our theorem, together with a result of Borozan and Cornuéjols [4], implies that all valid functions are dominated by functions of the form γ_K , for some lattice-free convex set K , and furthermore every inequality of $\text{corner}(B)$ is generated by some valid function. This proves the following.

Corollary 2. *Every minimal valid function is continuous and piecewise linear.*

This is related to a conjecture of Gomory and Johnson [6], stating that all valid functions for the infinite group problem are piecewise linear. This conjecture was disproved in [3], although the above corollary shows that the statement holds when the number of integer variables is fixed.

The proof technique used in the theorem can be applied in a more general setting. If C is a convex set whose interior contains \bar{x} but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, then the closed convex hull of $P(B) \setminus C$ is the intersection of $P(B)$ with the (unique) intersection cut defined by C . In other words, intersection cuts give a way to reconvexify $P(B) \setminus C$. In general, let P be

the linear relaxation of (1), and P_I be the convex hull of feasible solutions to (1). Given a fractional vertex v of P , one could find valid inequalities cutting off v as follows. Given a polyhedron C whose interior contains v but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, the closed convex hull of $P \setminus C$ is a polyhedron Q containing P_I but not v . Even though Q may have exponentially many more facets than P or C , it admits an extended formulation which is polynomial in the size of the representation of P and C [2]. Therefore, given C , one can separate v from Q in polynomial time. Conversely, the same construction as in the proof of the Theorem 1 shows that every facet of P_I cutting off v can be obtained from an appropriate convex set C . More precisely, given a facet-defining inequality for P_I $\alpha x \leq \beta$ cutting off v , there exists some polyhedron C , whose interior contains v but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, such that $\alpha x \leq \beta$ is facet-defining for the closed convex hull of $P \setminus C$.

References

- [1] E. Balas, Intersection cuts – A new type of cutting planes for integer programming, *Operations Research* 19 (1971) 19–39.
- [2] E. Balas, Disjunctive programming: properties of the convex hull of feasible points, GSIA Management Science Research Report MSRR 348, Carnegie Mellon University (1974), published as invited paper in *Discrete Applied Mathematics* 89 (1998) 1–44.
- [3] A. Basu, M. Conforti, G. Cornuéjols, G. Zambelli, A Counterexample to a Conjecture of Gomory and Johnson, manuscript (2008).
- [4] V. Borozan, G. Cornuéjols, Minimal Valid Inequalities for Integer Constraints, *Mathematics of Operations Research* 34 (2009) 538-546.
- [5] R.G. Gomory, Some polyhedra related to combinatorial problems, *Linear Algebra and Applications* 2 (1969) 451–558.
- [6] Gomory, R. E., Johnson, E. L., T-space and Cutting planes, *Mathematical Programming, Series B*, 96: 341-375, 2003.
- [7] L. Lovász, Geometry of Numbers and Integer Programming, *Mathematical Programming: Recent Developements and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989), 177-210.