On matrices with the Edmonds-Johnson property arising from bidirected graphs

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Abstract

In this paper we study totally half-modular matrices obtained from $\{0, \pm 1\}$-matrices with at most two nonzero entries per column by multiplying by 2 some of the columns. We give an excluded-minor characterization of the matrices in this class having strong Chvàtal rank 1. Our result is a special case of a conjecture by Gerards and Schrijver [6]. It also extends a well known theorem of Edmonds and Johnson [5].

1 Introduction

Given a polyhedron P, the *Chvátal rank* of P is the smallest number t such that the t-th Chvátal closure of P is integral. The *strong Chvátal rank* of a rational matrix A is the smallest number t such that the polyhedron defined by the system $b \leq Ax \leq c$, $l \leq x \leq u$ has Chvátal rank at most t for all integral vectors b, c, l, u (we refer the reader to [13] for an exposition on the subject). Matrices with strong Chvátal rank 0 are exactly the totally unimodular matrices. Matrices with strong Chvátal rank at most 1 are said to have the *Edmonds-Johnson property* (*EJ property*).

While the class of integral matrices with strong Chvátal rank 0 is well understood, no general characterization is known for integral matrices with the EJ property. Few classes of matrices with such property are known. Edmonds and Johnson [5] showed that any integral matrix in which the sum of the absolute values of the entries in each column is at most 2 has the EJ property (see [14] for a thorough survey). Gerards and Schrijver [7] proved that an integral matrix in which the sum of the absolute values of the entries in each row is at most 2 has the Edmonds-Jonson property if and only if it does not contain an odd-K_4 minor. Recent results of Conforti et al.[2] and Del Pia and Zambelli [4] imply that any matrix obtained from a totally unimodular matrix with at most two nonzero entries per row by multiplying by 2 some of the columns has the EJ property.

A vector or matrix A is *half-integral* if $2A$ is integral. An integral matrix A is said *totally half-modular* if, for each nonsingular square submatrix B of A, B^{-1} is half-integral. All

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the known classes of matrices with the EJ property are totally half-modular. Gerards and Schrijver [6] conjectured a characterization of the class of totally half-modular matrices with the EJ property in terms of minimal forbidden minors. We explain the conjecture next.

It is known [7] that the class of totally half-modular matrices with the EJ property is closed under the following operations:

(i) deleting or permuting rows or columns, or multiplying them by -1;

(ii) dividing by 2 an even row (i.e. a row where all entries are $0, \pm 2$);

(iii) pivoting on a $+1$ entry,

where pivoting on the top-left entry of $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ results in $\begin{pmatrix} -1 & g \\ f & D - fg \end{pmatrix}$ (here f is a column vector and g a row vector). We say that a matrix A' is a *minor* of A if it arises from A by a series of operations (i)-(iii), and A' is a *proper minor* of A if A' is a minor of A but A is not a minor of A'. The following totally half-modular matrices are minimal forbidden minors for the EJ property,

$$A_3 := \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix} \quad , \quad A_4 := \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}.$$

That is, A_3 and A_4 do not have the EJ property, but all their proper minors do. Gerards and Schrijver [6] conjectured that A_3 and A_4 are the only minor-minimal totally half-modular matrices without the EJ property.

Conjecture 1. *A totally half-modular matrix has the EJ property if and only if it has no minor equal to A_3 or A_4.*

The above conjecture seems to be extremely hard. Furthermore, the matrix A_3 does not appear as a forbidden minor in any of the classes of totally half-modular matrices for which Conjecture 1 has been proven so far. In order to make progress and to gain insight on the role of the minor A_3, we prove the conjecture for a special class of matrices. Conforti, Di Summa, Eisenbrand and Wolsey [1] proved that, if A is a matrix obtained from the node-edge incidence matrix $\bar{A}$ of a bipartite graph by multiplying by 2 some of the columns of $\bar{A}$, and if b is an integral vector, deciding if $Ax = b$ has a nonnegative integral solution is $\mathcal{NP}$-hard. Since incidence matrices of bipartite graphs are totally unimodular, such a matrix A is totally half-modular. Therefore, even characterizing which of the matrices in this class have the EJ property is interesting. Furthermore, we know that A_4 is never a minor of any of these matrices (this follows from the fact A_4 is obtained from the Fano matroid by multiplying a column by 2, and the fact that $\bar{A}$ cannot contain the Fano matroid as a minor since it is totally unimodular [15]). Thus, according to Conjecture 1, A_3 should be the only forbidden minor in this class.

In this paper we prove Conjecture 1 for a wider class of totally half-modular matrices. The following is the main result of our paper.

Theorem 1. *Let A be a totally half-modular matrix obtained by multiplying by 2 some of the columns of a $\{0, \pm 1\}$-matrix with at most two nonzero entries per column. The matrix A has the EJ property if and only if it does not contain A_3 as a minor.*

Note that, in the above theorem, the $\{0, \pm 1\}$-matrix corresponding to A does not need to be totally unimodular in order for A to be totally half-modular.

1.1 Bidirected graphs and minors

It will be convenient to state our result in terms of bidirected graphs.

A *bidirected graph* is a triple $G = (V(G), E(G), \sigma(G))$, where $V(G)$ is the set of the nodes of G, $E(G)$ is the set of the edges of G and $\sigma(G)$ is a *signing* of $(V(G), E(G))$, i.e. a map that assigns to each $e \in E(G)$ and $v \in e$ a *sign* $\sigma_{v,e}(G) \in \{+1, -1\}$. The edges in $E(G)$ are of three types: *ordinary edges*, having two distinct endnodes, *half-edges*, having only one endnode, and *loops*, having two identical endnodes. Let $E_0(G)$, $H(G)$ denote the sets of ordinary edges, half-edges, and loops, respectively. Parallel edges are allowed. For convenience, we define $\sigma_{v,e}(G) := 0$ if $v \notin e$. When it is clear from the context, we write E, σ, E_0, H and L instead of $E(G)$, $\sigma(G)$, $E_0(G)$, $H(G)$ and $L(G)$. The *incidence matrix* of G is the $|V| \times |E|$ matrix $A_G = (a_{v,e})$ such that $a_{v,e} = \sigma_{v,e}$ for all $e \in E \setminus L$, $a_{v,e} = 2\sigma_{v,e}$ for all $e \in L$. Given a bidirected graph G and a subset F of $E_0(G)$, we denote by $A(G, F)$ the matrix obtained from A_G by multiplying by 2 the columns relative to edges in F.

Given $U \subseteq V(G)$, we denote by $\delta_G(U)$ (or $\delta(U)$ when there is no ambiguity) the set containing the edges E that have exactly one endnode in U (in particular, half-edges and loops belong to $\delta_G(U)$ if their endnode is in U). The *subgraph of G induced* by U is the bidirected graph $G' = (U, E', \sigma')$ where E' is the set of edges of G whose endnodes are all in U and σ' is the restriction of σ to E'.

Paths and *cycles* in G are defined in the standard way in the undirected graph (V, E_0). In particular, cycles have always length at least 2. The *odd edges* of G are the edges $vw \in E_0$ such that $\sigma_{v,vw} = \sigma_{w,vw}$. A cycle or path Q in G is *even* if the number of odd edges in it is even, *odd* otherwise. Note that a cycle Q is even if and only if the sum of the signs on the edges in Q is divisible by 4 (i.e. $\sum_{vw \in E(Q)} (\sigma_{v,vw} + \sigma_{w,vw}) \equiv_4 0$).

A bidirected graph is said *bipartite* if it does not contain any odd cycle. (Note that, when $E = E_0$ and all edges are odd, this notion coincides with the usual definition of bipartite graph.) By a theorem of Heller and Tompkins [9], $G = (V, E, \sigma)$ is bipartite if and only V can be partitioned into sets V_1, V_2 such that, for every $e \in E_0$, e has one endnode in V_1 and the other in V_2 if e is odd, and e has both endnodes in either V_1 or V_2 if e is even.

We will show in Lemma 4 that a matrix $A(G, F)$ is totally half-modular if and only if (G, F) satisfies the following.

Cycles condition: *no odd cycle of G contains edges in F.* (1)

Next we restate the notion of minor of a matrix $A(G, F)$ in terms of operations on the pair (G, F).

Switching signs. Given a node $v \in V$, the signing σ' obtained from σ by setting $\sigma'_{v,e} = -\sigma_{v,e}$ for all $e \in E$ is said to be obtained by *switching signs on the node v*.

Given $e \in E$, the signing σ' obtained from σ by setting $\sigma'_{v,e} = -\sigma_{v,e}$ for all $v \in V$, is said to be obtained by *switching signs on the edge e*.

Deletion. Given a node $v \in V$, the pair (G', F') obtained from (G, F) by *deleting node v* is defined as follows. $V(G') = V \setminus \{v\}$, $E(G')$ contains all edges of $E(G)$ not incident to v and, for each edge $vw \in E_0(G)$, $E(G')$ contains a loop on w if $vw \in F$, or a half-edge on

w otherwise. We will identify such new loops and half-edges in G' with the corresponding edges incident to v in G. The signing on the edges of G' coincides with σ on $G \setminus v$, while $F' = F \cap E_0(G')$. (Note that our definition of node deletion is non-standard, since we do not remove all the edges incident to v, but we replace them with loops or half-edges.)
Given a subset of nodes $U \subseteq V$, the pair (G', F') is obtained from (G, F) by *deleting the nodes in* U if (G', F') is obtained from (G, F) by deleting one by one the nodes in U. Note that G' may be different from the subgraph of G induced by $V \setminus U$.
Given an edge $e \in E$, (G', F') is obtained from (G, F) by *deleting edge* e if $G' = (V, E \setminus \{e\}, \sigma')$ and $F' = F \setminus \{e\}$, where σ' coincides with σ on $E \setminus \{e\}$.
Contraction. Let $e = vw \in E_0(G)$ and possibly after switching signs assume $\sigma_{v,e} \neq \sigma_{w,e}$. We say that (G', F') is obtained from (G, F) by *contracting edge* e if G' is the bidirected graph obtained by replacing the nodes v, w with one new node $r \notin V$, by deleting all the edges vw such that $\sigma_{v,vw} \neq \sigma_{w,vw}$, by replacing each edge vw such that $\sigma_{v,vw} = \sigma_{w,vw}$ by a loop in r with sign $\sigma_{v,vw}$, by replacing each edge uv, $u \neq w$, or uw, $u \neq v$, in $E_0(G)$ by an edge ur in $E(G')$, by replacing each half-edge (resp. loop) on v or w by a half-edge (resp. loop) in r, and by letting the signing in G' coincide with σ on $E(G')$. Let $F' = F \cap E_0(G')$. We will identify each edge of G' incident to r with the original edge of G.
Note that, if (G, F) satisfies the cycles condition (1), then contracting one by one the edges of an odd cycle C results in a new loop on the node obtained by the contraction of C.

Given a pair (G, F) satisfying the cycles condition (1), a pair (G', F') is a *minor* of (G, F) if it is obtained from the latter through some of the following operations:

(O1) Switching signs on a node or on an edge of G;

(O2) Deleting a node or an edge in (G, F);

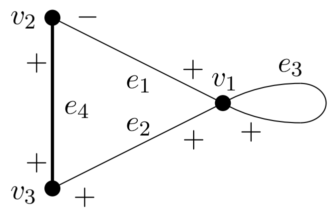
(O3) Contracting an edge $e = vw$ in $E_0(G) \setminus F$;

(O4) Contracting an edge $e = vw$ in F such that $\delta(v) \subseteq F \cup L(G)$.

We observe that the class of pairs (G, F) such that $A(G, F)$ is half-modular and has the EJ property is closed under taking minors. Clearly operations (O1),(O2) correspond to multiplying by -1 or removing rows and columns of $A(G, F)$. Assuming that (G, F) satisfies the cycles condition (1), operation (O3) corresponds to pivoting on the entry (v, e) in $A(G, F)$ and removing the row corresponding to v and the column corresponding to e, while operation (O4) corresponds to dividing by 2 the row of $A(G, F)$ corresponding to v (which is even because $\delta(v) \subseteq F \cup L$), pivoting on the entry (v, e), and then removing the row corresponding to v and the column corresponding to e.

Let $\mathscr{G}_4 = (G_4, F_4)$ be defined as follows: $V(G_4) = \{v_1, v_2, v_3\}$, $E(G_4) = \{e_1, e_2, e_3, e_4\}$, with $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_1v_1$, $e_4 = v_2v_3$, $F_4 = \{e_4\}$, and G_4 has $+1$ sign on all edges, except $\sigma_{v_2,e_1} = -1$. See Figure 1.

Note that $\mathscr{G}_4$ satisfies the cycles condition (1). One can verify that the matrix $A(\mathscr{G}_4)$ contains A_3 as a minor (pivot on the $+1$ entry (v_1, e_1) and delete the column corresponding to e_1). Thus, if a pair (G, F) satisfying the cycles condition contains $\mathscr{G}_4$ as a minor, then $A(G, F)$ does not have the EJ property.



$$A(\mathscr{G}_4) = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

Figure 1: Representation of $\mathscr{G}_4$ and corresponding matrix $A(\mathscr{G}_4)$. Boldfaced edges represent edges in F_4.

In the remainder of the paper, we denote by $\mathscr{C}$ the family of pairs (G, F), where G is a bidirected graph, $F \subseteq E_0(G)$ and (G, F) satisfies the cycles condition and does not contain $\mathscr{G}_4$ as a minor. We will prove the following.

Theorem 2. *Given a pair (G, F) that satisfies the cycles condition, $A(G, F)$ has the EJ property if and only if (G, F) does not contain $\mathscr{G}_4$ as a minor.*

We show that Theorem 2 implies Theorem 1. Indeed, let A be a totally half-modular matrix obtained by multiplying by 2 some of the columns of a $\{0, \pm 1\}$-matrix with at most two nonzero entries per column. If A contains A_3 as a minor, then A does not have the EJ property, because A_3 does not have the EJ property. Vice versa, assume A does not contain A_3 as a minor, and let (G, F) be a pair such that $A = A(G, F)$. Since $A(\mathscr{G}_4)$ contains A_3 as a minor, (G, F) does not contain $\mathscr{G}_4$ as a minor. Thus, by Theorem 2, A has the EJ property.

Theorem 2 extends a theorem of Edmonds and Johnson [5], mentioned in the introduction, stating that incidence matrices of bidirected graphs have the EJ property.

In Section 2 we show that we can reduce ourselves to studying systems of the form $Ax = c$, $x \geq 0$, and we describe the irredundant nontrivial Chvátal inequalities for such systems. Section 3 describes structural properties of the pairs $(G, F) \in \mathscr{C}$, while Section 4 introduces the concept of balanced bicoloring of the edges of (G, F) and discusses when elements in $\mathscr{C}$ admit such a bicoloring. The results of Sections 3 and 4 are needed in the proof of Theorem 2, given in Section 5.

2 Chvátal closure

We show that, to prove Theorem 2, we can reduce ourselves to studying systems in standard forms.

Lemma 3. *If, for every (G, F) in $\mathscr{C}$ and every $c \in \mathbb{Z}^{E(G)}$, the system*

$$\begin{aligned} A(G, F)\, x &= c \\ x &\geq 0. \end{aligned} \tag{2}$$

has Chvátal rank at most 1, then $A(G, F)$ has the EJ property for every (G, F) in $\mathscr{C}$.

Proof. Let us assume that (2) has Chvátal rank at most 1 for every (G, F) in $\mathscr{C}$ and every integral vector c. Given $(G, F) \in \mathscr{C}$, let b, c, l, u be integral vectors. Let $A := A(G, F)$. We need to show that the polyhedron $P := \{x : b \le Ax \le c,\, l \le x \le u\}$ has Chvàtal rank at most 1. Observe first that, if we define $\tilde{b} = b - Al, \tilde{c} = c - Al, \tilde{u} = u - l$, the polyhedron $\tilde{P} := \{x : b' \le Ax \le c',\, 0 \le x \le u'\}$ is the translate of P by $-l$, i.e. $\tilde{P} = P - l$. Since l is integral, it follows that the first Chvàtal closure of P is integral if and only if the first Chvàtal closure of $\tilde{P}$ is integral. Therefore we may assume that $l = 0$, thus $P = \{x : b \le Ax \le c,\, 0 \le x \le u\}$.

By a standard argument, it can be shown that P has Chvàtal rank 1 if and only if the polyhedron $\tilde{P} := \{(x, s) : Ax + s = c,\, 0 \le x \le u,\, 0 \le s \le c - b\}$ has Chvàtal rank 1. Observe that the constraint matrix (A, I) of the system $Ax + s = c$ is of the form $A(\tilde{G}, F)$, where $\tilde{G}$ is the bidirected graph obtained from G by introducing a half-edge with sign $+1$ on every node of G.

Thus, it suffices to show that, for every $(G, F) \in \mathscr{C}$, for every $c \in \mathbb{Z}^{V(G)}$, $u \in \mathbb{Z}^{E(G)}$, and for all $I \subseteq E(G)$, the polyhedron $\{x \in \mathbb{R}_+^{E(G)} : A(G, F)x = c,\, x_e \le u_e,\, e \in I\}$ has Chvátal rank at most 1.

The proof is by induction on $|I|$, where by assumption the statement holds for $|I| = 0$. Let $(G, F) \in \mathscr{C}$, $c \in \mathbb{Z}^{V(G)}$, $u \in \mathbb{Z}^{E(G)}$, and $I \subseteq E(G)$ such that $I \neq \emptyset$. Let $P := \{x \in \mathbb{R}_+^{E(G)} : Ax = c,\, x_e \le u_e,\, e \in I\}$ and let $\bar{x}$ be a point in the first closure P' of P. We need to show that $\bar{x}$ is a convex combination of integral points in P.

Let $\bar{e} \in I$. Assume first that $\bar{e} \in E_0(G)$, say $\bar{e} = vw$. Let $(\tilde{G}, \tilde{\sigma})$ be the bidirected graph defined as follows; let $V(\tilde{G}) = V(G) \cup \{z\}$, where z is a new node, let $E(\tilde{G}) = E(G) \setminus \{\bar{e}\} \cup \{e_v, e_w\}$, where $e_v = vz$, $e_w = wz$, and let $\tilde{\sigma}_{z,e_v} = \tilde{\sigma}_{z,e_w} = +1$, $\tilde{\sigma}_{v,e_v} = \sigma_{v,\bar{e}}$, $\tilde{\sigma}_{w,e_w} = -\sigma_{w,\bar{e}}$. If $\bar{e} \notin F$, let $\tilde{F} = F$, else $\tilde{F} = F \cup \{e_v, e_w\}$. It can be easily verified that $(\tilde{G}, \tilde{F}) \in \mathscr{C}$. Define $\tilde{x}_{e_v} := \bar{x}_{\bar{e}}$, $\tilde{x}_{e_w} := u_{\bar{e}} - \bar{x}_{\bar{e}}$, and $\tilde{x}_e := \bar{x}_e$ for all $e \in E \setminus \{\bar{e}\}$. Finally, let $\tilde{c} := A(\tilde{G}, \tilde{F})\tilde{x}$. Observe that $\tilde{c}_w = c_w - \sigma_{w,\bar{e}} u_{\bar{e}}$, $\tilde{c}_z = u_{\bar{e}}$ if $\bar{e} \notin F$, while $\tilde{c}_w = c_w - 2\sigma_{w,\bar{e}} u_{\bar{e}}$, $\tilde{c}_z = 2u_{\bar{e}}$ if $\bar{e} \in F$. Furthermore, $\tilde{c}_t = c_t$ for all $t \in V(G) \setminus \{w\}$.

We prove that $\tilde{x}$ is in the first closure $\tilde{P}'$ of the polyhedron $\tilde{P} := \{y : A(\tilde{G}, \tilde{F})y = \tilde{c},\, y \ge 0,\, y_e \le u_e,\, e \in I \setminus \{\bar{e}\}\}$. Consider a valid inequality $\alpha y \le \beta$ for $\tilde{P}$, where α is an integral vector. We need to show that $\tilde{x}$ satisfies the corresponding Chvàtal inequality $\alpha y \le \lfloor \beta \rfloor$. By construction, the inequality $\alpha_{e_v} x_{\bar{e}} + \alpha_{e_w}(u_{\bar{e}} - x_{\bar{e}}) + \sum_{e \in E(G) \setminus \{\bar{e}\}} \alpha_e x_e \le \beta$ is valid for P. Since $\bar{x} \in P'$, it follows that $\bar{x}$ satisfies the Chvàtal inequality $(\alpha_{e_v} - \alpha_{e_w})x_{\bar{e}} + \sum_{e \in E(G) \setminus \{\bar{e}\}} \alpha_e x_e \le \lfloor \beta - \alpha_{e_v} u_{\bar{e}} \rfloor$. Since α and u are integral, $\lfloor \beta - \alpha_{e_v} u_{\bar{e}} \rfloor = \lfloor \beta \rfloor - \alpha_{e_v} u_{\bar{e}}$, therefore $\tilde{x}$ satisfies $\alpha y \le \lfloor \beta \rfloor$. Thus $\tilde{x} \in \tilde{P}'$. By induction, $\tilde{P}'$ is an integral polyhedron, thus $\tilde{x}$ is a convex combination of integral points in $\tilde{P}$. It follows that $\bar{x}$ is a convex combination of integral points in P.

