Decomposing Berge graphs containing no proper wheel, long prism or their complements

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Abstract

In this paper we show that, if G is a Berge graph such that neither G nor its complement $\overline{G}$ contains certain induced subgraphs, named proper wheels and long prisms, then either G is a basic perfect graph (a bipartite graph, a line graph of a bipartite graph or the complement of such graphs) or it has a skew partition that cannot occur in a minimally imperfect graph. This structural result implies that G is perfect.

1 Introduction

A graph is *perfect* if, in all its induced subgraphs, the size of a largest clique is equal to the chromatic number. A graph is *Berge* if it does not contain an odd hole or its complement. The strong perfect graph conjecture (SPGC) [1] states that Berge graphs are perfect. In 2001, Conforti, Cornuéjols and Vušković [6] suggested the following approach to solving the SPGC: show that all Berge graphs can be decomposed into four basic classes of perfect graphs (bipartite graphs, line graphs of bipartite graphs and their complements) using decompositions that cannot occur in minimally imperfect graphs (2-joins and certain kinds of skew partitions). Chudnovsky, Robertson, Seymour, Thomas [3] announced recently that they solved the SPGC using this approach. Conforti, Cornuéjols, Vušković, Zambelli obtained partial results discussed in the present paper and in [7]. There are

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three separate cases to decomposing Berge graphs G depending on whether they contain certain induced subgraphs called long prisms and proper wheels (to be defined later):

(i) Neither G nor its complement $\bar{G}$ contains a long prism or a proper wheel,

(ii) G contains a long prism,

(iii) G contains a proper wheel but neither G nor $\bar{G}$ contains a long prism.

This paper proves a decomposition theorem for Berge graphs that satisfy (i). Chudnovsky, Robertson, Seymour, Thomas [3] simultaneously proved a similar result. Furthermore, they obtained a decomposition of Berge graphs that satisfy (ii). Finally they also showed in [3] that no minimally imperfect Berge graph satisfies (iii), thus solving the SPGC. Our results on decomposing Berge graphs that satisfy (iii) are available in [7].

1.1 Notation and definitions

We refer to West [13] for standard terminology in graph theory. Let G be an undirected simple graph. We denote by $\overline{G}$ the complement graph of G. Given $X \subset V(G)$, we denote by $G[X]$ the subgraph of G induced by X and by $\overline{G}[X]$ the subgraph of $\overline{G}$ induced by X. Given two distinct nodes $x, y \in V(G)$ and an induced subgraph G' of G, we say that *x and y are twins with respect to G'* if they have the same neighbors in $G' \setminus \{x, y\}$. If x and y are adjacent (resp. not adjacent) we say that x and y are *true twins* (resp. *false twins*) with respect to G'.

Given a path $P = x_1, \ldots, x_n$ and $1 \leq i \leq j \leq n$, we denote with $P_{x_i x_j}$ the path $x_i, ..., x_j$ contained in P, and we say that $P_{x_i x_j}$ is a *sub-path* of P. The set $\{x_i \mid 1 < i < n\}$ is the *interior* of P. The nodes in the interior of P are also referred to as the *intermediate nodes of P*. The *length* of a path is its number of edges, and it is denoted by $|P|$. P is said to be *odd* (resp. *even*) if P has odd (resp. even) length.

A path in $\overline{G}$ is called a *co-path*. The *interior* and the *length of a co-path P* are, respectively, the interior and the length of $\overline{P}$. A set $X \subseteq V(G)$ is said to be *co-connected* if $\overline{G}[X]$ is a connected graph.

Given a set $X \subset V(G)$, a node $x \notin X$ and an edge $e = yz$ such that $y, z \notin X$, we say that x is *universal for X* if x is adjacent to every node of X, and we say that e *sees* X if both y and z are universal for X.

Given two disjoint sets of nodes A and B, a *direct connection between A and B* is a minimal chordless path (in term of its node set) $P = x_1, ..., x_n$ such that x_1 has a neighbor in A and x_n has a neighbor in B.

Definition 1 *Given two node disjoint triangles a_1, a_2, a_3 and b_1, b_2, b_3, a* long prism *is a graph induced by three node disjoint chordless paths $P^1 = a_1, \ldots, b_1$, $P^2 = a_2, \ldots, b_2$ and $P^3 = a_3, \ldots, b_3$, at least one of which has length greater than one, such that the only adjacencies between the nodes of distinct paths are the edges of the two triangles. We denote such a graph by $3PC(a_1a_2a_3, b_1b_2b_3)$.*

The following concepts were introduced in [5].

Definition 2 *A* wheel (H, v) *consists of a hole* H *and a node* v *not in* H *that has at least* 3 *neighbors in* H*. If* v *has* k *neighbors in* H*, we say that* (H, v) *is a* k-wheel*. The node* v *is called the* center *of the wheel. If* x *and* y *are neighbors of* v *in* H*, a sub-path of* H *with endnodes* x *and* y *is called a* sector *if it contains no neighbor of* v *in its interior.*

An odd wheel *is a wheel containing an odd number of triangles. An* even wheel *is a wheel which is not odd.*

A triangle-free wheel *is a wheel containing no triangle.*

A twin wheel *is a* 3*-wheel* (H, v) *such that* (H, v) *contains exactly* 2 *triangles.*

A line wheel *is a* 4*-wheel* (H, v) *such that* (H, v) *contains exactly* 2 *triangles and these triangles have only node* v *in common.*

A universal wheel *is a wheel* (H, v) *in which* v *is adjacent to every node of* H*.*

A proper wheel *is an even wheel which is not a triangle-free wheel, a twin wheel, a line wheel or a universal wheel.*

A *double line wheel* (H, u, v) is the graph induced by a hole H and two distinct nonadjacent nodes u and v not in H such that (H, u) and (H, v) are both line wheels and the edges of H that see u or v are distinct and these four edges alternate between those that see u and those that see v.

Definition 3 *A* cap (H, v) *consists of a hole* H *and a node* v *not in* H *that has exactly two neighbors* a *and* b *in* H*, and* a *and* b *are adjacent. We say that* v *is the* tip *of* (H, v) *while* a *and* b *are the* attachments *of* v *in* H*.*

A hole or an anti-hole is *big* if it has at least 6 nodes. If (H, v) is a wheel or a cap, we say that (H, v) is *big* if H is big.

A graph G has a *skew partition* if the nodes of $V(G)$ can be partitioned into nonempty sets A, B, C, D such that every node of A is adjacent to every node of B and there is no edge between C and D. We say that the set $A \cup B$ is a *skew cutset* of G that *separates* C from D. When A or B has cardinality one, $A \cup B$ is called a *star cutset*. Chvátal [4] conjectured that a minimally imperfect graph cannot have a skew partition. Recently, Chudnovsky, Robertson, Seymour and Thomas [3] proved the conjecture. Previously, Robertson, Seymour and Thomas [11] had shown that the skew-partition conjecture holds for special types of skew partitions. Namely, a skew partition (A, B, C, D) is *good* if either C contains a node universal for A, or A contains a node with no neighbors in C. For example, a star cutset defines a good skew partition.

Theorem 4 (Robertson, Seymour and Thomas [11]) *No minimal imperfect graph contains a good skew partition.*

Theorem 4 generalizes previous results due to Hoàng, who showed that no minimally imperfect graph contains a T-cutset (i.e. a skew cutset in which both C and D contain a node universal for A) or a U-cutset (i.e. a skew cutset in which C contains a node universal for A and a node universal for B).

A Berge graph is *basic* if it belongs to one of the following four classes of perfect graphs: bipartite graphs, line graphs of bipartite graphs and their complements.

In this paper we will prove the following.

Theorem 5 *Let G be a Berge graph such that neither G nor $\overline{G}$ contains a proper wheel or a long prism. Then either G is basic, or G contains a good skew partition.*

Note that, since basic graphs are perfect, Theorems 4 and 5 imply that the Strong Perfect Graph Conjecture holds for graphs containing no proper wheel, long prism or their complements.

To prove Theorem 5, we will sometimes use the following lemma, due to Roussel and Rubio [12] and proved independently also by Robertson, Seymour and Thomas [11], who divulged it as a useful tool in the study of Berge graphs and named it *The Wonderful Lemma*.

Lemma 6 (The Wonderful Lemma) (Roussel and Rubio [12]) *Let G be a Berge graph where $V(G)$ can be partitioned into a co-connected set S and an odd chordless path $P = u, u', \dots, v', v$ of length at least 3 such that u and v are both universal for S. Then one of the following holds:*

(i) *An odd number of edges of P see S.*

(ii) *$|P| = 3$ and $S \cup \{u', v'\}$ contains an odd chordless co-path between u' and v'.*

(iii) *$|P| \geq 5$ and there exist two nonadjacent nodes x, x' in S such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a chordless path.*

The original proof of the Wonderful Lemma (with a slightly different statement) can be found in [12]. A similar proof is contained in [7].