If $\bar{e} \in H(G)$ (resp. $\bar{e} \in L(G)$), where e is incident to a node v, define $(\tilde{G}, \tilde{\sigma})$ as follows. Let $V(\tilde{G}) = V(G) \cup \{z\}$, where z is a new node, let $E(\tilde{G}) = E(G) \setminus \{\bar{e}\} \cup \{\tilde{e}, \ell\}$, where $\tilde{e} = vz$ and ℓ is a half-edge on z (resp. a loop on z), let $\tilde{\sigma}_{z,\tilde{e}} = \tilde{\sigma}_{z,\ell} = +1$, $\tilde{\sigma}_{v,\tilde{e}} = \sigma_{v,\bar{e}}$. Let $\tilde{F} := F$ (resp. $\tilde{F} := F \cup \{\tilde{e}\}$). It can be easily verified that $(\tilde{G}, \tilde{F}) \in \mathscr{C}$. Define $\tilde{x}_{\tilde{e}} = \bar{x}_{\bar{e}}$, $\tilde{x}_\ell = u_{\bar{e}}$, and $\tilde{x}_e = \bar{x}_e$ for all $e \in E \setminus \{\bar{e}\}$. Finally, let $\tilde{c} := A(\tilde{G}, \tilde{F})\tilde{x}$. Observe that $\tilde{c}_z = u_{\bar{e}}$ (resp. $\tilde{c}_z = 2u_{\bar{e}}$), while $\tilde{c}_t = c_t$ for all $t \in V(G)$. One can show that $\tilde{x}$ is in the first closure $\tilde{P}'$ of the polyhedron $\tilde{P} := \{y : A(\tilde{G}, \tilde{F})y = \tilde{c},\, y \ge 0,\, y_e \le u_, e \in I \setminus \{\bar{e}\}\}$. The proof is similar to the previous case. As before, this implies that $\bar{x}$ is a convex combination of integral points in P. $\square$

Lemma 4. *Given a pair (G,F), the matrix $A(G,F)$ is totally half-modular if and only if (G,F) satisfies the cycles condition* (1)*.*

Proof. For the "if" direction, suppose G contains an odd cycle C such that $F' := E(C) \cap F \neq \emptyset$. Let $\Sigma = (\sigma_{v,e})_{v \in V(C),\, e \in E(C)}$. Since C is odd, all entries of Σ^{-1} are $\pm\frac{1}{2}$. The matrix $A(C, F \cap E(C))^{-1}$ is obtained from Σ^{-1} by multiplying by $\frac{1}{2}$ the rows corresponding to elements in F'. It follows that some of the entries of $A(C, F \cap E(C))^{-1}$ have value $\pm\frac{1}{4}$.

For "the only if" direction, assume (G,F) satisfies the cycles condition, and let $A := A(G,F)$. We may assume that G is connected, otherwise it suffices to prove the statement for each connected component of G. Since any submatrix A' of A is of the form $A' = A(G',F')$ for some pair (G',F') that satisfies the cycles condition, it suffices to show that, if A is square and nonsingular, then A^{-1} is half-integral. Suppose A is a $k \times k$ nonsingular matrix. Then $V(G) = \{v_1, \dots, v_k\}$ and $E(G) = \{e_1, \dots, e_k\}$. Since G is connected, we may assume that $e_1, \dots, e_{k-1}$ induce a spanning tree of G. Let $\Sigma := (\sigma_{v,e})_{v \in V, e \in E}$. The matrix A^{-1} is obtained from Σ by multiplying the rows corresponding to elements in $F \cup L(G)$ by $\frac{1}{2}$. If $e_k \in H(G) \cup L(G)$, then the matrix Σ is totally unimodular, thus Σ^{-1} is integral and A^{-1} is half-integral.

If $e_k \in E_0(G)$, then it is contained in the unique cycle C of G. If C is even, then Σ is singular, and so is A. Therefore C is odd. Up to permuting rows and columns, $\Sigma = \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$, where P is the incidence matrix of the cycle C. It can be readily verified that $\det(P) = \pm 2$ and R is totally unimodular, therefore P^{-1} is half-integral while R^{-1} is integral. Also, $\Sigma^{-1} = \begin{pmatrix} P^{-1} & -P^{-1}QR^{-1} \\ 0 & R^{-1} \end{pmatrix}$, therefore the first $|C|$ rows of Σ^{-1} are half-integral, while the other rows are integral. Since (G,F) satisfies the cycles condition, $E(C) \cap F = \emptyset$, therefore A^{-1} is obtained from Σ^{-1} by multiplying by $\frac{1}{2}$ some of the last $k - |C|$ rows. It follows that A^{-1} is half-integral. □

Let P be a polyhedron and let P' be its Chvátal closure. A Chvátal inequality $\alpha x \leq \beta$ for P is *nontrivial* if it is not valid for P, and is *irredundant* if it is not the sum of two inequalities that are valid for P' and that define faces of P' different from the one defined by $\alpha x \leq \beta$. Two inequalities $\alpha x \leq \beta$ and $\alpha' x \leq \beta'$ valid for P' are *equivalent* if they define the same face of P'. The proof of the next lemma is standard.

Lemma 5. *If A is a totally half-modular matrix and b, u are integral vectors, any irredundant nontrivial Chvátal inequality for $Ax = b$, $0 \leq x \leq u$ is equivalent to an inequality of the form $(\mu A + \gamma^0 - \gamma^u)x \geq \lceil \mu b - \gamma^u u \rceil$ such that μ, γ^0, γ^u have $0, \frac{1}{2}$ entries, $\mu A + \gamma^0 - \gamma^u$ is integral, and $\mu b - \gamma^u u$ is not integral.*

In the remaining of this paper, whenever Z is a set, $Y \subseteq Z$, and z is a vector in $\mathbb{R}^Z$, we denote by $z(Y) = \sum_{i \in Y} z_i$.

At some point in our proof of Theorem 2 it will be necessary to introduce upper bounds on the edges in $F \cup L(G)$. Hence in the following Lemma we describe the Chvátal inequalities for these more general systems.

Lemma 6. *Let (G, F) be a pair satisfying the cycles condition, $c \in \mathbb{Z}^V$, and $u \in \mathbb{Z}^E$. Let $\alpha x \geq \beta$ be an irredundant nontrivial Chvátal inequality for*

$$\begin{aligned} A(G,F)\,x &= c \\ x &\geq 0 \\ x_f \leq u_f,&\ f \in F \cup L. \end{aligned} \tag{3}$$

Then, for some $U \subseteq V(G)$ such that $c(U)$ is odd, $\alpha x \geq \beta$ is equivalent to

$$x(\delta(U) \setminus (F \cup L)) \geq 1. \tag{4}$$

Furthermore, for every $S \subset U$, $S \neq \emptyset$, there exists $vw \in E_0 \setminus F$ such that $v \in S$ and $w \in U \setminus S$.

Proof. Let $A = A(G, F)$. By Lemma 5, $\alpha x \geq \beta$ is equivalent to an inequality of the form $(\mu A + \gamma^0 - \gamma^u)x \geq \lceil \mu c - \gamma^u u \rceil$, where $\mu \in \{0, \frac{1}{2}\}^V$, $\gamma^0, \gamma^u \in \{0, \frac{1}{2}\}^E$, $\gamma^u_e = 0$ for all $e \in E \setminus (F \cup L)$, $\mu A + \gamma^0 - \gamma^u \in \mathbb{Z}^E$, and $\mu c - \gamma^u u \notin \mathbb{Z}$. Let $U := \{v \in V : \mu_v \neq 0\}$. Observe that all entries of μA are integer, except for the entries corresponding to edges in $\delta(U) \setminus (F \cup L)$, which have value $\pm\frac{1}{2}$. Hence $\gamma^0_e = \frac{1}{2}$ for every $e \in \delta(U) \setminus (F \cup L)$, $\gamma^0_e = 0$ for every other edge, and $\gamma^u_e = 0$ for every $e \in F \cup L$. Since $\mu c \notin \mathbb{Z}$, $c(U)$ is odd. Since $\lceil \mu c \rceil = \mu c + \frac{1}{2}$ and $\mu A x = \mu c$ for every x that satisfies (3), $\alpha x \geq \beta$ is equivalent to $\gamma^0 x \geq \frac{1}{2}$. Multiplying the latter by 2, one obtains (4).

Finally, suppose there exists $S \subset U$, $S \neq \emptyset$, such that all the edges between S and $U \setminus F$ are in F. Then $\delta(U) \setminus (F \cup L) = (\delta(S) \cup \delta(U \setminus S)) \setminus (F \cup L)$ and $(\delta(S) \cap \delta(U \setminus S)) \setminus (F \cup L) = \emptyset$. Also, since $c(U)$ is odd, by symmetry we may assume $c(S)$ is odd and $c(U \setminus S)$ is even. Hence $x(\delta(S) \setminus (F \cup L)) \geq 1$ is a Chvátal inequality, while $x(\delta(U \setminus S) \setminus (F \cup L)) \geq 0$ is implied by (3). The sum of the two latter inequalities is precisely (4), contradicting the assumption that $\alpha x \geq \beta$ is irredundant. □

We will refer to inequalities of the form (4) as *odd-cut inequalities (relative to U)*. When G is an undirected simple graph, $F = \emptyset$, and c is the vector of all 1s, the odd-cut inequalities reduce to the well known ones for the perfect matching polytope. The odd cut inequalities can be separated in polynomial time, since the separation problem reduces to a minimum weight odd-cut. Thus, using the reductions in the proof of Lemma 3, linear optimization over the first Chvátal closure of $b \leq A(G, F)x \leq c$, $l \leq x \leq u$, can be solved in polynomial time for all integral b, c, l, u whenever (G, F) has the cycles property. If $A(G, F)$ does not contain A_3 as a minor, by Theorem 1 linear optimization over the integer hull of $b \leq A(G, F)x \leq c$, $l \leq x \leq u$ is polynomial.

The following lemma will be useful in the proof of Theorem 2.

Lemma 7. *Let G be a bidirected graph, let $F \subseteq E_0$, and let $I \subseteq F \cup L$. If the system $A(G, F)x = c, x \geq 0$ has Chvátal rank at most 1 for every $c \in \mathbb{Z}^V$, then the system $A(G, F)x = c, x \geq 0, x_f \leq 1, \forall f \in I$ has Chvátal rank at most 1 for every $c \in \mathbb{Z}^V$.*

Proof. Let $A := A(G, F)$. Assume that the system $Ax = c, x \geq 0$ has Chvátal rank at most 1 for every integral vector c. Suppose by contradiction that there exists a fractional vertex $\bar{x}$ of the first closure of $\{x : Ax = c,\ x \geq 0,\ x_f \leq 1\ f \in I\}$. Let $\tilde{x}_e := \bar{x}_e$ for all $e \in E \setminus I$, $\tilde{x}_f := \bar{x}_f - \lfloor \bar{x}_f \rfloor$ for all $e \in I$. Let $\tilde{c} := A\tilde{x}$. Note that $\tilde{c}$ is integer. Since $I \subseteq F \cup L$, $\tilde{c}_v$

is congruent modulo 2 to c_v for all $v \in V$, therefore, for every $U \subseteq V$, $\tilde{c}(U)$ is odd if and only if $c(U)$ is odd. Thus, by Lemma 6, the odd-cut inequalities for $Ax = \tilde{c}, x \geq 0$ and for $Ax = c, x \geq 0, x_f \leq 1, f \in I$ are the same. Since $\tilde{x}_e = \bar{x}_e$ for every $e \in E \setminus (F \cup L)$, $\tilde{x}$ is a fractional vertex of the first closure of $\{x : Ax = \tilde{c}, x \geq 0\}$, a contradiction. $\square$

Given a set X of vectors, let $\text{span}\{X\}$ denote the linear space generated by the vectors in X. Given a set E and $R \subseteq E$, we denote by $\chi(R) \in \{0,1\}^E$ the characteristic vector of R. Given a graph $G = (V, E)$, a family $\mathscr{L}$ of subsets of V is called *laminar*, if and only if, for any $U, U' \in \mathscr{L}$ such that $U \cap U' \neq \emptyset$, it follows that $U \subseteq U'$ or $U' \subseteq U$.

The next lemma is used in the proof of Theorem 2. Its proof, which we do not report here, adopts standard uncrossing arguments (see for example [3, 8, 10, 11, 12]).

Lemma 8 (Uncrossing Lemma). *Let $G = (V, E)$ be a graph, let $c \in \mathbb{Z}^V$, $\bar{x} \in \mathbb{R}^E$ with $\bar{x} > 0$. Let $\mathscr{F} := \{U \subseteq V : c(U)$ odd and $\bar{x}(\delta(U)) = 1\}$. Then there exists a laminar subfamily $\mathscr{L}$ of $\mathscr{F}$ such that* $\text{span}\{\chi(\delta(U)) : U \in \mathscr{L}\} = \text{span}\{\chi(\delta(U)) : U \in \mathscr{F}\}$.

3 Structure of (G, F)

The purpose of this section is to derive structural properties of pairs $(G, F) \in \mathscr{C}$ that will be used in the proof of Theorem 2. We recall that a *cutset* of G is a set of nodes N such that $G \setminus N$ is not connected. A *cutnode* of G is a node v such that $\{v\}$ is a cutset. A *block* of G is maximal subgraph of G that does not have a cutnode. The following conditions will play an important role in our proof.

(C1): *No block of $G \setminus F$ contains two disjoint edges in F*;
(C2): *F is acyclic.*

Given a cycle C and a family $\{f_i, i \in I\}$ of chords of C, we say that $\{f_i, i \in I\}$ is a *family of non-crossing chords of C* if for every pair of chords $f_i, f_j, i, j \in I$, there exists a path in C between the two endnodes of f_i that contains both the endnodes of f_j.

Lemma 9. *Let $(G, F) \in \mathscr{C}$ that does not satisfy (C1). Then G is bipartite, $L(G) = \emptyset$, and F is a family of non-crossing chords of a cycle in $G \setminus F$.*

Proof. Let $f = vw$ and $f' = v'w'$ be two edges in F such that v, w, v', w' are distinct and in a same block B of $G \setminus F$. Clearly B is 2-connected. Let P_1 be a shortest path in $G \setminus F$ from f to f'. W.l.o.g. the extremes of P_1 are v and v'. Now let P_2 be a path in $G \setminus F$ from w' to w that does not pass through v. P_2 does not intersect P_1, as otherwise we can obtain $\mathscr{G}_4$ as a minor by deleting all edges in $E \setminus (E(P_1) \cup E(P_2) \cup \{vw, v'w'\})$ and by deleting node w', which contradicts $(G, F) \in \mathscr{C}$. Now let P_3 be a path in $G \setminus F$ from w to v that does not pass through v'. We observe that P_3 does not intersect P_1 and P_2 except on v and w. Indeed, if P_3 intersects P_1, then we obtain $\mathscr{G}_4$ as a minor by deleting all edges in $E \setminus (E(P_1) \cup E(P_3) \cup \{vw, v'w'\})$ and by deleting node w'; if P_3 intersects P_2, then we obtain $\mathscr{G}_4$ as a minor by deleting all edges in $E \setminus (E(P_2) \cup E(P_3) \cup \{vw, v'w'\})$ and by deleting node v'. Now let P_4 be a path in $G \setminus F$ from v' to w' that does not pass through v. Symmetrically, P_4 does not intersect P_1 or P_2 except on v' and w'. P_4 does not intersect P_3 either, otherwise we obtain $\mathscr{G}_4$ as a minor by deleting all edges in $E \setminus (E(P_1) \cup E(P_3) \cup \{vw, v'w'\})$, and by deleting node v. Hence $C := v, P_1, v', P_4, w', P_2, w, P_3, v$ is a cycle in $G \setminus F$, and f and f' are non-crossing chords of C.

We show that the edges in F are chords of C. Let $f'' = v''w'' \in F \setminus \{f, f'\}$. We show that f'' is a chord of C. If not, let P be a shortest path from an endnode of f'' to a node in C. W.l.o.g. the extreme of P in f'' is v'', and let u be the extreme of P in C. By symmetry, assume that $u \notin \{v, w\}$. The pair (G', F') obtained by deleting all edges in $E \setminus (E(C) \cup E(P) \cup \{vw, v''w''\})$ and by deleting w'' has $\mathscr{G}_4$ as a minor.

We show that the edges in F form a family of non-crossing chords of C. Suppose there exist $f, g \in F$ such that no path in C between the two endnodes of f contains both the endnodes of g. Thus there exists a subpath P of C between the endnodes of f that contains exactly one endnode v of g, where v is an internal node of P. Let w be the other endnode of g. The pair (G', F') obtained by deleting all edges in $E \setminus (E(P) \cup \{f, g\})$ and by deleting node w has $\mathscr{G}_4$ as a minor.

We show that $L = \emptyset$. If not, let $\ell \in L$, let P be a shortest path from the endnode of ℓ to C, and let u be the extreme of P in C. Let $f \in F$ such that $u \notin f$, and let P_f be the subpath of C between the endnodes of f such that $u \in V(P_f)$. The pair (G', F') obtained by deleting all edges in $E \setminus (E(P) \cup E(P_f) \cup \{f, \ell\})$ and by contracting all the edges in $E(P)$ has $\mathscr{G}_4$ as a minor.

We show that G is bipartite. If not, let $\bar{C}$ be an odd cycle. If there exist two different nodes $v, w \in V(\bar{C}) \cap V(C)$, it can be verified that there exists a path P in C from v to w containing edges in F. Hence the graph spanned by the edges in $E(\bar{C}) \cup E(P)$ contains an odd cycle with edges in F, contradicting $(G, F) \in \mathscr{C}$. Thus $|V(\bar{C}) \cap V(C)| \leq 1$. Let P be a shortest path from $\bar{C}$ to C, and let $f \in F$ so that no endnode of f is in P. The pair (G', F') obtained by deleting all edges in $E \setminus (E(\bar{C}) \cup E(C) \cup E(P) \cup \{f\})$ and by contracting all edges in $E(P) \cup E(\bar{C})$ has $\mathscr{G}_4$ as a minor. $\square$

A set $S \subseteq E(G)$ is a *star* if all edges in S are incident to one node v, called the *center of the star* S, and S does not contain parallel edges.

For $f = vw, f' = v'w'$ in F, we say that f' *is nested in* f if every path in $G \setminus F$ from v to w contains the nodes v', w'. We say that f and f' are *nested* if f' is nested in f or f is nested in f'.

Lemma 10. *Let $(G, F) \in \mathscr{C}$ that satisfies (C1) and (C2), and let B be a block of G such that $B \setminus F$ is connected and $E(B) \cap F \neq \emptyset$. One of the following holds.*

(i) B is bipartite and $E(B) \cap (F \cup L(G))$ is a star;

(ii) There exists an edge f in $E(B) \cap F$ such that all other edges in $E(B) \cap F$ are nested in f.

Proof. We may assume $|E(B) \cap F| \geq 2$ otherwise (ii) is trivially satisfied.

10.1. *Given two edges $f = vw, f' = v'w'$ in $E(B) \cap F$, one of the following holds:*

a) f and f' are adjacent, say $v = v'$, and for any two distinct nodes $s, t \in \{v, w, w'\}$ there exists a path in $B \setminus F$ between s and t that does not pass through $\{v, w, w'\} \setminus \{s, t\}$;

b) f and f' are nested;

c) one among v and w, say v, is a cutnode of $G \setminus F$ separating w from $\{v', w'\} \setminus \{v\}$.

Assume first that f and f' are adjacent, w.l.o.g. $v = v'$. By (C2), $w \neq w'$. If f, f' do not satisfy a), by symmetry every path in $B \setminus F$ from v to w passes through w', or every path in $B \setminus F$ from w to w' passes through v. In the first case f' is nested in f, thus case b) applies. In the second case v is a cutnode of $G \setminus F$ separating w from w', which means that case c) applies.

Thus we assume that all the nodes v, w, v', w' are pairwise different. Suppose that f, f' do not satisfy b). As $B \setminus F$ is connected, there is a path P from v to w in $B \setminus F$ that does not contain both v' and w'. P does not contain any node among v' and w', otherwise the pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(P) \cup \{f, f'\})$, and by deleting the endnode of f' that is not in $V(P)$ has $\mathscr{G}_4$ as a minor. Analogously, there exists a path P' from v' to w' in $B \setminus F$ that does not contain any node among v and w.

Let S be a shortest path in $B \setminus F$ with one extreme in $V(P)$ and the other extreme in $V(P')$. One extreme of S is an endnode of f, and the other extreme of S is an endnode of f'. If not, by symmetry, we may assume that one extreme of S is an internal node of P. The pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(P) \cup E(S) \cup E(P') \cup \{f, f'\})$, by contracting the edges in $E(S) \cup E(P')$, and by deleting one endnode of f' not in $V(S)$, has $\mathscr{G}_4$ as a minor. Thus w.l.o.g. the extremes of S are v, v'.

We show that f, f' satisfy c). If not, v is not a cutnode of $G \setminus F$ separating w from $\{v', w'\}$. Hence let S' be a shortest path in $B \setminus F$ with one extreme in $V(P)$ and the other in $V(P')$ that does not contain v. As above, one extreme of S' is an endnode of f, in this case w, and the other extreme of S' is an endnode of f'. We have that $V(S) \cap V(S') = \emptyset$, otherwise the pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(S) \cup E(S') \cup \{f, f'\})$ and by deleting w' has $\mathscr{G}_4$ as a minor. In particular the endnodes of S' are w, w'. Thus f and f' are chords of the cycle v, P, w, S', w', P', v in $G \setminus F$, thus they are contained in the same block of $G \setminus F$, contradicting (C1). $\diamond$

10.2. *If no two edges in $E(B) \cap F$ satisfy 10.1a), then statement (ii) holds.*

Let $f = vw$ be an edge in $E(B) \cap F$ that is not nested in any other edge of F. We show that all other edges in $E(B) \cap F$ are nested in f. Assume by contradiction that there exists an edge f' in $E(B) \cap F$ not nested in f. As f, f' do not satisfy 10.1a) or 10.1b), f, f' satisfy 10.1c). W.l.o.g. v is a cutnode of $G \setminus F$ separating w from $\{v', w'\} \setminus \{v\}$. Since B is 2-connected, there exists an edge $f'' = v''w''$ in $E(B) \cap F$ such that v'' is in the component of $G \setminus F \setminus \{v\}$ containing w, and w'' is in the component of $G \setminus F \setminus \{v\}$ containing $\{v', w'\} \setminus \{v\}$.

By assumption, f, f'' do not satisfy 10.1a). v'' is not a cutnode of $G \setminus F$ separating w'' from $\{v, w\} \setminus \{v''\}$, as there exists a path in $G \setminus F$ from v to w'' that does not contain v''. w'' is not a cutnode of $G \setminus F$ separating v'' from $\{v, w\}$, as there exists a path in $G \setminus F$ from v to v'' that does not contain w''. Thus f, f'' do not satisfy 10.1c). f'' is not nested in f, since no path in $G \setminus F$ from w to v contains w''. Hence by 10.1, f is nested in f'', contradicting the choice of f. $\diamond$

By 10.2, we may assume that there exist two edges $f = vw$ and $f' = vw'$ in $E(B) \cap F$ satisfying 10.1a). It follows that there exists a cycle, say H, in $B \setminus F$ passing through v, w

and w'; or there exist a node $z \neq v, w, w'$ and three paths in $B \setminus F$ from z to v, w and w', respectively, such that their union is a tree, say H.

We show that (i) holds. Suppose by contradiction that there exists an edge or loop $f'' \in E(B) \cap (F \cup L(G))$ such that $v \notin f''$. By (C2), we have that $f'' \neq ww'$.

Assume first that f'' has at most one endnode in H. Since B has no cutnode, there exists a path P from one endnode of f'' to H that does not contain v. If we choose f'' and P so that P is shortest possible, it follows that P does not contain any edge in F. Thus P is a path in $B \setminus F$, $V(P) \cup V(H)$ contains exactly one endnode of f'', and P does not contain both w, w', say $w' \notin V(P)$. One can now easily find a $\mathscr{G}_4$ minor in the graph spanned by the edges in $E(P) \cup E(H) \cup \{f, f''\}$, a contradiction.

Suppose then that f'' has two endnodes in H. In particular $f'' \in F$. If H is a cycle, then this contradicts (C1), since at least one among f and f' is disjoint from f'', and they are all contained in the same block of $G \setminus F$, since all their endnodes are in the cycle H. Thus H is a tree. A straightforward case analysis shows that the graph spanned by the edges $E(H) \cup \{f, f', f''\}$ contains $\mathscr{G}_4$ as a minor. Thus $E(B) \cap (F \cup L(G))$ is a star centered at v.

We only need to show that B is bipartite. Suppose by contradiction that there is an odd cycle C in B.

10.3. *Either v is a cutnode of $B \setminus F$ separating w from $V(C) \setminus \{v\}$, or w is a cutnode of $B \setminus F$ separating v from $V(C) \setminus \{w\}$.*

The cycle C does not contain both v and w, otherwise one can readily verify that the graph induced by $E(C) \cup \{f\}$ has an odd cycle containing f, contradicting that $(G, F) \in \mathscr{C}$. Suppose by contradiction that 10.3 does not hold. Then there exists a path P_w in $B \setminus F$ from w to a node in $V(C) \setminus \{v\}$ that does not contain v and a path P_v in $B \setminus F$ from v to a node in $V(C) \setminus \{w\}$ that does not contain w. If C contains exactly one among v and w, say v, then the graph induced by $E(C) \cup E(P_w) \cup \{f\}$ has an odd cycle containing f, a contradiction. Thus $V(C) \cap \{v, w\} = \emptyset$.

Let (G', F') be obtained from (G, F) by contracting all the edges of C. Let ℓ be the new loop obtained from contracting C. The subgraph of G' induced by the edges in $E(P_v) \cup E(P_w) \cup \{f, \ell\}$ contains $\mathscr{G}_4$ as a minor, a contradiction. $\diamond$

Suppose that v is a cutnode of $B \setminus F$. Since B does not have a cutnode, there must exist an edge in F not containing v, a contradiction. Thus, by 10.3, w is a cutnode of $B \setminus F$ separating v from $V(C) \setminus \{w\}$. Consider the path $P_1 \in B \setminus F$ between w and v that does not pass through w' and the path $P_2 \in B \setminus F$ between w and w' that does not pass through v, and let P be a shortest path between w and a node of C. Let (G', F') be obtained from (G, F) by contracting all the edges of C. Let ℓ be the new loop obtained from contracting C. The subgraph of G' induced by the edges in $E(P_1) \cup E(P_2) \cup \{f', \ell\}$ contains $\mathscr{G}_4$ as a minor, a contradiction. $\square$

In the proof of Theorem 2, we will be able to prove that the pair (G, F) satisfies the following.

(C3): *For every block B of G, each connected component of $B \setminus F$ has at least two nodes.*

Lemma 11. *Let $(G, F) \in \mathscr{C}$ that satisfies (C3) and let W be the set of edges in F with endnodes in distinct connected components of $G \setminus F$. Let B be a block of G such that $B \setminus F$ is not connected, let Q be a connected component of $B \setminus F$, and $\bar{Q}$ be the subgraph of G induced by $V(Q)$. Denote by $\bar{V}$ the set of nodes in Q incident to edges in $W \cap E(B)$. The following hold.*

(i) the nodes in $\bar{V} = \{v_1, \ldots, v_k\}$ can be ordered in such a way that v_i is a cutnode of $\bar{Q}$ separating v_{i-1} and v_{i+1}, $i = 2, \ldots, k-1$;

(ii) let $v_i w \in W \cap E(B)$ for some $i \in \{2, \ldots, k-1\}$. Then $\{v_i, w\}$ is a cutset of B separating v_{i-1} from v_{i+1};

(iii) for any $i, j \in \{1, \ldots, k\}$, $i \neq j$, there exists a path of length at least 2 in B from v_i to v_j that does not contain any node in $V(Q) \setminus \{v_i, v_j\}$.

Let $\Gamma(Q)$ be the subgraph of G induced by the nodes $v \in V(Q)$ for which there are paths in Q from v to v_1 and from v to v_k that do not pass through v_k and v_1, respectively. Then.

(iv) each edge in $L(G) \cup (W \setminus E(B))$ with one endnode in $\Gamma(Q)$ is incident to v_1 or v_k;

(v) $\Gamma(Q)$ is bipartite;

(vi) For any $f \in F \cap E(\Gamma(Q))$ and every $v \in F$, either $v \in \{v_1, v_k\}$, or v is a cutnode of $G \setminus F$ separating v_1 and v_k.

Proof. We first prove the following.

11.1. *Given pairwise distinct nodes $v, v', v'' \in \bar{V}$, one among v, v', v'' is a cutnode of Q separating the other two.*

Suppose by contradiction that there are three distinct nodes $v, v', v'' \in \bar{V}$ and paths $P_{v,v'}$ from v to v' in $Q \setminus v''$; $P_{v',v''}$ from v' to v'' in $Q \setminus v$; $P_{v,v''}$ from v to v'' in $Q \setminus v'$. As $v, v', v'' \in \bar{V}$, there exist edges $vw, v'w', v''w'' \in W \cap E(B)$.

We show that w, w', w'' are pairwise distinct, and that there exists a node $s \notin \{v, v', v''\}$ that is in at least two paths among $P_{v,v'}$, $P_{v',v''}$, $P_{v,v''}$. Suppose not.

Assume first that $w = w' = w''$. As (G, F) satisfies the condition (C3), there exists a node $\bar{w} \neq w$ in the connected component of $B \setminus F$ containing w. Since B is 2-connected, let P be a shortest path in $B \setminus w$ from $\bar{w}$ to $V(P_{v,v'}) \cup V(P_{v',v''}) \cup V(P_{v,v''})$, and let u be the extreme of P distinct from $\bar{w}$. W.l.o.g. $u \notin \{v, v'\}$, thus there exist paths $P_{u,v}$, from u to v, and $P_{u,v'}$, from u to v', so that $E(P_{u,v}), E(P_{u,v'}) \subseteq E(P_{v,v'}) \cup E(P_{v',v''}) \cup E(P_{v,v''})$, $E(P_{u,v}) \cap E(P_{u,v'}) = \emptyset$, and $|E(P_{u,v})|, |E(P_{u,v'}| \geq 1$. Since $\bar{w}$ and u are in different connected components of $B \setminus F$, the path P contains at least one edge in F. Let $\tilde{v}\tilde{w}$ be the edge in $F \cap E(P)$ so that node u and $\tilde{v}$ have minimum distance in P, and let $\tilde{P}$ be the subpath of P from u to $\tilde{v}$. The pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(P_{u,v}) \cup E(P_{u,v'}) \cup E(\tilde{P}) \cup \{vw, v'w, \tilde{v}\tilde{w}\})$, by deleting node $\tilde{w}$, and by contracting all edges of $\tilde{P}$, has $\mathscr{G}_4$ as a minor.
If exactly two of the nodes w, w' and w'' are identical, say $w = w''$, $w \neq w'$, then the pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(P_{v,v'}) \cup E(P_{v',v''}) \cup \{vw, v''w, v'w'\})$ and by deleting node w' has $\mathscr{G}_4$ as a minor.

It follows that w, w' and w'' are pairwise distinct. Assume that the paths $P_{v,v'}$, $P_{v',v''}$, $P_{v,v''}$ pairwise intersect only in their extremes. Then $E(P_{v,v'}) \cup E(P_{v',v''}) \cup E(P_{v,v''})$ induce a cycle C. Let P be a shortest path in $B \setminus v$ from w to $V(C) \cup \{w', w''\}$. By symmetry, we may assume that the nodes v' and w' are not in $V(P)$. Let C be the unique cycle in the graph spanned by the edges in $E(C) \cup E(P) \cup \{vw, v''w''\}$ that contains node v' and edge vw. The pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(C) \cup \{v'w'\})$ and by deleting node w' has $\mathscr{G}_4$ as a minor. Hence there exists a node $s \notin \{v, v', v''\}$ that is in at least two paths among $P_{v,v'}$, $P_{v',v''}$, $P_{v,v''}$.

It follows that the graph spanned by the edges in $E(P_{v,v'}) \cup E(P_{v',v''}) \cup E(P_{v,v''})$ contains three paths $P_{s,t}$, for $t = v, v', v''$, where these three paths pairwise intersect only in node s. For $t = v, v', v''$, we may assume that $V(P_{s,t}) \cap \bar{V} \subseteq \{s, t\}$, otherwise we may replace t with the node $\bar{t} \in V(P_{s,t}) \cap \bar{V}$, $\bar{t} \neq s$, that is closest to s in $P_{s,t}$. We consider two cases.

Case 1: $s \notin \bar{V}$. Since B is two connected, there exists a path from w' to $V(P_{s,v}) \cup V(P_{s,v'}) \cup V(P_{s,v''}) \cup \{w, w''\}$ in $B \setminus v'$. Let P be such a path such that $|E(P) \cap F|$ is smallest possible and, subject to that, so that $|E(P)|$ is smallest possible. Let u be the extreme of P different from w', and let u' be the node adjacent to u in P. W.l.o.g. $u \in V(P_{s,v'}) \cup V(P_{s,v}) \cup \{w\}$. We show that $u \in V(P_{s,v'})$ and $uu' \in F$. If not, let C be the unique cycle in the graph spanned by the edges in $E(P_{s,v}) \cup E(P_{s,v'}) \cup E(P_{s,v''}) \cup E(P) \cup \{vw, v'w'\}$, and let $\bar{P}$ be the shortest path from v'' to C. Since $u \in V(P_{s,v})$ or $uu' \notin F$, the extreme of $\bar{P}$ in C is incident in C to two edges in $E_0 \setminus F$. Thus the pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(C) \cup E(\bar{P}) \cup \{v''w''\})$, by contracting all the edges in $E(\bar{P})$, and by deleting node w'', has $\mathscr{G}_4$ as a minor.

Thus $u \in V(P_{s,v'})$ and $uu' \in F$. Since $u \in V(P_{s,v'}) \setminus \{v'\}$, $u \notin \bar{V}$, thus $uu' \notin W$, and so $u' \in V(Q)$. As Q is connected, let R be a shortest path in Q from u' to $V(P_{s,v}) \cup V(P_{s,v'}) \cup V(P_{s,v''})$. Since R contains no edge in F, the extreme of R distinct from u' must be v', otherwise $E(P) \setminus \{uu'\} \cup E(R)$ induces a path P' from w' to $V(P_{s,v}) \cup V(P_{s,v'}) \cup V(P_{s,v''})$ in $B \setminus v'$, and $E(P') \cap F = (E(P) \cap F) \setminus \{uu'\}$, a contradiction to the minimality of P. Let C be the unique cycle in the graph spanned by the edges in $E(P_{s,v'}) \cup E(R) \cup \{uu'\}$. Note that C contains the edge $uu' \in F$ and the node v', and that both edges incident to v' in C are in $E_0 \setminus F$. Thus the pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(C) \cup \{v'w'\})$ and by deleting node w' has $\mathscr{G}_4$ as a minor.

Case 2: $s \in \bar{V}$. Since B is 2-connected, let P be the shortest path in $B \setminus \{s\}$ with extremes in two distinct sets among $V(P_{s,v}) \cup \{w\}$, $V(P_{s,v'}) \cup \{w'\}$, $V(P_{s,v''}) \cup \{w''\}$. W.l.o.g. P has one extreme in $V(P_{s,v}) \cup \{w\}$, and the other in $V(P_{s,v'}) \cup \{w'\}$. By the minimality of P, $V(P) \cap (V(P_{s,v''}) \cup \{w''\}) = \emptyset$. Let C be the unique cycle in the graph spanned by the edges in $E(P_{s,v}) \cup E(P_{s,v'}) \cup E(P) \cup \{vw, v'w'\}$. If $E(C) \cap F \neq \emptyset$, then the pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(C) \cup E(P_{s,v''}) \cup \{v''w''\})$, by contracting all the edges in $E(P_{s,v''})$, and by deleting node w'', has $\mathscr{G}_4$ as a minor. It follows that P has both extremes in $V(P_{s,v}) \cup V(P_{s,v'})$, and that $E(P) \cap F \neq \emptyset$. In particular, P is a path in Q. If the extremes of P are v and v', then $E(P) \cup E(P_{sv}) \cup E(P_{sv'})$ induces a cycle in Q containing $s, v, v' \in \bar{V}$, which we already showed is not possible. Thus, by symmetry, we may assume that the extreme of P in $P_{sv'}$ is a node $s' \neq v'$. If we let $P_{s'v'}$ and $P_{s's}$ be the paths in $P_{sv'}$ from s' to s and v', respectively, then $(V(P_{s'v}) \cup V(P_{s'v'}) \cup V(P_{s's})) \cap \bar{V} = \{s, v, v'\}$, which is precisely Case 1. $\diamond$

Since Q is connected, by statement 11.1 there exists a path P in Q such that $\bar{V} \subseteq V(P)$. Furthermore, if we let $v_1, \ldots, v_k$ be the nodes in $\bar{V}$ in the order they appear in P, it follows that v_i is a cutnode of Q separating $\{v_1, \ldots, v_{i-1}\}$ and $\{v_{i+1}, \ldots, v_k\}$, $i = 2, \ldots, k-1$.

(i)(ii) Let $v_i w \in W \cap E(B)$ for some $i \in \{2, \ldots, k-1\}$. It suffices to show that $\{v_i, w\}$ is a cutset of B separating v_{i-1} and v_{i+1}, since in this case v_i must be a cutnode of $\bar{Q}$ separating v_{i-1} and v_{i+1}, because $w \notin V(Q)$. Suppose by contradiction that there exists a path R from v_{i-1} to v_{i+1} in $B \setminus \{v_i, w\}$. Note that $E(R)$ cannot be contained in $E(Q)$, therefore $E(R) \cap F \neq \emptyset$. Let e_1, e_2 be the two edges in $E(P)$ incident to v_i. Let C be the unique cycle in the graph spanned by the edges in $E(R) \cup E(P)$ containing v_i. Then C contains also e_1, e_2 and $E(C) \cap F \neq \emptyset$The pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(C) \cup \{v_i w\})$ and by deleting node w has $\mathscr{G}_4$ as a minor.

(iii) It is sufficient to prove that for $i = 1, \ldots, k-1$, for every edge $v_i w \in W \cap E(B)$ there exists a path in B from w to v_{i+1} that does not contain any node in $V(Q) \setminus \{v_{i+1}\}$. In fact, the last edge of such path is in $W \cap E(B)$, and the statement follows by induction. Let $\bar{P}$ be a shortest path from w to v_{i+1} in $B \setminus \{v_i\}$. We show that $\bar{P}$ contains no node in $V(Q) \setminus \{v_{i+1}\}$. Otherwise, let $v_t \in V(Q) \setminus \{v_{i+1}\}$ be the closest node in $\bar{P}$ to w. Let P_1 be the subpath of $\bar{P}$ from w to v_t, and P_2 be the subpath of $\bar{P}$ from v_t to v_{i+1}. Note that $t > i+1$ since, by (ii), $\{v_i, w\}$ is a cutset of B separating v_{i+1} from v_t, but $v_i, w \notin V(P_2)$. Given $v_{i+1}w' \in W \cap E(B)$, $\{v_{i+1}, w'\}$ is a cutset of B separating v_i from v_t, thus $w' \in V(P_1)$. The path from w to v_{i+1} spanned by $E(P_1) \cup \{v_{i+1}w'\}$ is shorter than $\bar{P}$, a contradiction.

(iv) Suppose $f = vw$ is an edge in $L(G) \cup (W \setminus E(B))$ such that v is in $\Gamma(Q)$ but $v \neq v_1, v_k$. By (iii) there exists a path $P_{1,k}$ in B from v_1 to v_k that does not contain any node in $V(Q) \setminus \{v_1, v_k\}$. Note that $E(P_{1,k}) \cap F \neq \emptyset$. By definition of $\Gamma(Q)$, there exist a path P_1 from v to v_1 and a path P_k from v to v_k in $G \setminus F$ that do not pass through v_k and v_1, respectively. The pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(P_{1,k}) \cup E(P_1) \cup E(P_k) \cup \{vw\})$ and by deleting node w if $v \neq w$ has $\mathscr{G}_4$ as a minor.