Remark 7 *In a Berge graph containing no long prism, only the first two outcomes of the Wonderful Lemma are possible. Indeed, if P has length at least 5 and there exist two nonadjacent nodes x, x' in S such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a chordless path, then $V(P) \cup \{x, x'\}$ induces a long prism.*

We will also use the fact that the following structure contains an odd hole and therefore cannot occur in a Berge graph. Given a triangle a_1, a_2, a_3 and a node y distinct from a_1, a_2, a_3, a $3PC(a_1a_2a_3, y)$ is a graph induced by three chordless paths $P^1 = a_1, \dots, y$, $P^2 = a_2, \dots, y$ and $P^3 = a_3, \dots, y$, having no common nodes other than y and such that the only adjacencies between nodes of $P^i \setminus y$ and $P^j \setminus y$, for $i, j \in \{1, 2, 3\}$ distinct, are the edges of the triangle. Also, at most one of the paths P^1, P^2, P^3 is an edge.

2 Proof of Theorem 5

Let G be a Berge graph such that neither G nor $\overline{G}$ contains a proper wheel or a long prism.

2.1 Double line wheels

Lemma 8 *If G contains a double line wheel (H, u, v), then either G is the line graph of a bipartite graph or it contains a good skew partition.*

Proof: If $|H| > 6$, let ab and cd be the edges of H that see u and assume, by symmetry, that a and c are endnodes of a sector, denoted by H_{ac}, of length at least 4. Let st be the edge of H_{ac} that sees v. W.l.o.g. $a \neq s, t$. Then G contains a long prism $3PC(uab, stv)$.

Hence $|H| = 6$. Let $H = (a_1, ..., a_6, a_1)$ and assume that u is adjacent to a_1, a_2, a_4 and a_5 and that v is adjacent to a_2, a_3, a_5 and a_6. Let $H' = (a_1, u, a_4, a_3, v, a_6, a_1)$, $Q = (a_1, v, u, a_6, a_2, a_5, a_1)$ and $Q' = (a_2, a_5, a_3, u, v, a_4, a_2)$. Then H' is a 6-hole and (H', a_2, a_5) is a double line wheel, while Q and Q' are both 6-anti-holes and (Q, a_3, a_4) (Q', a_1, a_6) are both double line wheels in $\overline{G}$.

For $x \in V \setminus (V(H) \cup \{u, v\})$, we examine the adjacencies between x and (H, u, v). Since, as we just observed, the complement of a double line wheel is a double line wheel, then, by going to the complement, we can assume that x has at most four neighbors in (H, u, v).

Claim 1: If x has at most four neighbors in (H, u, v), then one of the following holds, up to the symmetries of (H, u, v):

(i) x has no neighbor in (H, u, v);
(ii) x is true or false twin of one of the nodes in (H, u, v) w.r.t. (H, u, v);
(iii) The only neighbors of x in (H, u, v) are a_1, a_3, a_4 and a_6; (1)
(iv) The only neighbors of x in (H, u, v) are a_1 and a_6;
(v) The only neighbors of x in (H, u, v) are a_1, a_2 and u.

Proof of Claim 1: Since G does not contain any proper wheel, then (H, x) can be a universal wheel, a twin wheel, a line wheel or $V(H) \cup x$ induces a triangle-free-graph or a cap. If (H, x) is a universal wheel, then x has more than four neighbors in (H, u, v). Assume that (H, x) is a twin wheel and let $N_H(x) = \{a_{i-1}, a_i, a_{i+1}\}$. Then x is adjacent to u if and only if a_i is a neighbor of u, otherwise, if C is the hole obtained from H by replacing a_i with x, then (C, u) is a proper wheel or an odd wheel. Similarly, x is adjacent to v if and only if a_i is a neighbor of v. But then x is a twin of a_i w.r.t. (H, u, v). Assume now that (H, x) is a line wheel. Since x has already four neighbors in (H, u, v), either x is a false twin of u or v or (iv) holds.
Assume next that $V(H) \cup x$ induces a cap. By symmetry, we can assume that x is adjacent to a_1. Assume first that x is adjacent to a_1 and a_6. If x is adjacent to both u, v, then (Q', x) is an odd wheel in $\overline{G}$. If x is adjacent to exactly one of u, v, say u, then x, u, a_2, v, a_6 induces a 5-hole. So (iv) must hold. Assume now that x is adjacent to a_1 and a_2. Then (v) must hold since, otherwise, (Q', x) is a proper wheel in $\overline{G}$.
Finally, assume that $V(H) \cup x$ induces a triangle-free graph. By symmetry we can assume that $V(H') \cup x$ also induces a triangle-free graph. If x has no neighbor in (H, u, v), (i) holds. Thus, by symmetry, we can assume that x is either adjacent to a_1 or to a_2. If x is

adjacent to a_1, then x is not adjacent to a_2, a_6 and u, and (Q, x) must be a twin wheel in $\overline{G}$, hence x is a twin of a_6 w.r.t. (H, u, v). Assume then that x is adjacent to a_2 and, by symmetry and by the previous case, assume x is not adjacent to a_1, a_3, a_4 and a_6. Also, x is not adjacent to a_5, else there is a 5-hole $(a_1, a_2, x, a_5, a_6, a_1)$. Hence (Q, x) must be a line wheel in $\overline{G}$, so x is adjacent to v but not to u, but now (Q', x) is a proper wheel in $\overline{G}$. This completes the proof of Claim 1.

We say that a graph G' is an *extended multi line wheel* if G' can be partitioned into sets $A_1, ..., A_6$, U, V and W with the property that every node in A_i is adjacent to every node in A_{i+1} (where the indices are taken modulo 6), every node in U (resp. V) (resp. W) is adjacent to every node in A_1, A_2, A_4 and A_5 (resp. A_2, A_3, A_5, A_6) (resp. A_1, A_3, A_4, A_6) and these are the only edges with endnodes in different sets of the partition. All the sets, except at most W, are nonempty.

Since G contains a double line wheel (H, u, v), then G contains an extended multi line wheel G' such that $a_i \in A_i$, $u \in U$ and $v \in V$. Assume G' is maximal (in terms of its node set) with this property.

Claim 2:

- Every node of Type (1)(iii) w.r.t. (H, u, v) belongs to W.
- If a node x of Type (1)(ii) w.r.t. (H, u, v) does not belongs to G', then x is a true twin of a node of degree 3 in (H, u, v), say a_1, and x is of Type (v) w.r.t. (H^*, u, v) for some 6-hole H^* obtained from H by replacing a_6 by a node $a_6^* \in A_6$.
- If a node x is of Type (1)(iv) w.r.t. (H, u, v), adjacent to a_1 and a_6, then x is universal for $A_1 \cup A_6$ and has no neighbor in $G' \setminus A_1 \cup A_6$.
- If a node x is of Type (1)(v) w.r.t. (H, u, v), adjacent to a_1, a_2 and u, then x is universal for $A_1 \cup A_2 \cup U$ and has no neighbor in $A_3 \cup A_4 \cup A_5 \cup V$.

Proof of Claim 2: By construction, every node of G' must be a twin of a node of (H, u, v) w.r.t. (H, u, v) or must be of Type (1)(iii). Suppose that some node x of Type (1)(iii) does not belong to W. Then either x is not adjacent to some node y in A_1, A_3, A_4 or A_6 or x is adjacent to some node y in A_2, A_5, U or V. Let (H^*, u^*, v^*) be the double line wheel obtained by adding y and removing the corresponding node of (H, u, v). Now x contradicts Claim 1 in (H^*, u^*, v^*) or in its complement. So the first part of Claim 2 holds. Now suppose that some node x of Type (1)(ii) does not belong to G'. By symmetry we can assume that x is a twin of a_1 or a_2 w.r.t. (H, u, v). If x is a twin of a_1, then either x is not adjacent to some node y in A_2, A_6 or U or x is adjacent to some node y in A_3, A_4, A_5 or V. Constructing (H^*, u^*, v^*) as above, we obtain a contradiction of Claim 1 unless y is in A_6. If x is a twin of a_2, constructing (H^*, u^*, v^*) as above, we obtain a contradiction of Claim 1 in all cases. The last two statements of Claim 2 follow similarly. This completes the proof of Claim 2.

By Claim 2, the nodes of $G \setminus G'$ partition into two sets X and Y as follows: X contains the nodes of $G \setminus G'$ that have no neighbor in $V(G')$ or are of Type (1)(iv) or (v) w.r.t. at least one double line wheel (H^*, u^*, v^*) where $H^* = a_1^*, \ldots, a_6^*, a_1^*$ with $a_i^* \in A_i$, $u^* \in U$, $v^* \in V$. The set Y contains the remaining nodes of $G \setminus G'$. In the complement graph $\overline{G}$, the nodes of Y have either no neighbor in $V(G')$ or are of Type (1)(iv) or (v) w.r.t. at least one double line wheel (Q^*, a_3^*, a_4^*) where $Q^* = a_1^*, v^*, u^*, a_6^*, a_2^*, a_5^*, a_1^*$ with $a_i^* \in A_i$, $u^* \in U$, $v^* \in V$.