(v) Suppose that there exists an odd cycle C in $\Gamma(Q)$. If $v_1, v_k \notin V(C)$, then by contracting all the edges of C results in a loop ℓ that is not incident to v_1 or v_k, and we obtain $\mathscr{G}_4$ as a minor as in the proof of (iv). W.l.o.g. we assume $v_1 \in V(C)$. By definition of $\Gamma(Q)$ there exists a path (possibly of length 0) between C and v_k in $\Gamma(Q) \setminus F$ that does not pass through v_1. Let P_k be one such path of minimum length. By (iii) there exists a path $P_{1,k}$ in B from v_1 to v_k that does not contain any node in $V(Q) \setminus \{v_1, v_k\}$. Note that $V(P_K) \cap V(P_{1,k}) = \{v_k\}$. As C is odd, there exists a path P_C in C so that the graph spanned by the edges in $E(P_C) \cup E(P_k) \cup E(P_{1,k})$ is an odd cycle $\bar{C}$. Note however that $E(\bar{C}) \cap F \neq \emptyset$, contradicting $(G, F) \in \mathscr{C}$.

(vi) Let $f = vw \in F \cap E(\Gamma(Q))$. By contradiction assume that $w \neq v_1, v_k$ and w is not a cutnode of $G \setminus F$ separating v_1 and v_k. Suppose first that $v \neq v_1, v_k$. Given two paths in $G \setminus F$ from v, to v_1 and v_k, respectively, that do not contain w, we obtain $\mathscr{G}_4$ as a minor as in the proof of (iv). Hence we assume, w.l.o.g., that $v = v_1$. Let P_v (resp. P_w) be a path in $G \setminus F$ from v_k to v (resp. w) that does not pass through w (resp. v). Let $v_k w' \in W$. The pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(P_v) \cup E(P_w) \cup \{vw, v_k w_k\})$ and by deleting node w' has $\mathscr{G}_4$ as a minor. $\square$

Given two adjacent edges $uw, vw \in W$, $u \neq v$, such that $\sigma_{w,uw} \neq \sigma_{w,vw}$, we say that (G', F') is obtained from (G, F) by *shrinking* uw and vw if $V(G') = V(G)$, $E(G') = E(G) \setminus$

$\{uw, vw\} \cup \{uv\}$, $F' = F \setminus \{uw, vw\} \cup \{uv\}$, and the signing σ' on $E(G')$ is defined by $\sigma'_{u,uv} = \sigma_{u,uw}$, $\sigma'_{v,uv} = \sigma_{v,vw}$, $\sigma'_{z,e} = \sigma_{z,e}$ for every $e \in E(G') \setminus \{uv\}$, $z \in e$.

Observe that (G', F') satisfies the cycles condition. Indeed, given a cycle C in G' that contains uv, the corresponding cycles (one if $w \notin V(C)$, two if $w \in V(C)$) in G obtained from C by replacing uv with the two edges wu, wv, are even because they contain edges in F. Since $\sigma_{u,uw} + \sigma_{w,uw} + \sigma_{w,vw} + \sigma_{v,vw} \equiv_4 \sigma'_{u,uv} + \sigma'_{v,uv}$, C is even.
However, (G', F') may contain the minor $\mathscr{G}_4$. We say that two edges uw, vw in W are *shrinkable* if the graph obtained from (G, F) by shrinking uw and vw does not contain $\mathscr{G}_4$ as a minor.

Lemma 12. *Let $(G, F) \in \mathscr{C}$ that satisfies (C3). Let B be a block of G such that $B \setminus F$ is not connected. If some node w in B is incident to at least two edges in $W \cap E(B)$, then there exist two shrinkable edges in $W \cap E(B)$ incident to w.*

Proof. We say that two adjacent edges $wu, wv \in W \cap E(B)$, $u \neq v$, are *consecutive* if there is no edge $rw \in W \cap E(B)$ such that $\{r, w\}$ is a cutset of B separating u and v. If $wu \in W \cap E(B)$ and w is incident to other edges in $W \cap E(B)$, then there exists at least one edge $wv \in W \cap E(B)$ so that wu, wv are consecutive. We start by proving the following claim.

12.1. *Let uw, vw be consecutive edges in $W \cap E(B)$ and let (G', F') be obtained by shrinking uw, vw. Suppose that (G', F') contains $\mathscr{G}_4$ as a minor. Then there exists a cycle C in B such that, up to switching the roles of u and v, $v, w \in V(C)$, $u \notin V(C)$, v is incident to two edges in $E(C) \setminus F$, w is incident to at least one edge in $E(C) \cap F$ and $\{v, w\}$ is a cutset of B.*

Since (G', F') contains $\mathscr{G}_4$ as a minor, in G' there is a cycle C that contains at least one edge in F', a node $c \in V(C)$ incident to two edges in $E(C) \setminus F'$, and a path P from c to a node d such that $V(P) \cap V(C) = \{c\}$, $E(P) \cap F' = \emptyset$, and d is either incident to an edge $f = dt$ (possibly $t = d$) in $F' \cup L(G')$ such that $t \notin V(C) \cup V(P)$, or it belongs to an odd cycle H such that $(V(C) \cup V(P)) \cap V(H) = \{d\}$. Since (G, F) does not contain $\mathscr{G}_4$ as a minor and $uv \in F'$, then $uv \in E(C) \cup \{f\}$ and $w \in V(C) \cup V(P) \cup \{t\}$ (if d is incident to $f = dt \in F'$), or $uv \in E(C)$ and $w \in V(C) \cup V(P) \cup V(H)$ (if d belongs to the odd cycle H).

If $uv \in E(C)$, then $u, v \in V(B)$ implies $V(C) \subseteq V(B)$. Otherwise, if $uv = dt$, w.l.o.g. $v = d$, and $w \in V(C) \setminus \{c\}$, otherwise the graph spanned by the edges in $E(C) \cup E(P) \cup \{vw\}$ contains $\mathscr{G}_4$ as a minor. Thus in this case $v, w \in V(B)$ implies $V(C) \cup V(P) \subseteq V(B)$. Note that in both cases $V(C) \subseteq V(B)$.

Let Q be the connected component of $B \setminus F$ containing c, and let $\bar{Q}$ be the subgraph of G induced by $V(Q)$. Let $\bar{V}$ be the set of nodes of $\bar{Q}$ incident to some edge in $W \cap E(B)$. As c is incident to two edges in $E(C) \setminus F'$, let $\bar{C}$ be the shortest subpath of C containing c as an internal node and with endnodes, say c' and c'', $c' \neq c''$ that are incident in G with edges in $W \cap E(B)$. Note that such path $\bar{C}$ must exist, otherwise $uv \notin E(C)$, thus $V(C) \cup V(P) \subseteq V(B)$, and so $V(C) \cup V(P) \subseteq V(Q)$, in which case $f = uv$ and $w \in V(C) \cup V(P)$, implying that w and one among u, v belong to $V(Q)$, contradicting the fact that $uw, vw \in W$. Furthermore, $c', c'' \in \bar{V}$.

We show that d is incident to the edge $f = dt$ and that $f = uv$. If not, then $uv \in E(C)$. If $w \in V(C) \setminus \{c\}$, then the edges in $E(C) \setminus \{uv\} \cup \{uw, vw\}$ form two cycles in G. Let C' be the one passing through c. Note that $E(C') \cap F \neq \emptyset$, c is incident to two edges in $E(C') \setminus F$, and $V(C') \cap (V(P) \cup \{t\}) = \{c\}$ (or $V(C') \cap (V(P) \cup V(H)) = \{c\}$). Thus the graph spanned by the edges in $E(C') \cup E(P) \cup \{f\}$ (or $E(C') \cup E(P) \cup E(H)$) contains $\mathscr{G}_4$ as a minor, a contradiction. Thus $w \in V(P) \cup \{t\}$ (if d is incident to $f = dt \in F'$) or $w \in V(P) \cup V(H)$ (if d belongs to the odd cycle H). By Lemma 11(iii), there exists a path S in B from c' to c'' that contains no node in $V(Q) \setminus \{c', c''\}$. The subgraph of G spanned by the edges in $E(\bar{C}) \cup E(S) \cup E(P) \cup \{f\}$ (or by $E(\bar{C}) \cup E(S) \cup E(P) \cup E(H)$) contains $\mathscr{G}_4$ as a minor, unless d is incident to $f = dt \in F$ and $t \in V(S) \setminus \{c', c''\}$. In particular, since $d \in V(Q)$ and $t \notin V(Q)$, $dt \in W \cap E(B)$ and $c', c'', d \in \bar{V}$. By Lemma 11(i) one among c', c'', d is a cutnode of $\bar{Q}$ separating the other two. The only possibility is that $d = c$ and d is a cutnode of $\bar{Q}$ separating c' and c''. So P has length zero. Since $w \in V(P) \cup \{t\}$, then $w \in \{d, t\}$. By Lemma 11(ii), $\{d, t\}$ is a cutset of B separating c' and c'', thus $\{d, t\}$ separates u and v, but this contradicts the choice of wu, wv to be consecutive.

Thus d is incident to the edge $f = dt$ and $f = uv$. W.l.o.g., $v = d$, and we saw that $w \in V(C) \setminus \{c\}$, and $V(C) \cup V(P) \cup \{u\} \subseteq V(B)$. Moreover w is incident to at least one edge in $E(C) \cap F$, otherwise the graph spanned by $E(C) \cup \{uw\}$ contains $\mathscr{G}_4$ as a minor. By Lemma 11(i), one among c', c'', v is a cutnode of $\bar{Q}$ separating the two others. The only possibility is that $v = c$, and v is a cutnode of $\bar{Q}$ separating c' and c''. By Lemma 11(ii), this implies that $\{v, w\}$ is a cutset of B separating c' and c''. $\diamond$

12.2. *Let uw, vw be two consecutive edges in $W \cap E(B)$. If $\{v, w\}$ is a cutset of B separating two nodes r' and r'' such that $wr', wr'' \in E(B) \setminus F$, then uw, vw are shrinkable.*

Since B is 2-connected, there exist paths P' and P'' in $B \setminus w$ from v to r' and r'', respectively. Let Q be the connected component of $G \setminus F$ containing w and $\bar{V}$ be the set of nodes in Q incident to edges in $W \cap E(B)$. Since $vw \in W \cap E(B)$ and $r', r'' \in V(Q)$, P' and P'' contain some nodes c' and c'', respectively, in $\bar{V}$, such that the subpaths of P' and P'' from r' to c' and from r'' to c'', respectively, are in Q. By Lemma 11(ii), $\{w, u\}$ is a cutset of B separating c' and c'', and so $u \in V(P') \cup V(P'')$.

Let V_u (resp. V_v) be the set of nodes in the connected component of $B \setminus \{v, w\}$ (resp. $B \setminus \{u, w\}$) containing u (resp. v), and let $V_{u,v} := V_u \cap V_v$. We show that w is not adjacent to any node in $V_{u,v}$. Suppose by contradiction that there exists an edge ws with $s \in V_{u,v}$. Clearly $ws \notin W \cap E(B)$, otherwise by Lemma 11(ii), $\{w, s\}$ is a cutset of B separating u and v, contradicting the fact that the edges uw and vw are consecutive. Hence $s \in V(Q)$. Let $B_{u,v}$ be the subgraph of B induced by the nodes in $V_{u,v} \cup \{u, v\}$. Note that $B_{u,v}$ is connected. Let s' be the first node incident with edges in $W \cap E(B)$ in a path from s to u in $B_{u,v}$. As $s \in V(Q)$ and $u \notin V(Q)$, $s' \in \bar{V}$. Moreover, $c', c'' \notin V_{u,v}$, thus $s' \notin \{c', c''\}$ Then s', c' and c'' are three distinct nodes in $\bar{V}$ but none is a cutnode of Q separating the other two, contradicting Lemma 11(i).

Let (G', F') be the pair obtained from (G, F) by shrinking uw, vw. Suppose by contradiction that (G', F') contains the minor $\mathscr{G}_4$. By 12.1, there exists a cycle C in B such that, up to switching the roles of u and v, we have $v, w \in V(C)$, $u \notin V(C)$ and v is incident to two edges in $E(C) \setminus F$. Since $ws \notin E(G)$ for all $s \in V_{u,v}$ and $u \notin V(C)$, each node in $V(C) \setminus \{v, w\}$ is contained in the connected component of $B \setminus \{v, w\}$ not containing u. It follows that $V(C) \cap V(P') = \{v\}$. Since P' contains an edge in F, because $c' \in V(Q)$ and $u \notin V(Q)$, the graph spanned by the edges in $E(P') \cup E(C)$ contains $\mathscr{G}_4$ as a minor, contradicting $(G, F) \in \mathscr{C}$. $\diamond$

Let $w \in V(B)$ be a node incident to at least two edges in $W \cap E(B)$. Suppose by contradiction that no two edges in $W \cap E(B)$ incident to w are shrinkable. By 12.2, for all edges $e = vw \in W \cap E(B)$ such that $\{v, w\}$ is a cutset of B, there exists at least one connected component H of $B \setminus \{v, w\}$ such that $wr \notin E_0 \setminus F$ for all $r \in H$. Let H_e be the smallest such component, and let $\bar{e} = \bar{v}w$ be in $W \cap E(B)$ such that $\{\bar{v}, w\}$ is a cutset of B and $H_{\bar{e}}$ is smallest possible. Note that one such edge exists by 12.1. Denote by $\bar{G}$ the subgraph of G induced by $H_{\bar{e}} \cup \{\bar{v}, w\}$. By construction, no node of $H_{\bar{e}}$ is in the connected component of $G \setminus F$ containing w. Since B is 2-connected, w has at least a neighbor in $H_{\bar{e}}$ distinct from $\bar{v}$, say $u \in V(\bar{G})$. It follows that $uw \in W \cap E(B)$.

We show that uw and $\bar{v}w$ are the only edges in $E(\bar{G})$ adjacent to w. If not, then there exist $u' \in H_{\bar{e}}$ such that $u'w \in W$, $u' \neq \bar{v}, u$, and uw, $u'\bar{w}$ are consecutive. By 12.1 and by to symmetry, $\{u, w\}$ is a cutset of B, thus one of the connected components of $B \setminus \{u, w\}$ is contained in $H_{\bar{e}}$, contradicting the definition of $\bar{e}$.

Hence uw and $\bar{v}w$ are the only edges in $\bar{G}$ incident to w. In $G \setminus \{uw\}$ every path from u to w passes through $\bar{v}$, thus by 12.1 there exists a cycle C passing through $\bar{v}$ and w and not through u such that the two edges in C incident to $\bar{v}$ are not in F and w is incident to at least one edge in $E(C) \cap F$. Hence $V(C) \subseteq V(B) \setminus H_{\bar{e}}$. since $\bar{G} \setminus \{w, \bar{v}\}$ is connected by definition of $\bar{G}$, and since w is not a cutnode of B, the graph $\bar{G} \setminus \{w\}$ is connected, so there exists a path P in $\bar{G} \setminus \{w\}$ from u to $\bar{v}$. We observe that $E(P) \cap F = \emptyset$, otherwise the graph spanned by the edges in $E(C) \cup E(P)$ contains $\mathscr{G}_4$ as a minor, a contradiction.

Since $\bar{v}w \in W \cap E(B)$, each of the two disjoint paths in C from $\bar{v}$ to w contains an edge in $W \cap E(B)$. Let $\bar{C}$ be the shortest subpath of C containing $\bar{v}$ as an internal node and with endnodes that are incident in G to edges in $W \cap E(B)$. Let Q be the connected component of $G \setminus F$ containing $\bar{v}$ and let $\bar{V}$ be the set of nodes of $V(Q)$ incident to an edge in $W \cap E(B)$. It follows that $\bar{v}, u, c', c'' \in \bar{V}$. Note however that $E(\bar{C}) \cup E(P)$ contain three disjoint paths in Q, all of length at least one, from $\bar{v}$ to u, c', c'' respectively, contradicting Lemma 11(i). $\square$

4 Balanced bicolorings

The following concept will be crucial in the proof of Theorem 2. Given (G, F), where $F \subseteq E_0$, we say that a partition (R, B) of $E(G)$ in two (possibly empty) sets, referred to as *colors*, is a *balanced bicoloring* of (G, F), if for every $v \in V(G)$, we have

$$\sum_{vw \in R \setminus (F \cup L(G))} \frac{\sigma_{v,vw}}{2} + \sum_{vw \in R \cap (F \cup L(G))} \sigma_{v,vw} = \sum_{vw \in B \setminus (F \cup L(G))} \frac{\sigma_{v,vw}}{2} + \sum_{vw \in B \cap (F \cup L(G))} \sigma_{v,vw}. \quad (5)$$

Lemma 13. *Let G be a bidirected graph and $F \subseteq E_0(G)$. If (G, F) has a balanced bicoloring, then it satisfies the following* parity conditions.

a) $|\delta_G(v) \setminus (F \cup L(G))|$ is even for every $v \in V(G)$;

b) For every component Q of $G \setminus F$ such that $H(Q) = \emptyset$, $|\delta_G(V(Q))|$ is congruent modulo 2 to the number of odd edges in $E_0(Q) \setminus F$.

Proof. a) Given $v \in V(G)$, $\sum_{vw \in R \cap (F \cup L)} \sigma_{v,vw} - \sum_{vw \in B \cap (F \cup L)} \sigma_{v,vw} \in \mathbb{Z}$, thus by (5) also $\frac{1}{2}(\sum_{vw \in R \setminus (F \cup L)} \sigma_{v,vw} - \sum_{vw \in B \setminus (F \cup L)} \sigma_{v,vw}) \in \mathbb{Z}$. Hence $|\delta_G(v) \setminus (F \cup L)|$ is even.
b) Let Q be a component of $G \setminus F$ such that $H(Q) = \emptyset$. By (5),

$$\sum_{v \in V(Q)} \left(\sum_{vw \in R \setminus (F \cup L)} \frac{\sigma_{v,vw}}{2} + \sum_{vw \in R \cap (F \cup L)} \sigma_{v,vw} - \sum_{vw \in B \setminus (F \cup L)} \frac{\sigma_{v,vw}}{2} - \sum_{vw \in B \cap (F \cup L)} \sigma_{v,vw} \right) = 0. \tag{6}$$

The edges that contribute to the sum in (6) can abe partitioned into $\delta(V(Q))$, $E_0(Q) \cap F$, and $E_0(Q) \setminus F$. Since $H(Q) = \emptyset$, $\delta(V(Q)) \subseteq F \cup L$. Thus edges in $\delta(V(Q))$ and odd edges in $E_0(Q) \setminus F$ $E_0(Q) \setminus F$ contribute ± 1 to the sum, while edges in $E_0(Q) \cap F$ and even edges in $E_0(Q) \setminus F$ contribute $0, \pm 2$. As the sum in (6) equals zero, the total number of edges contributing ± 1 to the sum must be even, thus $|\delta_G(V(Q))|$ is congruent modulo 2 to the number of odd edges in $E_0(Q) \setminus F$. $\square$

The main goal of this section is to prove the following lemma.

Lemma 14. *Let $(G, F) \in \mathscr{C}$ satisfying (C3). If (G, F) satisfies the parity conditions a) and b) of Lemma 13, then (G, F) has a balanced bicoloring.*

The next lemma gives a useful way to construct balanced bicolorings.

A *trail* in a bidirected graph (G, F) is an alternating sequence T of nodes and edges $T = (e_0), v_1, e_1, \ldots, v_{k-1}, e_{k-1}, v_k, (e_k)$ – starting either with the node v_1 or with the half-edge e_0 on v_1, and ending either with the node v_k or with the half-edge e_k on v_k – such that, for $i = 1, \ldots, k-1$, $e_i = v_i v_{i+1}$, and where the edges are all distinct. The edges $e_1, \ldots, e_k$ can be either ordinary edges or loops. Trail T is *closed* if it its first and last element are nodes v_1, v_k, respectively, and $v_1 = v_k$. Note that nodes can be repeated and, if e_h is a loop in the trail, then $v_h = v_{h+1}$. A *sub-trail* of T is a subsequence $T' = v_i, e_i, v_{i+1}, \ldots, v_{j-1}, e_{j-1}, v_j$, where $1 \le i \le j \le k$.

We denote by $V(T)$ and $E(T)$ the sets of nodes and edges in T, and define $E_0(T)$, $L(T)$, and $H(T)$ accordingly. We remark that the set $E_0(T)$ can be partitioned into a path P between v_1 and v_k and cycles.