Claim 3: Let $X_{1,2}$ be the set of nodes of X that are universal for $A_1 \cup A_2 \cup U$ and possibly adjacent to nodes of A_6 but to no other nodes of G'. Then there exists a node of A_6 that has no neighbor in $X_{1,2}$.

Proof of Claim 3: Suppose not. Since every node of A_6 has a neighbor in $X_{1,2}$ and every node of $X_{1,2}$ has a non-neighbor in A_6, there must exist $r, s \in A_6$ and $t, z \in X_{1,2}$ such that rt and sz are edges but rz and st are not. Indeed, it is immediate to verify that the statement is true if $|A_6| \leq 2$ or $|X_{1,2}| \leq 2$. By induction on $|A_6| + |X_{1,2}|$, given $z \in X_{1,2}$, either we are done by applying the inductive hypothesis to A_6 and $X_{1,2} \setminus z$, or A_6 contains a node s with no neighbors in $X_{1,2} \setminus z$ so z and s are adjacent. Let r be a non-neighbor of z in A_6 and t be a neighbor of r in $X_{1,2}$, then rt and sz are edges but rz and st are not.

If neither rs nor zt is an edge, consider the 6-hole $L = r, t, a_2, z, s, a_5, r$. Then (L, u) is a proper wheel. If exactly one of rs and zt is an edge, there is a 5-hole r, t, a_2, z, s, r or r, t, z, s, a_5, r. If both rs and zt are edges, the nodes in $\{r, s, t, z, a_2, a_3, a_4, a_5\}$ induce a long prism. This proves Claim 3.

Claim 4: If $x_i, x_j \in X$ are universal for A_i, A_{i+1} and for A_j, A_{j+1}, respectively, where $1 \leq i < j \leq 5$, and possibly to other nodes of G', then x_i and x_j are in different connected components of $G[X]$.

Proof of Claim 4: Suppose not. Choose a pair $x_i, x_j \in X$, $i < j$, with a shortest path P connecting them in $G[X]$. By the choice of P, the internal nodes of P have no neighbor in G'. By Claim 2, there exists a double line wheel (H^*, u^*, v^*) where $H^* = a_1^*, \ldots, a_6^*, a_1^*$ with $a_i^* \in A_i$, $u^* \in U$, $v^* \in V$, such that x_i and x_j are both of Type (1)(iv) or (v) w.r.t. (H^*, u^*, v^*). If $j - i \geq 2$, the nodes of $H \cup P$ induce a long prism. If $j = i + 1$, it is sufficient to consider the cases $j = 2$ and $j = 3$ by symmetry. If $j = 2$, the nodes of $V(P) \cup \{a_2, a_3, a_4, a_5, u, v\}$ induce a long prism. If $j = 3$, the nodes of $V(P) \cup \{a_2, a_4, a_5, u, v\}$ induce a long prism. This proves Claim 4.

Assume that Y is nonempty. By symmetry Y contains a node y universal for $A_2 \cup A_3 \cup A_5 \cup A_6$. Furthermore, if y is of Type (1)(iv) in $\overline{G}$, we can assume that, in G, y is universal for $A_1 \cup A_4$ and has no neighbor in $U \cup V$. If y is of Type (1)(v) in $\overline{G}$, we can assume that, in G, y has no neighbor in $A_1 \cup A_4 \cup V$. Finally, if all the nodes of Y are universal for G', choose y to be any node of Y. Let A be the co-connected component of Y containing y and let B be the set of nodes $A_2 \cup A_3 \cup A_5 \cup A_6$ together with the nodes of $G \setminus G'$ that are universal for A. By Claim 4 applied to $\overline{G}$, A is universal for

$A_2 \cup A_3 \cup A_5 \cup A_6$. Clearly, A is universal for $Y \setminus A$. Therefore, by Claim 4 applied to G, $A \cup B$ is a skew cutset separating V from $A_1 \cup A_4 \cup U$. By Claim 3 applied to $\overline{G}$, if y is of Type (1)(v) in $\overline{G}$, at least one node of U is universal for A. And if y is not of Type (1)(v) in $\overline{G}$, the nodes of A_1 are universal for A. So $A \cup B$ is a good skew cutset.

By the argument above applied to $\overline{G}$, if X is nonempty then G has a good skew partition. Hence we may assume that X and Y are both empty. If any of the sets $A_1, ..., A_6$, U, V, W has cardinality greater than one, then G has a star cutset. So, if G has no good skew partition, G' is a multi line wheel. □

2.2 Line wheels

Lemma 9 *If G contains a line wheel, then G is either the line graph of a bipartite graph, or it contains a good skew partition.*

Proof: By Lemma 8, if G contains a double line wheel then we are done. Hence we can assume that G does not contain a double line wheel or its complement.

Assume that G contains a line wheel (H, v). Let ab and cd be the two edges of H that see v. Assume, w.l.o.g., that a and c are the endpoints of a sector of (H, v), denoted by H_{ac} while b and d are the endpoints of the other sector of (H, v), denoted by H_{bd}. Let A be a maximal co-connected set in $G \setminus H$ such that $v \in A$ and A sees both ab and cd (note that the nodes in A are either centers of line wheels w.r.t. H or they are universal for H). Let B be the set of nodes that are universal for A in $G \setminus (A \cup \{b, c\})$. If $H_{ac} \setminus a$ and $H_{bd} \setminus d$ lie in distinct connected components of $G \setminus (A \cup B)$, then let C be the connected component containing $H_{ac} \setminus a$ and $D = V(G) \setminus (A \cup B \cup C)$. Then (A, B, C, D) is a skew partition, $c \in C$ and $b \in D$ are both universal for A, hence $A \cup B$ is a T-cutset and we are done.

We will show that it must be the case that $H_{ac} \setminus a$ and $H_{bd} \setminus d$ lie in distinct connected components of $G \setminus (A \cup B)$. Assume not. Then there exists a path $P = x_1, \ldots, x_n$ in $G \setminus (V(H) \cup A \cup B)$ such that x_1 has a neighbor in $H_{ac} \setminus a$, x_n has a neighbor in $H_{bd} \setminus d$ and no intermediate node of P has a neighbor in $H \setminus \{a, d\}$.

Claim 1: a and d cannot both have a neighbor in $P \setminus \{x_1, x_n\}$.

Proof of Claim 1: Assume not and let x_i and x_j $(2 \leq i, j \leq n-1)$ be two nodes at minimum distance in P such that x_i is adjacent to a and x_j is adjacent to d. Then $F = a, x_i, P_{x_i x_j}, x_j, d$ is an odd path, otherwise $C = (a, F, d, c, H_{ac}, a)$ would be an odd hole. Since a and d are both universal for A and no node in $F \setminus \{a, d\}$ is universal for A, then, by Lemma 6 and Remark 7, F has length 3 and there exists an odd chordless co-path Q between x_i and x_j contained in $A \cup \{x_i, x_j\}$. Thus $C = (b, x_i, Q, x_j, b)$ is an odd anti-hole, a contradiction. This completes the proof of Claim 1.

Claim 2: No node of P has a neighbor in both H_{ac} and H_{bd}.

Proof of Claim 2: By contradiction, assume x_i $(1 \leq i \leq n)$ has a neighbor in both H_{ac} and H_{bd}.

Case 1: x_i has two nonadjacent neighbors in H.

Then (H, x_i) must be either a universal wheel, a twin wheel, a line wheel or $H \cup x_i$ is a triangle-free graph. If x_i sees both ab and cd, then, since x_i is not universal for A, $A \cup x_i$ is a co-connected set that sees both ab and cd, contradicting the maximality of A. Let s and s' be the neighbors of x_i in H_{ac} which are closest to, respectively, a and c in H_{ac}. Let t and t' be the neighbor of x_i in H_{bd} which are closest to, respectively, d and b in H_{bd}. Suppose that s, s', t, t' are all contained in $\{a, b, c, d\}$. Since x_i does not see both or exactly one of ab, cd, we can assume w.l.o.g. that $s = a$ and $t' = d$. Hence $(a, b, H_{bd}, d, x_i, a)$ is an odd hole. Therefore we can assume w.l.o.g. that s is in the interior of H_{ac}. Suppose $t = b$. Then s' is in the interior of H_{ac}, otherwise $(b, H_{bd}, d, c, x_i, b)$ is an odd hole. Hence, by symmetry, we can assume that $t \neq b$. Let H_{as} and H_{dt} be, respectively, the paths between a and s and d and t in $H \setminus b$.