We say that the trail T is *balanced* if either both extremes of T are half-edges, or T is a closed trail such that $|L(T)|$ is congruent modulo 2 to the number of odd edges in $E(T)$.

Lemma 15. *Let (G, F) be a pair in $\mathscr{C}$ such that $G \setminus F$ is connected. Suppose that there exists a family $\mathscr{T}$ of balanced trails in $G \setminus F$ such that $\{E(T), T \in \mathscr{T}\}$ defines a partition of $E(G) \setminus F$, and such that, for every $f \in F$, there exists $T \in \mathscr{T}$ such that $V(T)$ contains both endnodes of f.*

Then there exists a balanced bicoloring (R, B) of (G, F) with the following property: for any $T \in \mathscr{T}$ and any subtrail $T' = v_i, e_i, \ldots, e_{j-1}, v_j$ of T such that e_i and e_{j-1} are loops, e_i and e_{j-1} have the same color if and only if $\sum_{h=i+1}^{j-1} (\sigma_{v_h, e_{h-1}} + \sigma_{v_h, e_h})$ is a multiple of four.

Proof. Let $T_1, \ldots, T_h$ be the elements in $\mathscr{T}$. Since for every $f \in F$ there exists $T \in \mathscr{T}$ such that $V(T)$ contains both endnodes of f, we may partition F into sets $F_1, \ldots, F_h$ so that every edge in F_i has both endnodes in $V(T_i)$, $i = 1, \ldots, h$. If there exists a balanced bicoloring (R_i, B_i) of the edges of $E(T_i) \cup F_i$ for $i = 1, \ldots, h$ as in the statement, then $R := \cup_{i=1}^{h} R_i$,

$B := \cup_{i=1}^{h} B_i$ define a balanced bicoloring of (G, F) as in the statement. In particular, we may assume that $\mathscr{T}$ consists of only one element $T = (e_0), v_1, e_1, \ldots, e_{k-1}, v_k, (e_k)$ (where the extremes of T may be the half-edges e_0, e_k on v_1 and v_k, or the nodes v_1 and v_k).

We show next that (G, F) has a balanced bicoloring (R, B) as in the statement, and with the additional property that given any subtrail $T' = v_i, e_i, \ldots, e_{j-1}, v_j$ of T such that $v_{i+1}, \ldots, v_{j-1}$ are not incident to edges in F, e_i and e_{j-1} have the same color if and only if $\sum_{h=i+1}^{j-1}(\sigma_{v_h,e_{h-1}} + \sigma_{v_h,e_h})$ is a multiple of four.

We proceed by induction on $|F|$. If $F = \emptyset$, define a bicoloring (R, B) of $E(G)$ as follows; two consecutive edges e_j and e_{j+1} in T have the same color if and only if $\sigma_{v_j,e_j} \neq \sigma_{v_j,e_{j+1}}$. Since T is balanced, it follows that (R, B) is a balanced bicoloring of $E(G)$. Furthermore, given any subtrail $T' = v_i, e_i, \ldots, e_{j-1}, v_j$ of T, a simple counting argument shows that e_i and e_{j-1} have the same color if and only if $\sum_{h=i+1}^{j-1}(\sigma_{v_h,e_{h-1}} + \sigma_{v_h,e_h})$ is a multiple of four. Thus (R, B) satisfies the inductive hypothesis.

We now assume $F \neq \emptyset$. For every $f \in F$, let $j(f)$ be the minimum index in $\{1, \ldots, k\}$ such that the subtrail of T from v_1 to $v_{j(f)}$ contains both endnodes of f. In particular $v_{j(f)}$ is an endnode of f. Let $i(f)$ be the largest index such that $i(f) < j(f)$ and $v_{i(f)}$ is the endnode of f distinct from $v_{j(f)}$. Note that the subtrail $T(f)$ of T from $i(f)$ to $j(f)$ does not contain any endnode of f except the two extremes. By the choice of $i(f)$ and $j(f)$ the first edge $e_{i(f)}$ and the last edge $e_{j(f)-1}$ in $T(f)$ are ordinary edges.

Let $f, g \in F$ with $i(f) \neq i(g)$, and assume by symmetry that $i(f) < i(g)$. We show that either $j(f) \leq i(g)$ or $j(g) \leq j(f)$. If not, then $i(f) < i(g) < j(f) < j(g)$. By the choice of $j(g)$, the node $v_{j(g)}$ does not appear in $T(f)$. Therefore, the pair (G', F') obtained by deleting all edges in $E(G) \setminus (E(T(f)) \cup \{f, g\})$, deleting node $v_{j(g)}$, and contracting all edges in $E(T(f)) \setminus \{e_{i(f)}, e_{j(f)-1}\}$, has $\mathscr{G}_4$ as a minor .

Choose $f \in F$ such that $j(f) - i(f)$ is smallest possible. By induction, there exists a balanced bicoloring (R', B') of $E(G) \setminus \{f\}$. Possibly by switching sign on the endnodes of f, we may assume that the sign of f on both endnodes is $+1$. Let $i := i(f)$, $j := j(f)$, $T' = T(f)$. By the previous argument, no node v_h, $i < h < j$, is an endnode of an edge in F. We next note that T' does not contain any loop and there is no odd cycle contained in $E(T')$. Indeed, if T' contains a loop, then such loop must be on a vertex in $V(T')$ distinct from v_i, v_j, while any cycle in $E(T')$ does not contain any of v_i, v_j. Therefore, we obtain $\mathscr{G}_4$ as a minor by deleting all edges in $E(G) \setminus (E(T') \cup \{f\})$ and contracting all edges in $E(T')$ except for e_i, e_{j-1} (note that, if $E(T')$ contains an odd cycle, after contracting this becomes a loop). The edges in $E(T')$ can therefore be partitioned into a path P from i to j and even cycles. Furthermore, since (G, F) satisfies the cycles condition, the cycle defined by P and f is even. This shows that $(\sigma_{v_i,e_i} + \sigma_{v_i,f}) + (\sigma_{v_j,e_{j-1}} + \sigma_{v_j,f}) + \sum_{h=i+1}^{j-1}(\sigma_{v_h,e_{h-1}} + \sigma_{v_h,e_h})$ is a multiple of four. We assume that $\sigma_{v_i,e_i} = \sigma_{v_j,e_{j-1}} = 1$, the other cases being similar. In this case, it follows that $\sum_{h=i+1}^{j-1}(\sigma_{v_h,e_{h-1}} + \sigma_{v_h,e_h})$ is a multiple of four, thus by inductive hypothesis e_i and e_{j-1} have the same color in (R', B'), say color R'. We claim that the bicoloring (R, B) defined by $R = (R' \triangle E(T')) \cup \{f\}$ and $B = B' \triangle E(T')$ is balanced. We need to show that (5) holds for every $v \in V(G)$. If $v \neq v_i, v_j$, then the condition holds because it was verified also by (R', B'). Thus we only need to verify (5) for $v = v_i$ and $v = v_j$. We consider the case $v = v_i$, the remaining case being identical. Observe that the only edge in $E(T')$ incident to v_i is e_i. Thus the only edge incident to v_i that has changed color is e_i, which had color R'

and now has color B. Therefore, the left-hand-side of (5) decreases by $1/2$ because of e_i, and it increase by 1 because of f which has color R, while the right-hand-side increases by $1/2$ because of e_i. This shows that (R, B) is balanced.
Finally, (R, B) satisfies the inductive hypothesis because of the inductive hypothesis on (R', B'), and because no loop changed color. □

Proof of Lemma 14. We prove the statement by double induction, first on $|V(G)|$, and then on $|E(G)|$. By property (C3), $|V(G)| \geq 2$. We can assume that G is connected, otherwise by induction we can bicolor each of the connected components.

14.1. *If (G, F) does not satisfy (C1), then it has a balanced bicoloring.*

By Lemma 9, G is bipartite, $L(G) = \emptyset$, and F is a family of non-crossing chords of a cycle C in $G \setminus F$. Note that the trail $T_0 := C$ is balanced because it contains no loops and because C is even since G is bipartite. Note that every edge in F has both endnodes in C. By parity property a) and because $L(G) = \emptyset$, every node of $V(G)$ is incident to an even number of edges in $E(G) \setminus (E(C) \cup F)$, thus $E(G) \setminus (E(C) \cup F)$ can be partitioned into cycles and trails whose extremes are both half-edges. Let $T_1, \dots, T_k$ be such a partition in cycle and trails. Since G is bipartite, all cycles are even, thus all trails $T_1, \dots, T_k$ are balanced. By Lemma 15 applied to the family $\mathcal{T} = \{T_0, \dots, T_k\}$, (G, F) has a balanced bicoloring. $\diamond$

14.2. *If G contains a cycle C such that $E(C) \subseteq F$, then (G, F) has a balanced bicoloring.*

Let $G' = G \setminus E(C)$ and $F' = F \setminus E(C)$. Clearly $(G', F') \in \mathcal{C}$ and it satisfies (C3) and the parity conditions, so by induction it has a balanced bicoloring (R', B'). Since no odd cycle in (G, F) has an edge in F, C is an even cycle, thus $E(C)$ can be partitioned into two sets (R'', B'') such that for every node $v \in V(C)$, the two edges e, e' incident to v in C have the same color if and only if $\sigma_{v,e} \neq \sigma_{v,e'}$. Thus $R := R' \cup R''$, $B := B' \cup B''$, define a balanced bicoloring of (G, F). $\diamond$

By the above two claims, we may assume that (G, F) satisfies (C1) and (C2).

14.3. *If G has a cutnode, then (G, F) has a balanced bicoloring.*

Let u be a cutnode of (G, F). Then there exist two connected subgraphs G_1, G_2 of G, both with at least two nodes, such that $V(G_1) \cap V(G_2) = \{u\}$, $V(G_1) \cup V(G_2) = V(G)$, $E(G_1) \cap E(G_2) = \emptyset$, $E(G_1) \cup E(G_2) = E(G)$. Let $F_1 := E(G_1) \cap F$ and $F_2 := E(G_2) \cap F$. Then (G_1, F_1) and (G_2, F_2) are in $\mathcal{C}$ and they both satisfy condition (C3). For $i = 1, 2$, let Q_i be the connected component of $G_i \setminus F_i$ containing u. Note that all components of $G_i \setminus F_i$ satisfy condition b) except, possibly, Q_i, and all nodes of G_i satisfy a) except, possibly, u.

If (G_1, F_1) and (G_2, F_2) satisfy conditions a) and b), then by induction there exist balanced bicolorings of (R_1, B_1), (R_2, B_2) of (G_1, F_1) and (G_2, F_2), thus $R := R_1 \cup R_2$, $B := B_1 \cup B_2$ defines a balanced bicoloring of (G, F).

If one of (G_1, F_1) and (G_2, F_2) does not satisfy condition a), then $|\delta_{G_1}(u) \setminus (F_1 \cup L(G_1)|$ and $|\delta_{G_2}(u) \setminus (F_2 \cup L(G_2)|$ are both odd. For $i = 1, 2$, let $(\bar{G}_i, F_i)$ be obtained from (G_i, F_i) by appending a half-edge h_i on node u, with sign $+1$. Observe that $(\bar{G}_i, F_i)$ satisfies condition a), and it trivially satisfies condition b). By induction, there exist a balanced bicoloring

(R_i, B_i) of $(\bar{G}_i, F_i)$, $i = 1, 2$. Assuming that $h_1 \in R_1$ and $h_2 \in B_2$, then $R = R_1 \setminus \{h_1\} \cup R_2$, $B = B_1 \cup B_2 \setminus \{h_2\}$ defines a balanced bicoloring of (G, F).

Lastly, assume that (G_1, F_1) and (G_2, F_2) satisfy condition a), but one of the two, say (G_1, F_1), does not satisfy condition *b*). In particular, $H(Q_1) = \emptyset$. Let $(\bar{G}_1, F_1)$ be obtained from (G_1, F_1) by appending two half-edges h, h' on node u, both with sign $+1$. Clearly $(\bar{G}_1, F_1)$ is in $\mathcal{C}$, and it satisfies (C3) and the parity conditions. Thus $(\bar{G}_1, F_1)$ has a balanced bicoloring (R, B). Note that h, h' have the same color, say R, otherwise $(R\setminus\{h, h'\}, B\setminus\{h, h'\})$ is a balanced bicoloring of (G_1, F_1), which by Lemma 13 contradicts the fact that (G_1, F_1) violates b). Let $(\bar{G}_2, F_2)$ be obtained from (G_2, F_2) by appending a loop ℓ on node u, with sign $+1$. Clearly $(\bar{G}_2, F_2)$ satisfies condition (C3) and the parity condition a). We will argue that $(\bar{G}_2, F_2)$ is in $\mathcal{C}$ and satisfies condition b); this will imply that $(\bar{G}_2, F_2)$ has a balanced bicoloring (R_2, B_2), say with $\ell \in B$, and thus $R = R_1 \setminus \{h, h'\} \cup R_2$, $B = B_1 \cup B_2 \setminus \{\ell\}$ defines a balanced bicoloring of (G, F).

To show that $(\bar{G}_2, F_2) \in \mathcal{C}$, it suffices to show that $(\bar{G}_2, F_2)$ is a minor of (G, F). First we prove that $F_1 \cup L(G_1) \neq \emptyset$ or (G_1, F_1) contains an odd cycle C. Indeed, if $F_1 \cup L(G_1) = \emptyset$, then $G_1 = Q_1$, and so G_1 has an odd number of odd edges. Since $E(G_1) = E_0(G_1)$ and all nodes in G_1 have even degree, $E(G_1)$ is the disjoint union of cycles, at leats one of which must be odd because G_1 has an odd number of odd edges.
Consider a shortest possible path P in $G_1 \setminus F_1$ from u to either an edge $f \in F \cup L(G_1)$ or to an odd cycle C. Then $(\bar{G}_2, F_2)$ can be obtained from (G, F) as a minor by contracting the edges in P, and possibly deleting the endnode of f not in P, if f is not a loop, or contracting all the edges in the odd cycle C.

We finally show that $(\bar{G}_2, F_2)$ satisfies property b). Let $\bar{Q}_2$ be the component of $\bar{G}_2 \setminus F$ induced by $V(Q_2)$. Note that $E(\bar{Q}_2) = E(Q_2) \cup \{\ell\}$. If $H(Q_2) \neq \emptyset$, then $\bar{Q}_2$ satisfies b). If $H(Q_2) = \emptyset$, then the connected component Q of G induced by $V(Q_1) \cup V(Q_2)$ has no half-edges, therefore $|\delta(V(Q)) \cap (F \cup L(G))|$ plus the number of odd edges in $E_0(Q) \setminus F$ is even. Since $|\delta_{G_1}(V(Q_1)) \cap (F_1 \cup L(G_1))|$ plus the number of odd edges in $E(Q_1) \setminus F_1$ is odd, it follows that $|\delta_{\bar{G}_2}(V(\bar{Q}_2)) \cap (F_2 \cup L(\bar{Q}_2))|$ plus the number of odd edges in $E(\bar{Q}_2) \setminus F_2$ is even. Thus $\bar{G}_2$, satisfies b). $\diamond$

By the above claim, we may assume that G does not have any cutnode. Thus G is a block. Since (G, F) satisfies a), $|H(G)|$ is even, say $|H(G)| = 2k$.

Case 1: $G \setminus F$ is connected. If $k = 0$, then, by property a), there exists a closed trail T in $G \setminus F$ such that $E(T) = E(G) \setminus F$. As (G, F) satisfies b), T satisfies the hypotheses of Lemma 15. Thus (G, F) has a balanced bicoloring. We assume $k \geq 1$. Furthermore, we may assume that $F \neq \emptyset$, otherwise by property a) the edges of G can be partitioned into k trails whose extremities are half-edges of G, and by Lemma 15 (G, F) has a balanced bicoloring. By Lemma 10, we need to consider two cases.

i) (G, F) satisfies Lemma 10(i). Let $h_1, \dots, h_{2(k-1)}$ be $2(k-1)$ half-edges of G, and let $v_1, \dots, v_{2(k-1)}$ be the corresponding endnodes. Since in this case G is bipartite, there exists a partition V_1, V_2 of $V(G)$ such that every odd edge has one endnode in V_1 and one in V_2 and every even edge has both endnodes in either V_1 or V_2. Consider the bidirected graph $\bar{G}$ obtained from G by introducing a dummy node u and replacing the half-edges $h_1, \dots, h_{2(k-1)}$ with the edges $uv_1, \dots, uv_{2(k-1)}$. We let $\sigma_{v_i, uv_i} = \sigma_{v_i, h_i}$, $\sigma_{u, uv_i} = \sigma_{v_i, h_i}$ if $v_i \in V_1$, $\sigma_{u, uv_i} = -\sigma_{v_i, h_i}$ if $v_i \in V_2$, $i = 1, \dots, 2(k-1)$. Observe that, by construction, $\bar{G}$ is bipartite. Note also that (G,F) does not contain $\mathscr{G}_4$ as a minor because F is a star centered at a node v, all loops of $\bar{G}$ are incident to v, and $\bar{G}$ does not contain any odd cycle. It follows that $(G,F) \in \mathcal{C}$. Since $\bar{G}$ has only two half-edges, there exits a trail T in $G \setminus F$ whose extremes are the two half-edges and such that $E(T) = E(G) \setminus F$. It follows from Lemma 15 that (G,F) has a balanced bicoloring.

ii) (G,F) satisfies Lemma 10(ii). Let $f = vw \in F$ such that any other edge in F is nested in f. Let P be a path in $G \setminus F$ between v and w. Then P contains all endnodes of edges in F. One can verify that the edges of $E(G) \setminus F$ can be partitioned in trails $T_1, \ldots, T_k$ such that all extremities are half-edges and such that $E(P) \subseteq E(T_1)$. It follows from Lemma 15 that (G,F) has a balanced bicoloring.

Case 2: $G \setminus F$ is not connected. Let W be the set of edges in F with endnodes in distinct connected components of $G \setminus F$.

If there is $w \in V(G)$ incident to at least two edges in W, then by Lemma 12 there exist two shrinkable edges $e', e'' \in W$ incident to w, say $e' = uw$, $e'' = vw$. Up to switching sign on wu, we may assume that $\sigma_{w,uw} \neq \sigma_{w,vw}$. Let (G', F', σ') be obtained from (G,F) by shrinking e', e'', and let $\bar{e} = uv$ be the new edge. It follows immediately that (G', F') satisfies (C3), a), and b), thus by induction (G', F') has a balanced bicoloring (R', B'). Assuming $\bar{e} \in R'$, it follows that $R := R' \cup \{e, e'\} \setminus \{\bar{e}\}$ and $B := B'$ define a balanced bicoloring of (G,F).

Thus we may assume that W is a matching in G. By switching signs on the endnodes of the edges in W, we may assume that, for all $vw \in W$, $\sigma_{v,vw} = \sigma_{w,vw} = +1$.

Let $Q_1, \ldots, Q_t$ be the connected components of $G \setminus F$. For $i = 1, \ldots, t$, let F_i be the set of edges of F with both endnodes in $V(Q_i)$, and let $\bar{V}_i = \{v_1^i, \ldots, v_{k_i}^i\}$ be the set of nodes in $V(Q_i)$ that are incident to some edge in W. Let $\bar{G}$ be the graph obtained from G by replacing each edge vw in W with two loops ℓ_v and ℓ_w on v and w, both with sign $+1$. For $vw \in W$, we refer to ℓ_v, ℓ_w, as the "new loops" of $\bar{G}$, and denote by $\bar{L}$ such set. For $i = 1, \ldots, t$, let W_i be the set of new loops with one endnode in $V(Q_i)$, that is, $W_i = \{\ell_v : v \in \bar{V}_i\}$. Note that $\bar{G}$ is not connected, and its connected components are the graphs $\bar{Q}_i := (V(Q_i), E(Q_i) \cup F_i \cup W_i)$, $i = 1, \ldots, t$. Also, for every $v \in \bar{V}_i$, there is exactly one new loop on v. Note that $(\bar{Q}_i, F_i)$ is in $\mathcal{C}$, since it is the pair obtained from (G,F) by deleting all nodes in $V(G) \setminus V(Q_i)$.

By Lemma 11(i), the nodes in $\bar{V}_i$ can be ordered so that v_j^i is a cutnode in $\bar{Q}_i$ separating v_{j-1}^i and v_{j+1}^i, $i = 1, \ldots, t$, $j = 2, \ldots, k_i - 1$. Let P^i be a path from v_1^i to $v_{k_i}^i$ in Q_i. Note that P^i passes through $v_2^i, \ldots, v_{k_i-1}^i$.

14.4. *For every $v \in V(Q_i)$, there exists a path in Q_i from v to v_1^i that does not pass through $v_{k_i}^i$ and a path in Q_i from v to $v_{k_i}^i$ that does not pass through v_1^i.*

Suppose not. Since Q_i is connected, we may consider the shortest path P from v to $\{v_1^i, v_{k_i}^i\}$. Up to symmetry, P does not contain $v_{k_i}^i$, and its extremes are v and v_1^i. Since G is 2-connected, there exists a shortest path P' in G from v to $V(P^i) \setminus \{v_1^i\}$ that does not pass through v_1^i. Note that, since no intermediate node of P' is an element of $\bar{V}_i$, then P' does not cross any edge of W, thus P' is entirely contained in $\bar{Q}_i$. Let u be the endnode of P' in $V(P^i) \setminus \{v_1^i\}$, and let P'' be the path contained in P^i from u to v_1^i. Let w be the node in $V(P) \cap V(P')$ that is closest to v_1^i in P, and let $\bar{P}$ be the path contained in P between v_1^i and w, and $\bar{P}'$ be the path contained in P' between u and w. Note that $w \neq v_1^i$, because P' does not pass through v, and that $\bar{P}'$ contains an edge in F, otherwise there exists a path from v to $v^i_{k_i}$ in Q_i that does not pass through v^i_1. Thus $v^i_1, \bar{P}, w, \bar{P}', u, P'', v^i_1$ form a cycle C such that $E(C) \cap F \neq \emptyset$, and the two edges of C incident to v^i_1 are not elements of F. It follows that the graph induced by $E(C) \cup \ell_{v^i_1}$ has a $\mathscr{G}_4$ minor, contradicting the fact that $\bar{Q}_i \in \mathcal{C}$. $\diamond$

By 14.4 and by Lemma 11(iv)(v)(vi), it follows that $\bar{Q}_i$ is bipartite, every loop of $\bar{Q}_i$ that is not an element of W_i is incident to either v^i_1 or $v^i_{k_i}$, and every edge in F_i has both endnodes in P^i.

We observe that, if $\bar{Q}_i$ has no half-edges, then $|L(\bar{Q}_i)|$ must be even. Indeed, by condition b), if there are no half-edges in $E(\bar{Q}_i)$ then $|L(\bar{Q}_i)|$ is congruent modulo 2 to the number of odd edges in $E_0(\bar{Q}_i) \setminus F$. By condition a) every node of $V(Q_i)$ is incident to an even number of edges in $E_0(\bar{Q}_i) \setminus F$, therefore $E_0(\bar{Q}_i) \setminus F$ can be partitioned into cycles. Since $\bar{Q}_i$ is bipartite, each of these cycles is even, therefore the number of odd edges in $E_0(\bar{Q}_i) \setminus F$ is even.

For $j = 1, \dots, k_i - 1$, denote by P^i_j the path contained in P^i from v^i_j to v^i_{j+1}. Note that, since W is a matching, $v^i_j \neq v^i_{j+1}$, thus P^i_j has length at least one.

14.5. *For $i = 1, \dots, t$, there exists a balanced bicoloring (R_i, B_i) of $(\bar{Q}_i, F_i)$ such that, for $j = 1, \dots, k_i - 1$, the loops $\ell_{v^i_j}$ and $\ell_{v^i_{j+1}}$ have the same color if and only if path P^i_j has an odd number of odd edges.*

Note that $\bar{T}^i := v^i_1, \ell_{v^i_1}, v^i_1, P^i_1, v^i_2, \ell_{v^i_2}, v^i_2, P^i_2, v^i_3, \dots, v^i_{k_i-1}, P^i_{k_i-1}, v^i_{k_i}, \ell_{v^i_{k_i}}, v^i_{k_i}$ is a trail that contains all loops in W_i. Since all the elements of $L(Q_i) \setminus W_i$ are incident to v^i_1 or $v^i_{k_i}$, there exists some trail T^i in $\bar{Q}_i \setminus F$ such that $\bar{T}^i$ is a subtrail of T^i, every loop of $\bar{Q}_i$ is in T^i, and T^i is either closed or its extremes are half-edges. Furthermore, we can choose T^i so that, if $\bar{Q}_i$ has some half-edge, then both extremes of T^i are half-edges. We argue that T^i is a balanced trail. Indeed, if T^i is closed, then $E(T^i)$ is the disjoint union of loops and cycles, and each of these cycles is even because $\bar{Q}_i$ is bipartite. It follows that, if T^i is closed, then $E(T^i)$ has an even number of odd edges. Since $|L(\bar{Q}_i)|$ is even and $L(\bar{Q}_i) \subseteq E(T^i)$, it follows that T^i is balanced.

Observe that, since (G, F) satisfies condition a), every node in $\bar{Q}_i$ is incident to an even number of edges in $E(\bar{Q}_i) \setminus (E(T^i) \cup F)$, therefore $E(\bar{Q}_i) \setminus (E(T^i) \cup F_i)$ can be partitioned into trails whose extremes are half-edges and cycles, and all cycles must be even because $\bar{Q}_i$ is bipartite. It follows that there exists a family $\mathcal{F}_i$ of trails such that $T_i \in \mathcal{F}_i$ and such that $\{E(T) : T \in \mathcal{F}\}$ is a partition of $E(\bar{Q}_i) \setminus F_i$. Since all edges in F_i have both endnodes in $V(T^i)$, it follows from Lemma 15 that $(\bar{Q}_i, F_i)$ has a balanced bicoloring (R_i, B_i). Furthermore, since $\bar{T}^i$ is a subtrail of T^i, Lemma 15 ensures that we can choose (R_i, B_i) so that, for $j = 1, \dots, k_i - 1$, the loops $\ell_{v^i_j}$ and $\ell_{v^i_{j+1}}$ have the same color if and only if $\sigma_{v^i_j, \ell_{v^i_j}} + \sigma_{v^i_{j+1}, \ell_{v^i_{j+1}}} + \sum_{vw \in E(P^i_j)} (\sigma_{v,vw} + \sigma_{w,vw})$ is congruent to four. Since $\sigma_{v^i_j, \ell_{v^i_j}} + \sigma_{v^i_{j+1}, \ell_{v^i_{j+1}}} = 2$, because all new loops of $\bar{G}$ have sign $+1$, this is equivalent to the statement 14.5. $\diamond$

We finally show how to recombine the bicolorings (R_i, B_i) into a balanced bicoloring of (G, F). Note that $\bar{R} := R_1 \cup \ldots \cup R_t$, $\bar{B} = B_1 \cup \ldots \cup B_t$ define a balanced bicoloring of $(\bar{G}, F \setminus W)$.

Since G is connected and $G \backslash W$ has t components, there exist $\tilde{W} \subseteq W$ such that $|\tilde{W}| = t-1$ and $(G \setminus W) \cup \tilde{W}$ is connected. We may assume that, for every edge $vw \in \tilde{W}$, both new loops ℓ_v and ℓ_w in $\bar{G}$ have the same color in $(\bar{R}, \bar{B})$. We will show that, for every $vw \in W \setminus \tilde{W}$, both new loops ℓ_v and ℓ_w in $\bar{G}$ have the same color in $(\bar{R}, \bar{B})$. This concludes the proof because the bicoloring (R, B) defined by $(\bar{R}, \bar{B})$ by assigning to every $vw \in W$ the common color of ℓ_v and ℓ_w is balanced.

Let W^+ be the set of edges $vw \in W$ such that ℓ_v and ℓ_w have the same color in $(\bar{R}, \bar{B})$, and let $W^- = W \setminus W^+$. We need to show $W^- = \emptyset$. Suppose not. Note that $G \setminus W^-$ is connected, because $\tilde{W} \subseteq W^+$ and by the choice of $\tilde{W}$. Thus, for every $vw \in W^-$, there exists a path $P(v, w)$ between v and w in $E(P^1) \cup \ldots \cup E(P^t) \cup W^+$. Among all elements of W^-, choose $vw \in W^-$ and $P(v, w)$ so that $P(v, w)$ is shortest possible, and let $P := P(v, w)$. Let C be the cycle in (G, F) defined by P and by vw. Up to changing the indices, we may assume that $v \in V(Q_1)$, $w \in V(Q_h)$, and $P = v, \bar{P}^1, w_1, w_1v_2, \bar{P}^2, \ldots, w_{h-1}, w_{h-1}v_h, \bar{P}^h, w$, where $w_iv_{i+1} \in \tilde{W}$, $i = 1, \ldots, h-1$, and $\bar{P}^i$ is the path between v_i and w_i in P^i for $i = 1, \ldots, h$ (where $v_1 = v$, $w_h = w$). We notice that, for $i = 1, \ldots, h-1$, $V(P) \cap \bar{V}_i = \{v_i, w_i\}$. Indeed, suppose for some i there exists a node $u \in \bar{V}_i$ distinct from v_i and w_i. In particular, u is an intermediate node in $\bar{P}^i$, thus both edges incident to u in P are in $E(G) \setminus F$. Since $u \in \bar{V}_i$, there exists $u' \in V(G)$ such that $uu' \in W$. If $u' \notin V(P)$, then $\mathscr{G}_4$ is a minor of the graph defined by the cycle C and the loop obtained by deleting u'. If $u' \in V(P)$, then either $uu' \in W^-$, in which case the unique path in P from u to u' is shorter than P, contradicting our choice of $vw \in W^-$, or $uu' \in W^+$, in which case the path in $E(P) \cup \{uu'\}$ between v and w is shorter than P, contradicting the choice of P. By 14.5, for $i = 2, \ldots, h-1$, edges $w_{i-1}v_i$ and w_iv_{i+1} have the same color if and only if $\bar{P}_i$ has an odd number of odd edges, ℓ_v and w_1v_2 have the same color if and only if $\bar{P}^1$ has an odd number of odd edges, and ℓ_w and $w_{h-1}v_h$ have the same color if and only if $\bar{P}^h$ has an odd number of odd edges. Since ℓ_v and ℓ_w have distinct colors, and since we are assuming that all edges in W are odd, a simple parity argument shows that P has an even number of even edges. Since vw is an odd edge, it follows that the cycle C is odd, a contradiction since no odd cycle of G contains edges in F. $\square$

5 Proof of Theorem 2

For the "if" direction of the statement, assume (G, F) contains $\mathscr{G}_4$ as a minor. As observed in the introduction, A_3 is a minor of $A(\mathscr{G}_4)$, thus A_3 is a minor of $A(G, F)$ as well. Since A_3 does not have the EJ property, and since such property is closed under taking minors, it follows that $A(G, F)$ does not have the EJ property.

The remainder of the section is devoted to proving the "only if" direction. For any bidirected graph G, $F \subseteq E(G)$, and any $c \in \mathbb{Z}^{|V(G)|}$, let $P(G, F, c) := \{x \in \mathbb{R}_+^{E(G)} : A(G, F)x = c\}$, and let $P'(G, F, c)$ be its first closure. We will prove that, for every $(G, F) \in \mathscr{C}$ and every $c \in \mathbb{Z}^{|V(G)|}$, $P'(G, F, c)$ is an integral polyhedron. By Lemma 3, this will imply Theorem 2.

By contradiction, suppose that there exists a pair (G, F) in $\mathscr{C}$ and an integral vector c such that $P'(G, F, c)$ has a fractional vertex $\bar{x}$. Among all such counterexamples, choose $(G, F), c, \bar{x}$ such that the quadruple $(|V(G)|, |E_0(G)|, |E(G)|, \lfloor \sum_{e \in E(G)} \bar{x}_e \rfloor)$ is lexicographically minimal.

It is straightforward to verify that G must have at least two nodes. Throughout the proof, let $A := A(G, F)$, $E := E(G)$, $E_0 := E_0(G)$, $L := L(G)$, $H := H(G)$, $\delta(\cdot) := \delta_G(\cdot)$.

Most of the proof is devoted to showing that $\bar{x}_e = \frac{1}{2}$ for all $e \in E$. Afterwards, we will argue that (G, F) has a balanced bicoloring (R, B). This will conclude the proof of Theorem 2, since the points y and z defined by $y := \bar{x} + \frac{1}{2}\chi(R) - \frac{1}{2}\chi(B)$, $z := \bar{x} - \frac{1}{2}\chi(R) + \frac{1}{2}\chi(B)$, are integral points in $P(G, F, c)$ such that $\bar{x} = \frac{1}{2}(y + z)$, contradicting the fact that $\bar{x}$ is a vertex of $P'(G, F, c)$.

Given a node v, if G' is obtained from G by switching sign on node v and $c' \in \mathbb{R}^{V(G)}$ is defined by $c'_u = c_u$, $u \in V(G) \setminus \{v\}$, $c'_v = -c_v$, then $\bar{x}$ is a vertex of $P'(G', F, c')$ because, for every $U \subseteq V(G)$, $c(U)$ is odd if and only if $c'(U)$ is odd. So, if (G, F), c, $\bar{x}$ is a minimal counterexample, then also (G', F), c', $\bar{x}$ is a minimal counterexample. Hence, throughout the proof we will perform such switching whenever convenient.

Note that $F \neq \emptyset$, since, by the theorem of Edmonds and Johnson [5], $P'(G, \emptyset, c)$ is integral. Furthermore, G is connected; otherwise, let G' be a component of G such that $\bar{x}_e \notin \mathbb{Z}$ for some $e \in E(G')$, let $F' = F \cap E(G')$, and let $\bar{x}'$ and c' be the restrictions of $\bar{x}$ and c, respectively, to $E(G')$ and $V(G')$. Note that (G', F') is in $\mathscr{C}$ and that $|V(G')| < |V(G)|$, hence $P'(G', F', c')$ is integral. However, $\bar{x}'$ is a vertex of $P'(G', F', c')$, a contradiction.

Claim 1. $\bar{x}_e > 0$ *for every* $e \in E$.

If $\bar{x}_e = 0$ for some e in $E(G)$, let (G', F') be obtained from (G, F) by deleting e, and $\bar{x}' \in \mathbb{R}^{E(G')}$ be obtained from $\bar{x}$ by removing the component corresponding to e. The point $\bar{x}'$ is a fractional vertex of $P'(G', F', c)$, which contradicts our choice of (G, F) since $(G', F') \in \mathcal{C}$, $|V(G')| = |V|$, $|E_0(G')| \leq |E_0|$, and $|E(G')| < |E(G)|$. $\diamond$

Note that A has full rank, otherwise deleting a redundant constraint from $Ax = c$, which corresponds to deleting a node from (G, F), gives a smaller counterexample. Since $\bar{x}$ is a vertex of $P'(G, F, c)$, it must satisfy at equality $|E|$ linearly independent inequalities valid for $P'(G, F, c)$. By Claim 1 and Lemma 8, there exists a laminar family $\mathscr{L}$ of sets in $\{U \subseteq V :$ $c(u)$ odd $\}$ such that $|\mathscr{L}| = |E| - |V|$ and $\bar{x}$ is the unique solution of the system defined by the $|E|$ linearly independent equations

$$\begin{array}{rcll} Ax & = & c & \\ x(\delta(U) \setminus (F \cup L)) & = & 1 & \quad U \in \mathscr{L}. \end{array} \tag{7}$$

By Lemma 6, we can also assume the following.

$$\text{For every } S \subset U,\ S \neq \emptyset,\ \exists\, vw \in E_0 \setminus F \text{ such that } v \in S \text{ and } w \in U \setminus S. \tag{8}$$

Claim 2. *For every* $e \in E$, $0 < \bar{x}_e < 1$. *Furthermore, for every* $e \in E \setminus (F \cup L)$, *there exists* $U \in \mathscr{L}$ *such that* $e \in \delta(U)$.

By Claim 1, $\bar{x}_e > 0$ for every e in E. First we show that $\bar{x}_f < 1$ for any f in $F \cup L$. Let $f \in F \cup L$, and suppose $\bar{x}_f \geq 1$. Possibly by switching the signs on the endnodes of f, we can assume that f has a sign $+1$ on its endnodes. Let $\bar{x}'$ be obtained from $\bar{x}$ by decreasing by 1 the component corresponding to f and let c' be obtained from c by decreasing by 2 the component/s corresponding to the endnodes of f. Since $\lfloor\sum_{e\in E}\bar{x}'_e\rfloor < \lfloor\sum_{e\in E}\bar{x}_e\rfloor$, by minimality of $(G,F),c,\bar{x}$ the polyhedron $P'(G,F,c')$ is integral. Note that, for every $U\subseteq V$, $c'(U)$ is odd if and only if $c(U)$ is odd, thus the odd-cut inequalities for $Ax=c'$, $x\geq 0$ are exactly the odd-cut inequalities $Ax=c$, $x\geq 0$. Since variables indexed by elements in $F\cup L$ do not appear in the odd-cut inequalities, $\bar{x}'$ is a fractional vertex of $P(G,F,c')$, a contradiction.

We show next that, for all e in $E\setminus(F\cup L)$, there exists $U\in\mathscr{L}$ such that $e\in\delta(U)$. Suppose not. Then there exists $e\in E\setminus(F\cup L)$ such that $e\notin\delta(U)$ for all $U\in\mathscr{L}$.

We first consider the case where $e=vw\in E_0$. Possibly by switching signs on v we may assume that $\sigma_{v,e}\neq\sigma_{w,e}$. Let (G',F') be obtained from (G,F) by contracting e, let r be the node obtained from the contraction of vw, and let $A'=A(G',F')$. Let $\bar{x}'$ be the restriction of $\bar{x}$ to the components relative to edges in $E(G')$, and let c' be obtained from c by removing the components corresponding to v and w and introducing a component relative to r with value $c'_r=c_v+c_w$. Since (G',F') is in $\mathscr{C}$ and $|V(G')|<|V|$, the polyhedron $P'(G',F',c')$ is integral. Note that $\bar{x}'\in P(G',F',c')$. Furthermore, the odd-cut inequalities for $A'x'=c',x'\geq 0$ are precisely the odd-cut inequalities for $Ax=c$, $x\geq 0$ relative to sets $U\subseteq V$ that either contain both v and w or none of them. This shows that $\bar{x}'\in P'(G',F',c')$. Since the equation $(A'x')_r=c'_r$ is the sum of $(Ax)_v=c_v$ and $(Ax)_w=c_w$, the equations in $A'x=c'$ are linearly independent. For every $U\in\mathscr{L}$, either $v,w\in U$ or $v,w\notin U$, since $e\notin\delta(U)$. Thus $\bar{x}'$ satisfies at equality the $|E|-1$ linearly independent inequalities defined by $A'x'=c'$ and by the odd-cut inequalities corresponding to sets in $\mathscr{L}$. Therefore, since $|E|-1\geq|E(G')|$, $\bar{x}'$ is a vertex of $P'(G',F',c')$, so it is an integral point. It follows that $\bar{x}_e$ must be the only fractional entry in $\bar{x}$, which is impossible since $(A\bar{x})_v=c_v$ and c_v is integer.

If e is a half-edge on node $v\in V$, the column relative to e in the constraint matrix M of the system (7) is the vector of all zeros except in row A_v. Since the columns of M are linearly independent, e is the only half-edge of G on v. Analogously, there are no loops on v. Let (G',F') be obtained from (G,F) by deleting node v and let $A':=A(G',F')$. Let $\bar{x}'\in\mathbb{Z}^{E(G')}$ be the vector obtained from $\bar{x}$ by removing the component relative to e, and let $c'\in\mathbb{Z}^{V(G')}$ be obtained from c by removing the component corresponding to v. Since (G',F') is in $\mathscr{C}$ and $|V(G')|<|V|$, the polyhedron $P'(G',F',c')$ is integral. Note that A' is obtained from A by removing the row corresponding to v and the column relative to e, and that the odd-cut inequalities for $P(G',F',c')$ are the odd-cut inequalities for $P(G,F,c)$ relative to sets $U\subseteq V\setminus\{v\}$. Thus $\bar{x}'\in P'(G',F',c')$. For every $U\in\mathscr{L}$, $U\subseteq V\setminus\{v\}$ since $e\notin\delta(U)$, thus all odd-cut inequalities in (7) are valid for $P'(G',F',c')$. It follows that $\bar{x}'$ satisfies at equality the $|E|-1=|E(G')|$ linearly independent inequalities defined by $A'x'=c'$ and by the odd-cut inequalities in (7), thus it is a vertex of $P'(G',F',c')$. This implies that, $\bar{x}'$ is integral and $\bar{x}_e$ is the only fractional entry of $\bar{x}$, which is impossible since $(A\bar{x})_v=c_v$ and c_v is integer.