If the distance between s and t in H is even, then $F = a, H_{as}, s, x_i, t, H_{dt}, d$ is an odd chordless path (since a and d have odd distance in H). Since a and d are both universal for A and no node in $F \setminus \{a, d\}$ is universal for A, then by Lemma 6 and Remark 7 F has length 3, hence $F = a, s, x_i, d$ and there exists an odd chordless co-path Q between x_i and s contained in $A \cup \{x_i, s\}$. If b or c is adjacent to neither x_i nor s, then $C = (b, q, Q, r, b)$ or $C' = (c, q, Q, r, c)$ is an odd anti-hole, a contradiction. Hence both b and c are adjacent to x_i or s. Since b cannot be adjacent to s, then b is adjacent to x_i. x_i is adjacent to s, b and d in H, hence (H, x_i) must be a triangle-free wheel, therefore x_i is not adjacent to c, thus c is adjacent to s. But then $V(Q) \cup \{a, b, c, d\}$ induces the complement of a long prism, namely a $3PC(acx_i, dbs)$, a contradiction.

Therefore s and t must have odd distance in H. One can readily verify that this implies that (H, x_i) has to be a line wheel. Furthermore, given s' and t' the neighbors of x_i in H adjacent to, respectively, s and t, ss' must be an edge of H_{ac} and tt' must be an edge of H_{bd}. Since A contains node v and (H, v) is a line wheel, and since $A \cup x_i$ is co-connected, then let y be a node in A such that (H, y) is a line wheel and y has minimum distance from x_i in $\overline{G}[A \cup x_i]$. Let Q be a shortest co-path between x_i and y in $A \cup x_i$. If Q has length one, then (H, x_i, y) is a double line wheel, a contradiction. Hence Q has length strictly greater than one. Since x_i is not adjacent to both b and d, we may assume, w.l.o.g., that x_i is not adjacent to d. Since s is adjacent to neither b nor d, then, since every node of $Q \setminus \{x_i, y\}$ is universal for A, $C = (s, y, Q, x_i, d, s)$ is an anti-hole, hence Q must have odd length. If H has length greater than 6, then either H_{ac} or H_{bd} had length at least 4, hence there exists a node w in $H \setminus \{a, b, c, d, s, s', t, t'\}$. Since such a node is universal for $Q \setminus \{x_i, y\}$ and Q has odd length strictly greater than one, then $C = (w, x_i, Q, y, w)$ is an odd anti-hole, a contradiction. Hence H has length 6 and $H = (a, b, t, d, c, s)$, where $b = t'$ and $c = s'$. But then $V(Q) \cup \{a, d, s, t\}$ induces the complement of a long prism, namely a $3PC(sty, dax_i)$.

Case 2: x_i has only two neighbors in H and they are adjacent.

Then, w.l.o.g., $i = 1$ and x_1 is adjacent to both c and d. Then $n > 1$. Let x_j be the node of lowest index in $P \setminus x_1$ to have a neighbor in H. Node x_j cannot be adjacent to c by the definition of P. Suppose x_j is not adjacent to d. If x_j has a unique neighbor t in H, then there is a $3PC(cdx_1, t)$, if x_j has two nonadjacent neighbors in H, then there is a $3PC(cdx_1, x_j)$, hence x_j has exactly two neighbors t and t' in H and they are adjacent, but then there is a long prism, namely a $3PC(cdx_i, tt'x_j)$, a contradiction. So x_j is adjacent to d. Suppose that a has a neighbor in P. Let x_k be the node of lowest index adjacent to a. By Claim 1, $k = n$. Since x_n has a neighbor in H_{bd}, then by Case 1, x_n is adjacent to a and b but no other node of H. Hence the nodes of H together with a subset of the nodes of $P_{x_j x_n}$ induce a $3PC(abx_n, d)$. Therefore a has no neighbors in P. Let F be the shortest path between c and b in $P \cup H \setminus \{a, d\}$. Then F is an odd path, otherwise $H' = (c, F, b, a, H_{ac}, c)$ would be an odd hole. Since b and c are universal for A and no node in F is universal for A, then by Lemma 6 and Remark 7 F has length 3, hence $F = c, x_1, x_2, b$ and there exists an odd chordless co-path Q between x_1 and x_2 contained in $A \cup \{x_1, x_2\}$. But then $C = (a, x_1, Q, x_2, a)$ is an odd anti-hole, a contradiction. This concludes the proof of Claim 2.

By Claim 2, $n > 1$, x_1 has no neighbors in H_{bd} and x_n has no neighbors in H_{ac}. By Claim 1 and 2, we can assume that a has no neighbors in $P \setminus x_1$. Suppose that (H, x_1) is not a cap. Let t be the neighbor of x_1 closest to a in H_{ac} and t' be the neighbor of x_1 closest to c in H_{ac} (possibly $t = t'$). Let H_{at} and $H_{ct'}$ be, respectively, the paths between a and t and c and t' in H_{ac}. Since (H, x_1) is not a cap, then H_{at} and $H_{ct'}$ have the same parity. Let s be the neighbor of x_n closest to b in H_{bd}, and H_{bs} the path between s and b in H_{bd}. Since a has no neighbors in $P \setminus x_1$, then $H' = (a, H_{at}, t, x_1, P, x_n, s, H_{bs}, b, a)$ is a hole, hence H' has even length. Since H_{at} and $H_{ct'}$ have the same parity, then $F = c, H_{ct'}, t', x_1, P, x_n, s, H_{bs}, b$ is an odd chordless path. Since b and c are universal for A and no intermediate node in F is universal for A, then by Lemma 6 and Remark 7 F has length 3, hence $F = c, x_1, x_2, b$, and there exists an odd chordless co-path Q between x_1 and x_2 in $A \cup \{x_1, x_2\}$. If a is not adjacent to x_1, then $C = (a, x_1, Q, x_2, a)$ is an odd anti-hole. Analogously, if d is not adjacent to x_2, then $C' = (d, x_1, Q, x_2, d)$ is an odd anti-hole. Hence a is adjacent to x_1 and d is adjacent to x_2. But then $V(Q) \cup \{a, b, c, d\}$ induces the complement of a long prism, namely a $3PC(x_1bd, x_2ac)$, a contradiction.

Therefore (H, x_1) is a cap and, by symmetry, (H, x_n) is a cap. Let t and t' be the attachments of x_1 in H and s and s' be the attachments of x_n in H. If a has a neighbor in $P \setminus x_1$, then by Claim 1 no node of $P \setminus \{x_1, x_n\}$ is adjacent to d, hence there exists a $3PC(x_n ss', a)$, a contradiction. So a has no neighbors in $P \setminus x_1$ and, by symmetry, d has no neighbors in $P \setminus x_n$, but then $V(H) \cup V(P)$ induces a long prism, namely a $3PC(x_1tt', x_nss')$, a contradiction.

□

2.3 Big universal wheels

Lemma 9 implies that Theorem 5 holds whenever G or $\overline{G}$ contains a line wheel. Hence, from now on, we will assume that G and $\overline{G}$ do not contain any line wheel.

Lemma 10 *If G contains a big universal wheel, then G has a T-cutset.*

Proof: Assume G contains a universal wheel (H, x) and let A be a maximal co-connected set of $G \setminus V(H)$ such that every node in A is universal for $V(H)$. Consider a bicoloring of the nodes of H obtained by coloring the nodes of H red and blue in such a way that two nodes have the same color if and only if they have even distance in H. Let y be a node in $G \setminus (V(H) \cup A)$ that is not universal for A such that y has two nonadjacent neighbors in H. We will show that (H, y) is a triangle-free wheel and y is universal for either the red or the blue nodes of H. Let u be a node in A that is not adjacent to y. By the maximality of A, y is not universal for $V(H)$, hence y has two consecutive nonadjacent neighbors s and t in H. Let H_{st} be a path between s and t in H containing no neighbors of y. Then s and t have distance 2 in H_{st}, otherwise $H' = (y, s, H_{st}, t, y)$ is a big hole and (H', u) is a proper wheel (since u is adjacent to every node but y in H'). Hence (H, y) is not a twin wheel, so $H \cup y$ is a triangle-free graph in which every pair of consecutive neighbors of y in H has distance two in H. Hence y is either universal for the red or for the blue nodes of H. So we can partition the nodes in $G \setminus (V(H) \cup A)$ that have two nonadjacent neighbors in H and that are not universal for A into sets Δ_R and Δ_B, where every node in Δ_R (resp. Δ_B) is universal for the red (resp. blue) nodes of H and has no blue (resp. red) neighbor in H. Next, we will show that either Δ_R or Δ_B is empty. Assume not and let r and b be two nodes in Δ_R and Δ_B respectively. Let st and $s't'$ be two nonadjacent edges of H where s, s' are red and t, t' are blue. If r and b are not adjacent, then $H' = (r, s, t, b, t', s', r)$ is a 6-hole and (H', u) is a proper wheel or a line wheel for every node u in A that is not adjacent to r or b. So r and b are adjacent and, since neither of them is universal for A, then $\overline{G}[A \cup \{r, b\}]$ contains a chordless path Q between r and b. $\overline{G}[V(Q) \cup \{s, s', t, t'\}]$ is a long prism, namely a $3PC(rtt', bs's)$, a contradiction.