We now prove that, given e in $E\setminus(F\cup L)$, $\bar{x}_e<1$. Let $\bar{U}\in\mathscr{L}$ such that $e\in\delta(\bar{U})$. Note that $\bar{x}_e\leq 1$ since $\bar{x}(\delta(\bar{U})\setminus(F\cup L))=1$. Suppose, by contradiction, that $\bar{x}_e=1$. It follows that e is the only edge in $\delta(\bar{U})\setminus(F\cup L)$, and that the odd-cut inequality relative to $\bar{U}$ is $x_e\geq 1$. Possibly by switching signs on the endnode/s of e, we may assume that e has sign $+1$ on its endnode/s. Let (G',F) be obtained from (G,F) by deleting e, and let $A':=A(G',F)$.

Let c' be obtained from c by subtracting 1 to the entries relative to the endnode/s of e, and let $\bar{x}'$ be the vector obtained from $\bar{x}$ by removing the component corresponding to e. Since (G', F) is in $\mathscr{C}$, $|V(G')| = |V|$, $|E_0(G')| \leq |E_0|$, and $|E(G')| < |E|$, the polyhedron $P'(G', F, c')$ is integral.

We show that $\bar{x}' \in P'(G', F, c')$. Clearly $\bar{x}' \in P(G', F, c')$, so we need to show that it satisfies the odd-cut inequalities. Let $U \subseteq V(G')$ such that $c'(U)$ is odd and such that the odd-cut inequality $x(\delta_{G'}(U) \setminus (F \cup L)) \geq 1$ is not redundant for $P'(G', F, c')$. Since $\delta_{G'}(\bar{U}) \subseteq F \cup L(G')$, it follows from Lemma 6 that either $U \subseteq \bar{U}$ or $U \subseteq V \setminus \bar{U}$. If $e \notin \delta(U)$, then $\bar{x}'(\delta_{G'}(U) \setminus (F \cup L(G'))) = \bar{x}(\delta(U) \setminus (F \cup L)) \geq 1$. Assume $e \in \delta(U)$. Then $c(U) = c'(U) + 1$, which is even. If $U \subseteq \bar{U}$, then $c(\bar{U} \setminus U)$ is odd, hence $\bar{x}'(\delta_{G'}(U) \setminus (F \cup L)) = \bar{x}(\delta(\bar{U} \setminus U) \setminus (F \cup L)) \geq 1$. If $U \subseteq V \setminus \bar{U}$, then $c(\bar{U} \cup U)$ is odd, hence $\bar{x}'(\delta_{G'}(U) \setminus (F \cup L)) = \bar{x}(\delta(\bar{U} \cup U) \setminus (F \cup L)) \geq 1$. Thus $\bar{x}' \in P'(G', F, c')$.

Finally, since $\bar{x}' \in P(G', F, c')$ and $P(G', F, c')$ is integral, $\bar{x}'$ is a convex combination of integral vectors $y^1, \ldots, y^k \in P(G', F, c')$. Thus $\bar{x} = \binom{1}{\bar{x}'}$ is a convex combination of $\binom{1}{y^1}, \ldots, \binom{1}{y^k}$, which are integral points in $P(G, F, c)$, contradicting the fact that $\bar{x}$ is a fractional vertex of $P'(G, F, c)$. $\diamond$

Claim 3. *G does not contain a cycle in F (i.e. (G, F) satisfies condition (C2)).*

Suppose C is a cycle in F. Since $(G, F) \in \mathscr{C}$, the cycle C is even, hence the edges of C can be partitioned in two subsets R and B so that any two adjacent edges uv, uw in C are contained in the same side of the partition if and only $\sigma_{u,uv} \neq \sigma_{u,uw}$. Let $y := \bar{x} + \epsilon\chi(R) - \epsilon\chi(B)$ and $z := \bar{x} - \epsilon\chi(R) + \epsilon\chi(B)$, where $\epsilon = \min_{e \in E(C)} \bar{x}_e$. By Claim 2, $\epsilon > 0$. By the choice of R and B, it follows that $y, z \in P(G, F, c)$. Moreover, both y and z satisfy all the odd-cut inequalities for $Ax = c$, $x \geq 0$, since these only involve variables relative to edges in $E \setminus (F \cup L)$. Thus $y, z \in P'(G, F, c)$ and $\bar{x} = \frac{1}{2}(y + z)$, contradicting the fact that $\bar{x}$ is a vertex of $P'(G, F, c)$. $\diamond$

Claim 4. *Each node in V is incident to at least one edge in $E \setminus (F \cup L)$.*

By contradiction, let v be a node in V incident only with edges in $F \cup L$. Since $|V| \geq 2$ and G is connected, there exists an edge $f = vw$ in F incident to v. Possibly by switching sign on v, we may assume that $\sigma_{v,f} \neq \sigma_{w,f}$. Notice that c_v is even, otherwise the odd-cut inequality corresponding to the set $\{v\}$ is not satisfied.

Let (G', F') be obtained from (G, F) by contracting f (operation (O4)), let r be the node obtained from the contraction of vw, and let $A' := A(G', F')$. Let $\bar{x}'$ be the restriction of $\bar{x}$ to the component relative to edges in $E(G')$, and let c' be obtained from c by removing the components corresponding to v and w and introducing a new component relative to r with value $c'_r := c_v + c_w$.

Since (G', F') is in $\mathscr{C}$ and $|V(G')| < |V|$, the polyhedron $P'(G', F', c')$ is integral. We show that $\bar{x}' \in P'(G', F', c')$. Clearly $\bar{x}' \in P(G', F', c')$, so we need to show that it satisfies the odd-cut inequalities. Since c_v is even, c'_r has the same parity as c_w.

Let U' be a subset of $V(G') = V \setminus \{v, w\} \cup \{r\}$ such that $c'(U')$ is odd. If $r \notin U'$ then $c(U') = c'(U')$ and $\delta_{G'}(U') \setminus (F' \cup L(G')) = \delta(U') \setminus (F \cup L)$. If $r \in U'$, then, if we let $U := U' \setminus \{r\} \cup \{w\}$, $c(U)$ is odd and $\delta_{G'}(U') \setminus (F' \cup L(G')) = \delta(U) \setminus (F \cup L)$. It follows that every odd cut inequality for $P(G', F', c')$ is an odd cut inequality for $P(G, F, c)$, so $\bar{x}' \in P'(G', F', c')$.

By (8), $U \subseteq V \setminus \{v\}$ for every $U \in \mathscr{L}$, therefore all odd cut inequalities in (7) are valid for $P'(G', F', c')$ and they are satisfied at equality by $\bar{x}'$. Since the inequality $(A'x')_r = c'_r$ is the sum of $(Ax)_w = c_w$ and $(Ax)_v = c_v$, $\bar{x}'$ satisfies at equality the $|E| - 1 = |E(G')|$ linearly independent inequalities defined by $A'x = c'$ and by the odd-cut inequalities in (7). Hence $\bar{x}'$ is a vertex of $P(G', F', c')$, and it is therefore integral, contradicting Claim 2. $\diamond$

Claim 5. *If $G \setminus F$ is connected and $V \notin \mathscr{L}$, then $\bar{x}_e = \frac{1}{2}$ for all $e \in G$.*

Let U be a maximal set in the laminar family $\mathscr{L}$. Since $\mathscr{L}$ is laminar, for every $S \in \mathscr{L}$ either $S \subseteq U$ or $S \subseteq V \setminus U$. Since $V \notin \mathscr{L}$, $U \subset V$. As $G \setminus F$ is connected, there exists $e \in \delta(U) \cap (E_0 \setminus F)$. Let $e = vw$, where $v \in U$, and let (G', F) be obtained from (G, F) by deleting e and introducing half-edges h_v and h_w on v and w with signs $\sigma_{v,e}$ and $\sigma_{w,e}$, respectively. Let $A' := A(G', F)$. One can readily verify that (G', F) is in the class $\mathscr{C}$, $|V(G')| = |V|$, and $|E_0(G')| < |E_0|$, thus the polyhedron $P'(G', F, c)$ is integral. Now let $\bar{x}'$ be obtained from $\bar{x}$ by removing the component corresponding to e and introducing two components relative to h_v and h_w with $\bar{x}'_{h_v} = \bar{x}'_{h_w} = \bar{x}_e$. Clearly $\bar{x}' \in P(G', F, c)$. Each odd-cut inequality of the latter system is satisfied by $\bar{x}'$ since, for every $S \subseteq V$, $\bar{x}'(\delta_{G'}(S) \setminus (F \cup L(G'))) \geq \bar{x}(\delta(S) \setminus (F \cup L))$, where equality holds if and only if $|S \cap \{v, w\}| \leq 1$. Thus $\bar{x}' \in P'(G', F, c)$. Furthermore, for every $S \in \mathscr{L}$, $|S \cap \{v, w\}| \leq 1$, since either $S \subseteq U$ or $S \subseteq V \setminus U$. Thus $\bar{x}'$ satisfies at equality the odd-cut inequalities

$$x'(\delta_{G'}(S) \setminus (F \cup L(G'))) \geq 1 \quad \text{for every } S \in \mathscr{L}. \tag{9}$$

Since $\bar{x}'$ satisfies at equality $|E| = |E(G')| - 1$ linearly independent inequalities, $\bar{x}'$ lies on a face Q of dimension 1 of $P'(G', F, c)$, thus there exist two vertices y, z of $P'(G', F, c)$ in Q such that $\bar{x}' = \lambda y + (1 - \lambda) z$, where $0 \leq \lambda \leq 1$. Since $P'(G', F, c)$ is integral, the points y and z are integral and $0 < \lambda < 1$. Since $y, z \in Q$, y, z satisfy (9) at equality. By Claim 2, each edge $h \in E \setminus (F \cup L)$ is in $\delta(S)$ for some set $S \in \mathscr{L}$, thus each edge $h \in E(G') \setminus (F \cup L(G') \cup \{h_w\})$ is in $\delta(S)$ for some set $S \in \mathscr{L}$. Therefore $y_h, z_h \in \{0, 1\}$ for every $h \in E(G') \setminus (F \cup L(G') \cup \{h_w\})$.

Since $\bar{x}'_{h_v} = \bar{x}'_{h_w} = \bar{x}_e < 1$, we can assume that $y_{h_v} = 1$ and $z_{h_v} = 0$ and that precisely one among y_{h_w} and z_{h_w} is 0. Hence $\bar{x}_e = \lambda$. If $z_{h_w} = 0$, then $y_{h_w} = 1$ because $\bar{x}'_{h_w} = \lambda y_{h_w}$, thus if we define two points $\bar{y}, \bar{z} \in \mathbb{R}^E$ by $\bar{y}_h = y_h$, $h \in E \setminus \{e\}$, $\bar{y}_e = 1$, and $\bar{z}_h = z_h$, $h \in E \setminus \{e\}$, $\bar{z}_e = 0$, then $\bar{y}$ and $\bar{z}$ are integral points in $P(G, F, c)$ and $\bar{x} = \lambda \bar{y} + (1 - \lambda) \bar{z}$, contradicting the fact that $\bar{x}$ is a vertex of $P'(G, F, c)$. Therefore $y_{h_w} = 0$ and $z_{h_w} = k$ for some positive integer k. Since $\lambda = \bar{x}_e = \lambda y_{h_w} + (1 - \lambda) z_{h_w} = (1 - \lambda) k$, $\lambda = k/(k + 1)$. If $k = 1$, then all components of $\bar{x}$ are equal to $1/2$ and we are done. Thus we may assume that $k \geq 2$.

Note also that, since $z(\delta_{G'}(U) \setminus (F \cup L(G'))) = 1$ and $z_{h_v} = 0$, there exists $g \neq e$ in $\delta_{G'}(U) \setminus (F \cup L(G'))$ such that $z_g = 1$. Thus $\delta(U) \setminus (F \cup L) = \{e, g\}$ and $\bar{x}_g = 1 - \lambda = 1/(k + 1) < 1/2$. If $g \in E_0$, then by applying to g the same argument we used for e, we will obtain that $\bar{x}_g > 1/2$, a contradiction. Therefore $g \in H$. In particular, $\delta_{G'}(U) \cap E_0(G') \subseteq F$.

Let G'' be the bidirected graph obtained from G' by switching the sign of h_w. Let $A'' = A(G'', F)$, $c'' \in \mathbb{R}^V$ be defined by $c''_u = c_u$ for all $u \in V \setminus \{w\}$, and $c''_w = c_w - 1$. Clearly, (G'', F) is in the class $\mathscr{C}$ and $P'(G'', F, c'')$ is integral.

Let y'', z'' and $\bar{x}''$ be defined by $y''_h = y_h$, $z''_h = z_h$ and $\bar{x}''_h = \bar{x}_h$ for all $h \in E(G') \setminus \{e_w\}$, $y''_{h_w} = 1$, $z''_{h_w} = 1-k$ and $\bar{x}''_{h_w} = 1 - \bar{x}_e$. Observe that y'' and z'' are integral, they satisfy the system $A''x'' = c''$, and $\bar{x}'' = \lambda y'' + (1-\lambda) z''$. Since $y'' \geq 0$, it follows that $y'' \in P(G'', F, c'')$, and therefore $y'' \in P'(G'', F, c'')$. Since $z''_{h_w} < 0$, $z'' \notin P'(G'', F, c'')$.

We prove next that $\bar{x}'' \in P'(G'', F, c'')$. It suffices to show that $\bar{x}''$ satisfies all odd-cut inequalities for $P(G'', F, c'')$. Let $S \subseteq V$ such that $c''(S)$ is odd. If $w \notin S$, then $c''(S) = c(S)$ and $\bar{x}''(\delta_{G'}(S) \setminus (F \cup L(G'))) = \bar{x}(\delta(S) \setminus (F \cup L)) \geq 1$. Otherwise, since $\delta_{G'}(U) \cap E_0(G') \subseteq F$, it follows by (8) that $S \subseteq V(G') \setminus U$. Note that $c(U \cup S) = c(U) + c(S) = c(U) + c''(S) + 1$, hence $c(U \cup S)$ is odd. Since $\bar{x}''_{h_w} = 1 - \bar{x}_e = \bar{x}_g$, it follows that $\bar{x}''(\delta_{G'}(S) \setminus (F \cup L(G'))) = \bar{x}(\delta(U \cup S) \setminus (F \cup L)) \geq 1$.

Observe next that, for every $S \in \mathscr{L}$, $w \notin S$, otherwise $h_w \in \delta_{G'}(S)$ and $z(\delta_{G'}(S) \setminus (F \cup L(G'))) = 1$ would imply $z_{h_w} = 1 < k$. It follows that $\bar{x}''$ and y'' satisfy at equality the $|E| = |E(G'')| - 1$ constraints $A''x'' = c''$, $x''(\delta_{G''}(S)) \setminus (F \cup L(G'')) \geq 1$. It follows that $\bar{x}''$ and y'' both belong to a face Q' of $P'(G'', F, c'')$ of dimension 1. Recall that $\bar{x}'' = \lambda y'' + (1-\lambda) z''$, thus $\bar{x}''$ belongs to the line segment joining y'' and z''. Since $z'' \notin P'(G'', F, c'')$, it follows that there exists a vertex $\bar{z}$ of Q' in the line segment joining y'' and z''. Thus there exists $\bar{\lambda}$, $0 < \bar{\lambda} < 1$ such that $\bar{z} = \bar{\lambda} y'' + (1-\bar{\lambda}) z''$, and so $\bar{z}_g = 1 - \bar{\lambda}$ since $y''_g = 0$ and $z''_g = 1$. Note however that the point $\bar{z}$ should be integral, because it is a vertex of Q', and thus also a vertex of $P'(G'', F, c'')$, a contradiction. $\diamond$

Claim 6. *If G is bipartite, $G \setminus F$ is connected and $L = \emptyset$, then $\bar{x}_e = \frac{1}{2}$ for every $e \in E$.*

Since G is bipartite, it follows by a theorem of Heller and Tompkins [9] that the nodes in G can be partitioned into two subsets V_1, V_2 such that, for every $e = vw \in E_0$, v and w are in the same side of the bipartition if and only if $\sigma_{v,e} \neq \sigma_{w,e}$. By symmetry, we may assume $c(V_1) \geq c(V_2)$. For $i = 1, 2$, let H_i^+ and H_i^- be the sets of half-edges of G with endnode in V_i having, respectively, $+1$ and -1 sign.

Since $G \setminus F$ is connected, by Claim 5 we can assume that $V \in \mathscr{L}$. The odd-cut inequality relative to V is $x(H) \geq 1$, and it is satisfied at equality by $\bar{x}$. Since $L = \emptyset$, by summing the equations in $Ax = c$ corresponding to nodes in V_1 and subtracting the equations relative to nodes in V_2, we obtain $x(H_1^+ \cup H_2^-) - x(H_1^- \cup H_2^+) = c(V_1) - c(V_2)$.

Since $c(V)$ is odd, $c(V_1) - c(V_2) \geq 1$, thus $1 = \bar{x}(H) \geq \bar{x}(H_1^+ \cup H_2^-) - \bar{x}(H_1^- \cup H_2^+) \geq 1$, because $\bar{x} \geq 0$. It follows that $\bar{x}(H_1^- \cup H_2^+) = 0$, so $H_1^- \cup H_2^+ = \emptyset$ because $\bar{x} > 0$. So the equation $x(H) = 1$ can be obtained as a linear combination of the equations in $Ax = c$, contradicting the fact that the inequalities in (7) are linearly independent. $\diamond$

Given a star $\Delta \subseteq F \cup L$, let G^Δ be obtained from $G \setminus \Delta$ by introducing, for every node $v \in V$ incident to at least one edge of Δ, a loop ℓ_v on v, with sign $+1$ if $\sum_{f \in \Delta} \sigma_{v,f} \bar{x}_f \geq 0$ and sign -1 otherwise. Let L^Δ be the set of these new loops in G^Δ. Let $F^\Delta := F \setminus \Delta$ and $A^\Delta := A(G^\Delta, F^\Delta)$. Let $\bar{x}^\Delta \in \mathbb{R}^{E(G^\Delta)}$ be obtained from $\bar{x}$ by removing the components corresponding to the edges in Δ, and by setting, for every loop ℓ_v in L^Δ, $\bar{x}^\Delta_{\ell_v} = |\sum_{f \in \Delta} \sigma_{v,f} \bar{x}_f|$.

Claim 7. *Let $\Delta \subseteq F \cup L$ be a star centered at node $v_0 \in V$ with $\Delta \cap F \neq \emptyset$. If (G^Δ, F^Δ) does not contain $\mathcal{G}_4$ as a minor, then the following hold.*

(i) $\Delta \cap L = \emptyset$;

(ii) $G \setminus \Delta$ *is connected;*

(iii) $\bar{x}^\Delta = \lambda y + (1-\lambda)z$ *for some* $0 < \lambda < 1$*, where* y, z *are integral points in* $P(G^\Delta, F^\Delta, c)$ *satisfying* $y_e, z_e \leq 1\ \forall e \in E(G^\Delta) \setminus \{\ell_{v_0}\}$*. Moreover, for every* $U \in \mathscr{L}$*,* $|\delta(U) \setminus (F \cup L)| = 2$*;*

(iv) *If* $|\Delta| = 1$*, then* $\bar{x}$ *is half-integral.*

By assumption we have that (G^Δ, F^Δ) is in $\mathscr{C}$. Since $|V(G^\Delta)| = |V|$, and $|E_0(G^\Delta)| < |E_0|$, it follows that $P'(G^\Delta, F^\Delta, c)$ is integral.

The matrix A^Δ is obtained from A by deleting the columns relative to the edges in Δ, and by introducing columns relative to the loops in L^Δ. These columns are zero everywhere except for the entry relative to v, with value $2\sigma_{v,\ell_v}$. Observe that the space spanned by the columns of A^Δ contains the space spanned by the columns of A. Since A has full row-rank, it follows that A^Δ and A have rank $|V|$. The odd cut inequalities for $P(G, F, c)$ and for $P'(G^\Delta, F^\Delta, c)$ are the same, since they do not involve elements in $F \cup L$ and $F^\Delta \cup L(G^\Delta)$, therefore $\bar{x}^\Delta \in P'(G^\Delta, F^\Delta, c)$ and it satisfies the odd cut inequalities in (7) at equality. In particular, $\bar{x}^\Delta$ satisfies at equality $|E|$ linearly independent inequalities valid for $P'(G^\Delta, F^\Delta, c)$. This implies, $E(G^\Delta) \geq |E|$. Furthermore, $E(G^\Delta) > |E|$, otherwise $\bar{x}^\Delta$ is a vertex of $P'(G^\Delta, F^\Delta, c)$ and it is therefore integral, a contradiction.