Therefore we may assume, w.l.o.g., that every node in $G \setminus (V(H) \cup A)$ that has two nonadjacent neighbors in H and that is not universal for A is universal for the blue nodes of H and has no red neighbor in H. Let a be a red node of H and let b_1 and b_2 be its neighbors in H. Let B be the set of all nodes in $G \setminus (A \cup V(H)) \cup \{b_1, b_2\}$ that are universal for A. If a and $V(H) \setminus \{a, b_1, b_2\}$ lie in distinct connected components of $G \setminus (A \cup B)$, let C be the connected component of $G \setminus (A \cup B)$ containing a and $D = V(G) \setminus (A \cup B \cup C)$. Then (A, B, C, D) is a skew partition and, given a node t in $V(H) \setminus \{a, b_1, b_2\}$, then $t \in D$ and both a and t are universal for A, hence $A \cup B$ is a T-cutset. Hence we may assume that there exists a chordless path $P = x_1, \ldots, x_n$ in $G \setminus (A \cup B \cup V(H))$ such that x_1 is adjacent to a, x_n has a neighbor in $V(H) \setminus \{a, b_1, b_2\}$ and no intermediate node has a neighbor in $V(H) \setminus \{b_1, b_2\}$. Note that x_1 does not have two nonadjacent neighbors in H, hence $n > 1$. Also, no node in $P \setminus x_n$ has two nonadjacent neighbors in H. Note that b_1 and b_2 cannot both have neighbors in $P \setminus x_n$, otherwise let x_i and x_j be neighbors of b_1 and b_2, respectively, in $P \setminus x_n$ such that x_i and x_j are closest possible in $P \setminus x_n$. Then $H' = H \setminus a \cup V(P_{x_i x_j})$ is a hole and, for any node $u \in A$ that is not universal for $P_{x_i x_j}$, (H', u) is a proper wheel. Thus we may assume that b_1 has no neighbors in $P \setminus x_n$.

If x_n has only blue neighbors in H, then let t be the closest neighbor of x_n to b_1 and $H_{b_1 t}$ be the chordless path between b_1 and t in $H \setminus a$. $H' = (a, x_1, P, x_n, t, H_{b_1 t}, b_1, a)$ is a hole of even length, hence, since t and a have odd distance, P is an odd path. Let s be a neighbor of x_n distinct from b_1 and b_2. Then $F = a, x_1, P, x_n, s$ is an odd chordless path and $F \setminus \{a, s\}$ does not have any node universal for A. By Lemma 6 and Remark 7, F has length 3 and $A \cup \{x_1, x_2\}$ contains an odd chordless co-path Q between x_1 and x_2. Let w be a red node distinct from a. Then $C = (a, x_1, Q, x_2, a)$ is an odd anti-hole, a contradiction.

So x_n has a red neighbor in H, therefore x_n does not have two nonadjacent neighbors in H. Let t be the unique red neighbor of x_n in H. Since $|H| \geq 6$, t is not adjacent to b_1 or b_2, say b_i for $i = 1$ or 2. Let $H_{b_i t}$ be the path between b_i and t in $H \setminus a$. Since t and b_i have distinct colors, $H_{b_i t}$ has odd length, so $|H_{b_i t}| \geq 3$. If x_n has no neighbors in $H_{b_i t}$, then let u be a node of A that is not adjacent x_n. If b_i has no neighbors in P, then $H' = (t, H_{b_i t}, b_i, a, x_1, P, x_n, t)$ is a big hole and (H', u) is a proper wheel. Otherwise let x_j be the node of highest index in P adjacent to b_i. Then $H'' = (t, H_{b_i t}, b_i, x_j, P_{x_j x_n}, x_n, t)$ is a big hole and (H'', u) is a proper wheel. So x_n has exactly two neighbors s and t in H, s and t are adjacent and s is in $H_{b_i t}$, so $s \neq b_1, b_2$. So s and t have no neighbor in $P \setminus x_n$. Let x_j be a node of highest index with a neighbor in $\{a, b_2\}$. We already observed that x_j cannot have two nonadjacent neighbors in H. If x_j has a unique neighbor v in $\{a, b_2\}$, then there is a $3PC(x_n st, v)$. So $j = 1$ and x_1 is adjacent to b_2 and there is a long prism $3PC(x_1 a b_2, x_n ts)$, a contradiction. $\square$

2.4 Caps

By Lemmas 9 and 10, we may assume G and $\overline{G}$ do not contain any long prism or any big wheel except twin wheels and triangle-free wheels.

Lemma 11 *If G contains a big cap, then G has a good skew-partition.*

Before proving Lemma 11, we shall prove the following three lemmas.

Lemma 12 *Let Γ be a Berge graph. If Γ and $\overline{\Gamma}$ do not contain any big wheel (H, x) where x has more than $|H|/2$ neighbors in H, then Γ does not contain both a big hole and a big anti-hole.*

Proof: Assume, by contradiction, that Γ contains a hole H and an anti-hole A, where $n = |H| \geq 6$ and $m = |A| \geq 6$.

Assume first that $V(H) \cap V(A) \neq \emptyset$. It is immediate to verify that $|V(H) \cap V(A)| \leq 4$ and $V(H) \cap V(A)$ induces a chordless path or the complement of a chordless path. W.l.o.g., assume $P = \Gamma[V(H) \cap V(A)]$ is a chordless path, and let $k = |V(H) \cap V(A)|$.

By assumption, every node in $V(A) \setminus V(H)$ has at most $n/2$ neighbors in H, hence there are at most $(m-k)n/2$ edges between $V(H)$ and $V(A) \setminus V(H)$. Since P has k nodes and $k-1$ edges, and every node in A has exactly $m-3$ neighbors in A, then between $V(A) \cap V(H)$ and $V(A) \setminus V(H)$ there are exactly $k(m-3) - 2(k-1) = km - 5k + 2$ edges, hence there are at most $(m-k)n/2 - (km - 5k + 2)$ edges between $V(A) \setminus V(H)$ and $V(H) \setminus V(A)$.

Analogously, every node in $V(H) \setminus V(A)$ has at least $m/2$ neighbors in $V(A)$, hence there are at least $(n-k)m/2$ edges between $V(H)\setminus V(A)$ and $V(A)$. Also, there are exactly 2 edges between $V(A) \cap V(H)$ and $V(H) \setminus V(A)$, hence there are at least $(n-k)m/2-2$ edges between $V(A) \setminus V(H)$ and $V(H) \setminus V(A)$. Therefore

$$\frac{(n-k)m}{2} - 2 \le \frac{(m-k)n}{2} - km + 5k - 2$$

implying $n + m \le 10$, that is a contradiction since $n, m \ge 6$.

Hence we may assume that A and H are node disjoint. Every node in A has at most $n/2$ neighbors in H, hence there are at most $mn/2$ edges between $V(A)$ and $V(H)$. Every node in H has at least $m/2$ neighbors in A, hence there are at least $mn/2$ edges between $V(A)$ and $V(H)$, therefore there are exactly $mn/2$ edges between $V(A)$ and $V(H)$, every node in $V(A)$ has exactly $n/2$ neighbors in $V(H)$ and every node in $V(H)$ has exactly $m/2$ neighbors in $V(A)$. Let x be a node of A. If (H, x) is not a triangle-free wheel, then $H \cup x$ contains a hole H' of length at least 6 containing x, but H' and A have one node in common and we already showed that this is not possible. Hence, for every $x \in V(A)$, (H, x) is a triangle-free wheel. Let X and Y be the two stable sets of size $n/2$ partitioning $V(H)$, then for every $x \in V(A)$ either x is universal for X and has no neighbors in Y, or vice-versa. Since every node in H has neighbors in A and A is co-connected, there exist two non adjacent nodes x and y in A such that x is universal for X and has no neighbors in Y while y is universal for Y and has no neighbors in X. Let $x'y'$ and $x''y''$ be two nonadjacent edges of H with $x', x'' \in X$ and $y', y'' \in Y$, then $H' = (x, x', y', y, y'', x'', x)$ is a 6-hole and H' and A have two nodes in common, a contradiction. □

Note that, since G and $\overline{G}$ do not contain any big wheel except twin wheels and triangle-free wheels, then neither G nor $\overline{G}$ contains a big wheel (H, x) where x has more than $|H|/2$ neighbors in H. Hence, by Lemma 12, we may assume, w.l.o.g., that G contains no big anti-hole.