(i) Since the number of nodes incident to some element of Δ is $|\Delta \cap F| - 1$, it follows that $E(G^\Delta) = |E| - |\Delta| + |L^\Delta| = |E| - |\Delta \cap L| + 1$. Since $E(G^\Delta) > |E|$, it follows that $|\Delta \cap L| \leq 1$.

(ii) From the above, $|E(G^\Delta)| = |E| + 1$, therefore $\bar{x}^\Delta$ belongs to a face Q of dimension 1 of $P'(G^\Delta, F^\Delta, c)$. Suppose $G \setminus \Delta$ is not connected. Clearly also $G \setminus \Delta$ is not connected. Let G' be a connected component of G^Δ and let G'' be the union of all the other connected components of G^Δ. Let $F' = F^\Delta \cap E(G')$, $F'' = F^\Delta \cap E(G'')$, let $\bar{x}'$ and $\bar{x}''$ be the restriction of $\bar{x}^\Delta$ to the edges of G' and G'', respectively, and let c' and c'' be the restriction of c to $V(G')$ and $V(G'')$ respectively. Then $P'(G^\Delta, F^\Delta, c) = P'(G', F', c') \times P'(G'', F'', c'')$ (where "$\times$" indicates the cartesian product of two sets). In particular, $Q = Q' \times Q''$ where Q' and Q'' are faces of $P'(G', F', c')$ and $P''(G'', F'', c'')$, respectively. Since $\dim(Q) = \dim(Q') + \dim(Q'')$, either Q' or Q'' has dimension 0. Since $\bar{x}' \in Q'$ and $\bar{x}'' \in Q''$, $\bar{x}'$ is a vertex of Q' or $\bar{x}''$ is a vertex of Q''. Thus at least one among $\bar{x}'$ and $\bar{x}''$ are integral points. By Claim 4, $E(G') \setminus L^\Delta \neq \emptyset$ and $E(G'') \setminus L^\Delta \neq \emptyset$, thus there exists some edge $e \in E \setminus \Delta$ such that $\bar{x}_e$ is integer, contradicting Claim 2.

(iii) The point $\bar{x}^\Delta$ belongs to the polyhedron $\tilde{P} := P'(G^\Delta, F^\Delta, c) \cap \{x^\Delta \in \mathbb{R}^{E(G^\Delta)} : x^\Delta_e \leq \lceil \bar{x}^\Delta \rceil,\ e \in F^\Delta \cup L(G^\Delta)\}$. By Lemma 6, $\tilde{P}$ is the first Chvátal closure of the polyhedron defined by the system $A^\Delta x^\Delta = c, x^\Delta \geq 0, x^\Delta_f \leq 1, \forall f \in F^\Delta \cup L(G^\Delta) \setminus \{\ell_v\}$. By Lemma 7, $\tilde{P}$ is an integral polyhedron. Since $\bar{x}^\Delta$ belongs to a face of dimension 1 of $P'(G^\Delta, F^\Delta, c)$, $\bar{x}^\Delta$ belongs to a face $\tilde{Q}$ of dimension 1 of $\tilde{P}$. It follows that $\bar{x}^\Delta$ is a convex combination of two integral vertices y and z of $\tilde{Q}$, i.e. $\bar{x}^\Delta = \lambda y + (1-\lambda)z$ for some $0 < \lambda < 1$.

By Claim 2, $\lceil \bar{x}^\delta \rceil = 1$ for all $e \in F^\Delta \cup L(G^\Delta) \setminus \{\ell_{v_0}\}$, and each edge in $E(G^\Delta) \setminus (F^\Delta \cup L(G^\Delta))$ belongs to $\delta(U)$ for some $U \in \mathscr{L}$. Since y, z are in $\tilde{Q}$, they satisfy at equality all odd cut inequalities in (7). It follows that $y_e, z_e \in \{0, 1\}$ for every e in $E(G^\Delta) \setminus \{\ell_v\}$, and that $|\delta(U) \setminus (F \cup L)| = 2$ for every $U \in \mathscr{L}$.

(iv) Assume $|\Delta| = 1$. Then $\Delta = \{f\}$ for some $f = vw \in F$ and $E(G^\Delta) = E \setminus \{f\} \cup \{\ell_v, \ell_w\}$. Since $\bar{x}^\Delta_{\ell_v} = \bar{x}^\Delta_{\ell_w} = \bar{x}_f$, it follows that $\lceil \bar{x}^\Delta_{\ell_v} \rceil = \lceil \bar{x}^\Delta_{\ell_w} \rceil = 1$, therefore the points y, z defined in (iii) have all $0, 1$ components. Assume, by symmetry, that $y_{\ell_v} = 0$, and $z_{\ell_v} = 1$. Then $y_{\ell_w} = 1$ and $z_{\ell_w} = 0$, otherwise the points $\bar{y}, \bar{z} \in \mathbb{Z}^E$, obtained from y and z by replacing the two components relative to ℓ_v and ℓ_w with one component relative to f of value $\bar{y}_f = y_{\ell_v} = y_{\ell_w}$, $\bar{z}_f = z_{\ell_v} = z_{\ell_w}$, are in $P'(G, F, c)$ and $\bar{x} = \lambda\bar{y} + (1 - \lambda)\bar{z}$, a contradiction. It follows that $\bar{x}^\Delta_{\ell_v} = 1 - \lambda$ and $\bar{x}^\Delta_{\ell_w} = \lambda$. Since $\bar{x}^\Delta_{\ell_v} = \bar{x}_f = \bar{x}^\Delta_{\ell_w}$, $\lambda = 1/2$, thus $\bar{x}$ is half-integral. $\diamond$

Claim 8. *If $G \setminus F$ is connected, then $\bar{x}_e = 1/2$ for every e in E.*

By Claim 3, we know that (G, F) satisfies condition (C2). Suppose that this pair does not satisfy condition (C1). By Lemma 9, we have that $L = \emptyset$ and (G, F) is bipartite. Then, by Claim 6, $\bar{x}_e = 1/2$ for every e in E.

Assume that (G, F) satisfies condition (C1). Since $F \neq \emptyset$, let B be a block of G such that $B \cap F \neq \emptyset$. Block B must satisfy i) or ii) of Lemma 10. If it satisfies ii), then there exists an edge $f \in F$ such that every other edge in $E(B) \cap F$ is nested in f. If we let $\Delta := \{f\}$, it is easy to check that (G^Δ, F^Δ) does not contain $\mathcal{G}_4$ as a minor. Hence, by Claim 7(iv), $\bar{x}_e = 1/2$ for every e in E.

Thus we may assume that B satisfies Lemma 10(i). That is, $E(B) \cap (F \cup L)$ is the edge set of a star in B, centered at some node $v_0 \in V(B)$. Let $\Delta = E(B) \cap (F \cup L)$. It is easy to check that (G^Δ, F^Δ) is in $\mathscr{C}$. Hence by Claim 7(iii), $\bar{x}^\Delta = \lambda y + (1 - \lambda)z$ for some $0 < \lambda < 1$, where y and z are integral points in $P(G^\Delta, F^\Delta, c)$ such that $y_e, z_e \in \{0, 1\}$ for all $e \in E(G^\Delta) \setminus \{\ell_{v_0}\}$. It follows that $\bar{x}^\Delta_e \in \{\lambda, 1 - \lambda\}$ for all $e \in E(G^\Delta) \setminus \{\ell_{v_0}\}$, hence $\bar{x}_e \in \{\lambda, 1 - \lambda\}$ for every e in E, since for every edge in E there exists an edge in $E(G^\Delta) \setminus \{\ell_{v_0}\}$ with the same value. It suffices to show that $\lambda = 1/2$. Suppose by contradiction that $\lambda \neq 1/2$.

Define $\bar{y}, \bar{z} \in \{0, 1\}^E$ by $\bar{y}_e = \begin{cases} 1 & \text{if } \bar{x}_e = \lambda \\ 0 & \text{otherwise} \end{cases}$ and $\bar{z}_e = 1 - \bar{y}_e$ for all $e \in E$. By definition of $\bar{y}$ and $\bar{z}$, $\bar{x} = \lambda\bar{y} + (1 - \lambda)\bar{z}$. Furthermore, $(Ay)_u = (Az)_u = c_u$ for every $u \neq v_0$. We will show that $(A\bar{y})_{v_0} = (A\bar{z})_{v_0} = c_{v_0}$, thus showing that $\bar{y}, \bar{z} \in P(G, F, c)$, which contradicts the fact that $\bar{x}$ is a vertex.

We recall that, by Claim 7,

$$|\delta(U) \setminus (F \cup L)| = 2, \text{ for every set } U \in \mathscr{L}. \tag{10}$$

By Claim 5, $V \in \mathscr{L}$, otherwise $\bar{x}$ is half-integral. Since $\delta(V) \setminus L = H$, by (10) it follows that $|H| = 2$, say $H = \{h_1, h_2\}$, and that $\bar{x}_{h_1} + \bar{x}_{h_2} = 1$.

By (10), the constraint matrix M of the odd-cut inequalities $x(\delta(U) \setminus (F \cup L)) \geq 1$, $U \in \mathscr{L}$, has exactly two ones in every row. Therefore M is the edge-node incidence matrix of an undirected graph Γ, whose vertex set is $E \setminus (F \cup L)$ and where two elements $e_1, e_2 \in V(\Gamma)$ are adjacent if and only if there exists $U \in \mathscr{L}$ with $e_1, e_2 \in \delta(U)$. Note that Γ has no parallel edges since the inequalities in (7) are linearly independent. We show that there exists an edge $\bar{e} = vw$ in $E_0 \setminus F$ such that there is only one set $\bar{U}$ in $\mathscr{L}$ with $\bar{e} \in \delta(\bar{U})$. Suppose not. Then, by Claim 2, every element $e \in E_0 \setminus F$ has degree at least 2 in Γ. Assume first that Γ is acyclic. Since every node of Γ has degree at least two except for h_1, h_2, it follows that h_1, h_2 have degree one and that Γ is a path from h_1 to h_2. Since $V \in \mathscr{L}$, h_1 and h_2 are adjacent in Γ, thus Γ contains only one edge. This implies that $\mathscr{L} = \{V\}$. By Claim 2, there exists

$U \in \mathscr{L}$ such that $e \in \delta(U)$ for every $e \in E \setminus (F \cup L)$, thus $E \setminus (F \cup L) = \{h_1, h_2\}$. Since $G \setminus F$ is connected, G contains only one node, a contradiction since $F \neq \emptyset$.
It follows that Γ contains a cycle C. Let $e_1, \ldots, e_k \in V(\Gamma)$ be the nodes of Γ in C, and let $U_1, \ldots, U_k$ be the sets in $\mathscr{L}$ corresponding to the edges in C, say $\{e_i, e_{i+1}\} = \delta(U_i) \setminus (F \cup L)$, $i = 1, \ldots, k-1$, $\{e_1, e_k\} = \delta(U_k) \setminus (F \cup L)$. Thus $\bar{x}$ satisfy the equations $x_{e_i} + x_{e_{i+1}} = 1$, $i = 1, \ldots, k-1$, $x_{e_1} + x_{e_k} = 1$. Since these k equations are linearly independent, it follows that the unique solution is $x_{e_1} = \cdots = x_{e_k} = 1/2$. It follows that $\lambda = 1/2$ and $\bar{x}_e = 1/2$ for every $e \in E$, a contradiction.

Consider now $\bar{e} = vw \in E_0$ and $\bar{U} \in \mathscr{L}$ such that $\bar{e} \in \delta(\bar{U})$ and $\bar{e} \notin \delta(U)$ for every $U \in L$, $U \neq \bar{U}$. By switching signs on the endnodes of $\bar{e}$, we can assume that $\sigma_{v,\bar{e}} \neq \sigma_{w,\bar{e}}$. Now let (G', F') be obtained from (G, F) by contracting $\bar{e}$, and let r be the node obtained from the contraction of $\bar{e}$. Let $A' = A(G', F')$.

Let $\bar{x}'$ be the restriction of $\bar{x}$ to the components relative to $E(G')$, and let c' be obtained from c by removing the components corresponding to v and w and introducing a component relative to r with value $c'_r := c_v + c_w$. Since (G', F') is in $\mathscr{C}$ and $|V(G')| < |V|$, the polyhedron $P'(G', F', c')$ is integral. Clearly $\bar{x}' \in P(G', F', c')$. Furthermore, the odd-cut inequalities for $P(G', F', c')$ are exactly the odd-cut inequalities for $P(G, F, c)$ relative to sets $U \subseteq V$ such that either $v, w \in U$ or $v, w \notin U$, thus they are satisfied by $\bar{x}'$. It follows that $\bar{x}' \in P'(G', F', c')$. Furthermore, the equation $(A'x')_r = c'_r$ is the sum of $(Ax)_v = c_v$ and $(Ax)_w = c_w$, and, for every $U \in \mathscr{L} \setminus \{\bar{U}\}$, either $v, w \in U$ or $v, w \notin U$. It follows that $\bar{x}'$ satisfies at equality $|E| - 2 = |E(G')| - 1$ linearly independent inequalities valid for $P'(G', F', c')$.

It follows that $\bar{x}'$ is in a face of dimension 1 of $P'(G', F', c')$, thus there exist two vertices y' and z' of $P'(G', F', c')$ such that $\bar{x}' = \lambda' y' + (1-\lambda') z'$, for some $0 < \lambda' < 1$. Since $P'(G', F', c')$ is integral, y', z' are integral. By Claim 2, $y'_e, z'_e \in \{0, 1\}$ for every e in E. Since $\bar{x}^{\Delta}_{h_1} = \bar{x}'_{h_1}$ (possibly by switching the roles of y' and z'), it follows that $\lambda' = \lambda$. This implies that, for every $e \in E(G')$, $y'_e = \bar{y}_e$, $z'_e = \bar{z}_e$. Hence, $(A\bar{y})_u = (A\bar{z})_u = c_u$ for all $u \in V \setminus \{v, w\}$, and $(A\bar{y})_v + (A\bar{y})_w = (A'y')_r = c_v + c_w$, $(A\bar{z})_v + (A\bar{z})_w = (A'z')_r = c_v + c_w$. Without loss of generality we can assume that $v \neq v_0$. Since $(A\bar{y})_u = (A\bar{z})_u = c_u$ for every $u \neq v_0$, we deduce that $(A\bar{y})_w = c_v + c_w - (A\bar{y})_v = c_w$. Similarly, $(A\bar{z})_w = c_w$. Hence $\bar{y}, \bar{z} \in P(G, F, c)$, a contradiction. $\diamond$

Claim 9. *For every block B of G, every connected component of $B \setminus F$ has at least two nodes.*

Let B be a block of G such that a component of $B \setminus F$ consist of only one node, say $v \in V(B)$. Let $\Delta := \delta(v) \cap E(B) \cap F$. Since $\{v\}$ is a component of $B \setminus F$, one can easily show that $(G^{\Delta}, F^{\Delta}) \in \mathcal{C}$. This is contradicts Claim 7(ii). $\diamond$

Claim 10. *If $G \setminus F$ is not connected, then $\bar{x}_e = 1/2$ for every e in E.*

Let B be a block of G such that $B \setminus F$ is not connected. We denote by $Q_1, \ldots, Q_t$ the connected components of $B \setminus F$. Let W be the set of edges in F with endnodes in distinct components of $G \setminus F$, and let $\bar{V}_j$ be the set of nodes in Q_j that are incident to some edge in $W \cap E(B)$, $j = 1, \ldots, t$. By Claim 9, condition (C3) is satisfied, thus nodes in $\bar{V}_j = \{v^j_1, \ldots, v^j_{k_j}\}$ can be ordered in such a way that they satisfy the properties i) and ii) of Lemma 11.

For $j = 1, \dots, t$, let $Z_j = \{v_1^j, v_{k_j}^j\}$. We show next that there exists an edge $vw \in F$ such that $v \in Z_j$ and $w \in Z_{j'}$, where $1 \le j, j' \le t$, $j \ne j'$. By property ii) of Lemma 11, for every $f = vw \in W \cap E(B)$, $\{v, w\}$ is a node-cutset of B. Denote by C_f a connected components of $B \setminus F$ that has the smallest number of nodes. Choose $f = vw \in W \cap E(B)$ so that $|V(C_f)|$ is smallest possible. We claim that $v, w \in \cup_{j=1}^t Z_j$. Suppose not. Then, up to changing the indices, $v = v_i^1$ where $2 \le i \le k_1 - 1$. By symmetry, we may assume that $v_1^1 \in V(C_f)$. Since $v_1^1 \in \bar{V}_1$, there exists an edge $f' \in W \cap E(B)$ incident to v_1^1, say $f' = v_1^1 w'$. It follows that $w' \in V(C_f)$. Since $\{v_1^1, w'\}$ is a node-cutset of B, it follows that there exists a connected component of $B \setminus \{v_1^1, w'\}$ whose nodeset is contained in $V(C_f) \setminus \{v_1^1, w'\}$. This implies that $|V(C_{f'})| < |V(C_f)|$, contradicting the choice of f.

Thus, up to changing indices, $f = v_1^1 v_1^2$ is an edge in $W \cap E(B)$. Let $\Delta := \{f\}$. We claim that (G^Δ, F^Δ) does not contain $\mathscr{G}_4$ as a minor, which by Claim 7 implies that $\bar{x}_2 = \frac{1}{2}$ for all $e \in E$.

Let ℓ_1 and ℓ_2 be the new loops in G^Δ incident to v_1^1 and v_1^2 respectively. Suppose by contradiction that (G^Δ, F^Δ) contains $\mathscr{G}_4$ as a minor. Since (G, F) does not contain $\mathscr{G}_4$ as a minor, by symmetry we can assume that the loop of $\mathscr{G}_4$ is ℓ_1, and that v_1^2 is contained in the minor. Thus in G^Δ there exists a cycle C that passes through v_1^2 and that contains an edge in F, and a path P in $G \setminus F$ from v_1^1 to a node u of C such that $V(P) \cap V(C) = \{u\}$, where both edges in C incident to u are in $E_0 \setminus F$. It follows that $u \in V(Q_i)$.

Since $v_1^2 \notin V(Q_1)$ and $u \in V(Q_1)$, there exist i, i', $1 \le i < i' \le k_1$, such that $v_i^1, v_{i'}^1 \in \bar{V}(C)$ and such that C contains paths P_1, P_2 from u to v_i^1 and from u to $v_{i'}^1$, respectively, such that $V(P_1) \cap V(P_2) = \{u\}$ and such that P_1 and P_2 are contained in the subgraph $\bar{Q}_1$ of G induced by $V(Q_1)$. It follows that v_1^1 and $v_{i'}^1$ are in the same connected component of $\bar{Q}_1 \setminus \{v_i^1\}$, contradicting property i) of Lemma 11. $\diamond$

Claim 11. *The pair (G, F) satisfies the parity conditions of Lemma 13.*

By Claims 8 and 10, we have that $\bar{x}_e = \frac{1}{2}$ for every $e \in E$. Since $A\bar{x} = c$, it follows that $\bar{x}(\delta(v) \setminus (F \cup L))$ is an integer for every $v \in V$. Hence $|\delta(v) \setminus (F \cup L)|$ is even and parity condition a) is satisfied.

Given a connected component Q of $G \setminus F$ such that $H(Q) = \emptyset$, $c(V(Q))$ is even since $\delta(V(Q)) \setminus (F \cup L(Q)) = \emptyset$, otherwise $V(Q)$ defines an odd-cut inequality violated by $\bar{x}$. Since $A\bar{x} = c$, it follows that

$$c(V(Q)) = \frac{1}{2} \sum_{vw \in E_0(Q) \setminus F} (\sigma_{v,vw} + \sigma_{w,vw}) + \sum_{vw \in F \cap E(Q)} (\sigma_{v,vw} + \sigma_{w,vw}) + \sum_{\substack{vw \in \delta(V(Q)) \\ v \in V(Q)}} \sigma_{v,vw}.$$

Even edges of $E(Q)$ contribute 0 to the right-hand-side of the latter expression, each odd edge of $E(Q) \setminus F$ contributes ± 1, edges in F with both endnodes in $V(Q)$ contribute 0 or ± 2, while edges in $\delta(V(Q))$ contribute ± 1. Hence the number of odd edges in $E(Q)$ is congruent modulo 2 to $|\delta(V(Q))|$. $\diamond$

Claim 12. *(G, F) has a balanced bicoloring.*

It follows by Claims 9 and 11 and by Lemma 14. $\diamond$

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