Lemma 13 *Assume G contains a cap (H, x) and let a, b denote the attachments of x in H. Let $P = x_1, \ldots, x_n$ be a direct connection between x and $V(H) \setminus \{a, b\}$ contained in $G \setminus H$ such that no node of P is adjacent to a. Then x_1 is adjacent to b and no other node of P is adjacent to b.*

Proof: Assume first that b has no neighbors in P. If x_n has a unique neighbor t in H, then there is a $3PC(abx, t)$. If x_n has two nonadjacent neighbors in H, then there is a $3PC(abx, x_n)$. Hence x_n has exactly two neighbors t and t' in H and they are adjacent, but then either G contains a long prism $3PC(abx, tt'x_n)$, or $|H| = 4$, $n = 1$ and $V(H) \cup \{x, x_1\}$ induces an anti-hole of length 6. Therefore b has a neighbor in P. If $n = 1$ we are done, hence we may assume $n \ge 2$. Let t be the closest neighbor of x_n to a in H and H_{ta} be the path between a and t in $H \setminus b$, let $H' = (x, x_1, P, x_n, t, H_{ta}, a, x)$. Since $n \ge 2$, H' has length at least 6 and b is adjacent to a, x and some other node of P, so (H', b) is a big wheel that is not triangle-free, hence it must be a twin wheel, therefore b must be adjacent to x_1 and no other node of P. □

Lemma 14 *Assume G contains a connected set S, a chordless co-path $Q = y_1, ..., y_n$ disjoint from S such that y_1 and y_n have no neighbors in S and for every i, $2 \leq i \leq n-1$, y_i has a neighbor in S. Then $n \leq 4$.*

Proof: Assume, by contradiction, that $n \geq 5$. Since y_2 and y_3 have at least a neighbor in S and S is connected, there exists a chordless path $Q = y_2, q_1, \dots, q_m, y_3$ between y_2 and y_3 whose interior is contained in S. Q has even length, otherwise y_n, y_2, Q, y_3 is an odd hole, and Q has length at least 4, otherwise $Q = y_2, q_1, y_3$ and, given h the smallest index such that y_h is nonadjacent to q_1 (h is well defined since yn has no neighbors in S), then $y_1, P_{y_1 y_h}, y_h, q_1, y_1$ is an antihole of length at least 6, a contradiction. Hence $Q' = y_2, Q, y_3, y_1$ is an odd path of length at least 5 and $X = V(P) \setminus \{y_1, y_2, y_3\}$ is a co-connected set universal for y_1 and y_2. By Lemma 6 and Remark 7, the interior of Q' must contain a node universal for X, a contradiction since y_n has no neighbors in the interior of Q and y_3 is not adjacent to y_4. $\square$

Proof of Lemma 11:

Consider a big cap (H, v) and let a, b be the attachments of x in H. Let $a_0 = a$, P^0 be the path induced by $V(H) \setminus b$, $A_0 = \{a_0\}$ and $S_0 = V(P^0) \setminus a_0$.

Let $P^0, \dots, P^k$ be a sequence of chordless paths in G, where $P^i = x^i_1, \dots, x^i_{l_i}$. For every i, $1 \leq i \leq k$, let $a_i = x^i_1$, $A_i = A_{i-1} \cup a_i$ and $S_i = S_{i-1} \cup V(P^i) \setminus a_i$. Assume that the sequence $P^0, \dots, P^k$ satisfies the following properties:

1. P^i is a direct connection between x and S_{i-1} contained in $G \setminus (A_{i-1} \cup \{b\})$ such that no node in P^i is universal for A_{i-1},

2. For every i, $0 \leq i \leq k$, a_i is adjacent to b.

We will prove that, if $P = x_1, ..., x_n$ is a direct connection between x and S_k contained in $G \setminus (A_k \cup b)$ such that no node in P is universal for A_k, then x_1 is adjacent to b.

Note that this implies Lemma 11: obviously, P^0 is a sequence satisfying properties 1 and 2 above, hence we can consider a sequence $P^0, \dots, P^k$ ($k \geq 0$), that is the longest possible. Let $A = A_k$ and B be the set of all nodes in $V \setminus x$ that are universal for A. If $A \cup B$ is not a skew cutset that separates x from S_k, then there exists a direct connection $P = x_1, ..., x_n$ between x and S_k contained in $G \setminus (A_k \cup b)$ such that no node in P is universal for A_k. Since x_1 is adjacent to b, we can choose $P^{k+1} = P$. Now $P^0, \dots, P^{k+1}$ is a sequence satisfying properties 1 and 2, contradicting the maximality of k. Hence $A \cup B$ is a skew cutset that separates x from S_k. Let C be the connected component of $G \setminus (A \cup B)$ containing x and $D = V(G) \setminus (A \cup B \cup C)$. Then (A, B, C, D) is a skew partition and x is universal for A, hence it is a good skew partition.

Observe that, by construction, for every $0 \leq i \leq k$, A_i is a co-connected set and S_i is connected. Moreover, x has no neighbors in S_k and every node in A_i has a neighbor in S_i.

Claim 1: For every j, $1 \leq j \leq k$, and for every node $y \in S_j$, if y is universal for A_{j-1}, then y is the only neighbor of a_0 in $H \setminus b$.

Proof of Claim 1: By construction, for every i such that $1 \leq i \leq j$, no node in P^i is universal for A_{j-1}, hence y must be the node in P^0 adjacent to a_0.

Claim 2: For every i, $0 \leq i \leq k$, b does not see any edge of P^i.

Proof of Claim 2: The statement is trivial for $i = 0$ and it follows immediately by Lemma 13 for $i = 1$. Hence we may assume $i \geq 2$. Assume, by contradiction, that b sees an edge $x^i_j x^i_{j+1}$ of P^i. We will show that every node of A_{i-1} is adjacent to x^i_j or x^i_{j+1}. Assume not, then there exists $a_h \in A_{i-1}$ such that a_h is not adjacent to x^i_j and x^i_{j+1}. Let y be the neighbor of a_h in the chordless path $x, x^i_1, P^i_{x^i_1 x^i_{j-1}}, x^i_{j-1}$ closest to x^i_j in P^i (y is well defined since every node in A_k is adjacent to x) and let F be the path from y to x^i_j in $x, x^i_1, P^i_{x^i_1 x^i_j}, x^i_j$. Let F' be a chordless path between a_h and x^i_{j+1} in the graph induced by $S_{i-1} \cup (a_h \cup V(P^i_{x^i_{j+1} x^i_{l_i}}))$. By construction, no node in P^i except $x^i_{l_i}$ has a neighbor in S_{i-1}, hence $C = (a_h, y, F, x^i_j, x^i_{j+1}, F', a_h)$ is a big hole and b is adjacent to a_h, x^i_j and x^i_{j+1} in C, hence (C, b) is a big wheel that is neither a triangle-free nor a twin wheel, a contradiction.
Hence every node in A_{i-1} is adjacent to x^i_j or x^i_{j+1} but no node in P^i is universal for A_{i-1}, so there exists a chordless co-path $Q = y_1, \ldots, y_m$ in $G[A_{i-1}]$ such that y_1 is adjacent to x^i_j but not x^i_{j+1}, y_m is adjacent to x^i_{j+1} but not x^i_j and all the intermediate neighbors of Q are adjacent to both x^i_j and x^i_{j+1}. If $j > 1$, then x^i_j and x^i_{j+1} are not adjacent to x, hence $(x, x^i_{j+1}, y_1, Q, y_m, x^i_j, x)$ is a big anti-hole. Therefore $j = 1$, and $Q' = x, x^i_2, y_1, Q, y_m, x^i_1$ is a co-path of length at least 4. Let $S = (S_{i-1} \cup V(P^i)) \setminus \{x^i_1, x^i_2\}$. S is connected and neither x nor x^i_1 have neighbors in S, while, by construction, every intermediate node of Q' has a neighbor in S. Now Q' and S contradict Lemma 14. This completes the proof of Claim 2.

We will prove Lemma 11 by induction on k. If $k = 0$, then we are done by Lemma 13. Let us now assume, by induction, that the statement is satisfied for every big cap (H, x), and for every sequence $P_0, ..., P_j$ satisfying properties 1 and 2, whenever $j \leq k - 1$. Note that P must contain a node that is universal for A_{k-1}, otherwise $(V(P) \cup V(P^k)) \setminus a_k$ contains a direct connection P' from x to S_{k-1} and no node in P' is universal for A_{k-1} so, by induction, the first node of P', which is x_1, is adjacent to b and we are done. Let us assume, by contradiction, that x_1 is not adjacent to b.

Claim 3: No node of P is adjacent to a_k.

Proof of Claim 3: Assume, by contradiction, that a_k has a neighbor in P. Then, since by the argument above P contains a node universal for A_{k-1}, every node in A_k has a neighbor in P. For every $0 \leq i \leq k$, let $h(i)$ be the minimum index such that a_i is adjacent to $x_{h(i)}$ and let $h = \max_{0 \leq i \leq k} h(i)$. Since no node of P is universal for A_k, $h \geq 2$. If $h = 2$, then every node in A_k is adjacent to x_1 or x_2 but neither x_1 or x_2 are universal for A_k, hence A_k contains a chordless co-path $Q = y_1, \ldots, y_m$ such that y_1 is adjacent to

x_1 but not to x_2, y_m is adjacent to x_2 but not x_1 and every intermediate node of Q is adjacent to both x_1 and x_2. Therefore $Q' = x, x_2, y_1, Q, y_m, x_1$ is a co-path of length at least 4. Let $S = S_k \cup (V(P) \setminus \{x_1, x_2\})$. S is a connected set and neither x_1 nor x has a neighbor in S, while every intermediate node of Q' has a neighbor in S. Therefore Q' and S contradict Lemma 14. Hence we can assume $h \geq 3$. Let $a_j \in A_k$ be such that $h(j) = h$. Then $C = (a_j, x, x_1, P_{x_1x_h}, x_h, a_j)$ is a big hole. Since b is adjacent to both x and a_j, then (C, b) is either a cap or a twin wheel. If (C, b) is a cap, let F be a shortest path between x_h and b in $S_k \cup V(P_{x_hx_n}) \cup b$, then $C' = (x, x_1, P_{x_1x_h}, x_h, F, b, x)$ is a hole and a_j is adjacent to x, b and x_h in C', therefore (C', a_h) is a big wheel that is neither a triangle-free wheel nor a twin wheel. Hence (C, b) must be a twin wheel so b is adjacent either to x_1 or to x_h. In the former case we are done. Now assume that b is adjacent to x_h. $C' = (x, x_1, P_{x_1x_h}, x_h, b, x)$ is a big hole. Since every node in A_k has a neighbor in $P_{x_1x_h}$, then (C', a_i) must be a twin wheel for every $a_i \in A_k$, hence every node in A_k is adjacent to x_1 or x_h. Since no node in P is universal for A_k and A_k is co-connected, there exists two nonadjacent nodes a_s and a_t in A_k such that a_s is adjacent to x_1 and not to x_h, and a_t is adjacent to x_h and not to x_1. $C'' = (x, x_1, P_{x_1x_h}, x_h, a_t, x)$ is a big hole and (C'', a_s) is a cap where x, x_1 are the attachments of a_s in C''. By construction S_k contains a direct connection F form a_s to $C'' \setminus \{x_1, x\}$, but no node in S_k is adjacent to x or to x_1, hence F contradicts Lemma 13. This completes the proof of Claim 3.

Claim 4: b does not have any neighbor in P.

Proof of Claim 4: Assume by contradiction that x_j, for some $1 \leq j \leq n$, is adjacent to b. Let F be a chordless path between a_k and x_1 in $S_k \cup V(P)$. Since a_k has no neighbor in P, then P is a subpath of F and $C = (x, x_1, F, a_k, x)$ is a hole, b is adjacent to x, a_k and x_j in P but x_j is not adjacent to x (otherwise $j = 1$ and b is adjacent to x_1) and x_j is not adjacent to a_k (because, by Claim 3, a_k has no neighbors in P), hence (C, b) is a big wheel that is neither a twin wheel nor a triangle-free wheel, a contradiction.

Claim 5: $S_k \cup V(P)$ contains a chordless path $F = y_1, \ldots, y_{m+1}$ between x_1 and b such that a_k is adjacent to y_m and no other node in F and y_1 is universal for A_{k-1}.

Proof of Claim 5: Let $F = y_1, \ldots, y_{m+1}$ be a chordless path between x_1 and b in $S_k \cup V(P)$, where $y_1 = x_1$ and $y_{m+1} = b$. Note that, since b is not adjacent to x_1, then $C = (x, x_1, F, b, x)$ is a hole. Since b has no neighbor in P, then P is a subpath of F and y_m is in S_k. By Claim 1, y_m is not universal for A_{k-1} since y_m is adjacent to b.

Since P contains a node x_j universal for A_{k-1} and (C, a_i) is a wheel that is not triangle-free for each $a_i \in A_{k-1}$, every node in A_{k-1} must be adjacent to x_1. If y_m is adjacent to a_k we are done. Otherwise (C, a_k) is a cap where x, b are the attachments of a_k in C. Let $Z = z_1, \ldots, z_l$ be a direct connection form a_k to $V(C) \setminus \{x, b\}$, contained in $S_k \setminus V(F)$. Since no node in Z is adjacent to x, then by Lemma 13 z_1 is adjacent to b and no node in $Z \setminus z_1$ is adjacent to b. Hence, given y the closest neighbor of z_l to x_1 in F, $F' = x_1, F_{x_1y}, z_l, Z, z_1, b$ is a chordless path between x_1 and b in $S_k \cup V(P)$. Note that $F' = y'_1, \ldots, y'_{m'+1}$, where $y'_1 = x_1$ and $y'_{m'+1} = b$ and a_k is adjacent to $y'_{m'}$ and no other node in F'. Thus F' satisfies the statement of Claim 5.

Let j, $0 \leq j \leq k$, be the index such that $y_m \in V(P^j)$. Note that, since y_m and b are adjacent to a_k, then, by Claim 2, $y_m \neq x_2^k$, hence $j < k$. This implies that P^k consist of only one node, namely a_k.

Claim 6: a_k is universal for A_{k-2} and a_k is not adjacent to a_{k-1}.

Proof of Claim 6: If a_k is universal for A_{k-2}, then by construction a_k is not adjacent to a_{k-1}. Assume, by contradiction, that a_k is not universal for A_{k-2}. Then $(V(P^k) \cup V(P^{k-1})) \setminus \{a_{k-1}\}$ contains a direct connection $\tilde{P}^{k-1} = \tilde{x}_1^{k-1}, \ldots, \tilde{x}_{l'_{k-1}}^{k-1}$ from x to S_{k-2} such that no node in $\tilde{P}^{k-1}$ is universal for A_{k-2} (obviously, $\tilde{P}^{k-1}$ contains $P^k = a_k$ and $\tilde{x}_1^{k-1} = a_k$). Let $\tilde{a}_{k-1} = \tilde{x}_1^{k-1}$, $\tilde{A}_{k-1} = A_{k-2} \cup \{\tilde{a}_{k-1}\}$ and $\tilde{S}_{k-1} = S_{k-2} \cup V(\tilde{P}^{k-1})$. Let $\tilde{P} = \tilde{x}_1, \ldots, \tilde{x}_{n'}$ be a direct connection contained in $(V(P) \cup V(P^{k-1})) \setminus (\tilde{S}_{k-1} \cup \{a_{k-1}\})$ from x to $\tilde{S}_{k-1}$. By Claim 3 and by construction of P^{k-1}, $\tilde{P}$ does not contain any node universal for $\tilde{A}_{k-1}$. But $\tilde{x}_1 = x_1$, x_1 is not adjacent to b, contradicting the inductive hypothesis. This proves Claim 6.

Let h be the lowest index such that $2 \leq h \leq l_j$ such that x_h^j is adjacent to b (one such index exists since $y_m \in V(P^j) \setminus a_j$).

Claim 7: $h \geq 5$ and every node in A_{j-1} has a neighbor in $P^j_{x_2^j x_h^j}$.

Proof of Claim 7: By Claim 2, $h \geq 3$, hence $\tilde{H} = (b, x_1^j, P^j_{x_1^j x_h^j}, x_h^j, b)$ is a hole. We first show that every node in A_{j-1} has a neighbor in $P^j_{x_2^j x_h^j}$. Assume not, then there exists q, $0 \leq q \leq j-1$, such that a_q has no neighbor in $P^j_{x_2^j x_h^j}$. Let Z be a shortest path between a_q and x_h^j in $S_{j-1} \cup V(P^j_{x_h^j x_{l_j}^j})$. Then by construction no node in $P^j_{x_2^j x_{h-1}^j}$ has a neighbor in Z and x_1^j has no neighbor in $Z \setminus a_q$. If a_q is not adjacent to a_j, then $C = (x, a_j, P^j_{x_1^j x_h^j}, x_h^j, Z, a_q, x)$ is a big hole, otherwise $C' = (a_j, P^j_{x_1^j x_h^j}, x_h^j, Z, a_q, a_j)$ is a big hole. In both cases, either (C, b) or (C', b) is a big wheel that is neither a twin wheel nor a triangle-free wheel, a contradiction. To conclude the proof of Claim 6, we have only to show that $h \geq 5$. Note that h must be odd, otherwise $\tilde{H}$ is an odd hole. Assume then, by contradiction, that $h = 3$. Then, since every node in A_{j-1} is adjacent to x_2^j or x_3^j but no node in P^j is universal for A_{j-1}, A_{j-1} contains a chordless co-path $Q = q_1, \ldots, q_s$ such that q_1 is adjacent to x_2^j but not x_3^j, q_s is adjacent to x_3^j but not x_2^j, and every intermediate node of Q is adjacent to both x_2^j and x_3^j. But then $(x, x_3^j, q_1, Q, q_s, x_2^j, x)$ is a big anti-hole, a contradiction. This completes the proof of Claim 7.

Let $\tilde{H} = (b, x_1^j, P^j_{x_1^j x_h^j}, x_h^j, b)$. By Claim 7, $\tilde{H}$ is a big hole.

Claim 8: $j < k-1$.

Proof of Claim 8: We already observed that $j < k$. Assume, by contradiction, that $j = k-1$. Let $\tilde{P}^0 = \tilde{H} \setminus b$ and $\tilde{a}_0 = x_1^j$. Let $\tilde{P}^1 = \tilde{x}_1^1, \ldots, \tilde{x}_l^1$ be a direct connection between x and $V(\tilde{P}^0) \setminus \{\tilde{a}_0\}$ contained in $\{a_k\} \cup V(P^j_{x_h^j y_m})$. By construction, $\tilde{a}_0$ has no neighbors in $\tilde{P}^1$. Let $\tilde{a}_1 = \tilde{x}_1^1 = a_k$. Therefore the sequence $\tilde{P}^0$, $\tilde{P}^1$ satisfies properties 1 and 2 at the beginning of the proof. Let $\tilde{P} = \tilde{x}_1, \ldots, \tilde{x}_{n'}$ be a direct connection between x and $(V(\tilde{P}^0) \cup V(\tilde{P}^1)) \setminus \{\tilde{a}_0, \tilde{a}_1\}$ contained in $V(F) \cup V(P^j_{x^j_h y_m})$, where F is the path found in Claim 5. Obviously, $\tilde{x}_1 = x_1$, no node in $\tilde{P}$ is universal for $\{\tilde{a}_0, \tilde{a}_1\}$ and $\tilde{x}_1$ is not adjacent to b. If $k > 1$, then $\tilde{x}_1$ not adjacent to b contradicts the inductive hypothesis on k. So $k = 1$ and $\tilde{a}_0 = a_0$, $\tilde{H} = H$, $\tilde{P}^0 = P^0$, $\tilde{a}_1 = a_1$, $\tilde{P}^1 = P^1 = a_1$ and $\tilde{P} = P$. Then, by Claims 3 and 4, a_1 and b have no neighbors in P, by Claim 5 $S_2 \cup V(P)$ contains a chordless path $F = y_1, \ldots, y_{m+1}$ between x_1 and b such that a_1 is adjacent to y_m and no other node in F, $y_1 = x_1$ is adjacent to a_0 and no node other node in $F \setminus y_1$. Hence y_m must be the neighbor of b in $H \setminus a_0$, so a_1 is adjacent in H to b and y_m but not to a_0. If a_1 has no further neighbors in H, then $(x, a_0, P^0, y_m, a_1, x)$ is an odd hole, therefore (H, a_1) must be a twin wheel and a_1 is adjacent to the neighbor c of y_m in $H \setminus b$. Since y_m is the only neighbor of a_1 in F, then c is not a node of F, hence x_n is adjacent to y_m. $H' = (x, x_1, P, x_n, y_m, a_1, x)$ is a hole and (H', a_0) is a cap where x, x_1 are the attachments of a_0 in H'. $H \setminus \{a, b\}$ contains a direct connection P' from x to $V(H') \setminus \{x, x_1\}$ whose first node, that is the neighbor of a in $H \setminus b$, is not adjacent to x. By Lemma 13 the first node of P' must be adjacent to x_1, hence $n = 1$ and x_1 is adjacent in H to a, y_m and the neighbor of a in $H \setminus b$. Therefore (H, x_1) is a big wheel that is neither a triangle-free wheel nor a twin wheel, a contradiction. This completes the proof of Claim 8.

Claim 9: $j > 0$.

Proof of Claim 9: Assume $j = 0$, then y_m is the neighbor of b in $H \setminus a$. By Claim 8, $j < k - 1$, so by Claim 6 a_k is adjacent to a_0. Hence, in H, a_k is adjacent to a_0, b and y_m, so (H, a_k) is a twin wheel. Let $b' = a_k$, $H' = H \cup b' \setminus b$ is a big hole. (H', x) is a cap where the attachments of x in H' are a and b'. Note that $P^0 = H' \setminus b'$ and, by Claim 6, for every i, $0 \leq i \leq k-2$, a_i is adjacent to b'. Now P^{k-1} is a direct connection from x to S_{k-2} in $G \setminus (A_{k-2} \cup \{b'\})$ such that no node in P^{k-1} is universal for A_{k-2}, but $a_{k-1} = x^{k-1}_1$ is not adjacent to b', contradicting the inductive hypothesis. This completes the proof of Claim 9.

Assume that (H, x), $P^0, \ldots, P^k$, P and F are chosen so that j is largest possible, where the sequence $P^0, \ldots, P^k$ satisfies properties 1 and 2, P is a direct connection between x and S_k contained in $G \setminus (A_k \cup b)$ such that no node in P is universal for A_k and x_1 is not adjacent to b, and F satisfies Claim 5.

By Claim 7, the hole $\tilde{H}$ has length at least 6 and every node in A_{j-1} has a neighbor in $\tilde{H} \setminus \{a_j, b\}$. Let $\tilde{a}_0 = a_j$, $\tilde{P}^0 = \tilde{H} \setminus b$, $\tilde{S}_0 = V(\tilde{P}^0) \setminus b$ and $\tilde{A}_0 = \{a_0\}$. Since A_{j-1} is co-connected, there exists a bijection σ between $\{1, \ldots, j\}$ and $\{0, \ldots, j-1\}$ such that, if we define $\tilde{a}_i = a_{\sigma(i)}$ for every i, $1 \leq i \leq j$, and, for every $1 \leq q \leq j$, $\tilde{A}_q = \{\tilde{a}_i \,|\, 0 \leq i \leq q\}$, then for every q, $1 \leq q \leq j$, $\tilde{a}_q$ is not universal for $\tilde{A}_{q-1}$. Note that $\tilde{A}_j = A_j$ and every node in $\tilde{A}_j$ has a neighbor in $\tilde{S}_0$. For every i such that $1 \leq i \leq j$, we define $\tilde{S}_i = \tilde{S}_0$ and $\tilde{P}^i = \tilde{a}^i$.

For every i such that $j < i \leq k$, let $\tilde{a}_i = a_i$, $\tilde{A}_i = A_i$ and define recursively, for $i = j+1$ to k, the path $\tilde{P}^i$ and the set $\tilde{S}_i$ has follows: $\tilde{P}^i = \tilde{x}^i_1, \ldots, \tilde{x}^i_{l'_i}$ is a direct connection between x and $\tilde{S}_{i-1}$ contained in $V(P^i) \cup S_{i-1}$, while $\tilde{S}_i = (\tilde{S}_{i-1} \cup V(\tilde{P}^i)) \setminus \{\tilde{x}^i_1\}$. By construction,

$\tilde{S}_i \subseteq S_i$, P^i is a subpath of $\tilde{P}^i$ and $\tilde{x}_1^i = \tilde{a}_i$ is adjacent to b. Moreover, since $\tilde{P}^i$ is contained in $V(P^i) \cup S_{i-1}$, no node in $\tilde{P}^i$ is universal for $\tilde{A}_{i-1}$. Let $\tilde{P} = \tilde{x}_1, \ldots, \tilde{x}_{n'}$ be a direct connection from x to $\tilde{S}_k$ contained in $V(P) \cup S_k$. Since $\tilde{S}_k \subseteq S_k$, P is a subpath of $\tilde{P}$. Therefore $\tilde{x}_1 = x_1$ is not adjacent to b. Finally, since $\tilde{P}$ is contained in $V(P) \cup S_k$, no node in $\tilde{P}$ is universal for $\tilde{A}_k$. By Claims 3 and 4, $\tilde{a}_k$ and b have no neighbors in $\tilde{P}$ and by Claim 5 $\tilde{S}_k \cup V(\tilde{P})$ contains a chordless path $\tilde{F} = \tilde{y}_1, \ldots, \tilde{y}_{m'+1}$ between $\tilde{x}_1$ and b such that $\tilde{a}_k$ is adjacent to $\tilde{y}_{m'}$ and no other node in $\tilde{F}$ and $\tilde{y}_1$ is universal for A_{k-1}. Let j', $0 \leq j' \leq k$, be the index such that $\tilde{y}_{m'} \in V(\tilde{P}^{j'})$. By Claims 6-9, $1 \leq j' \leq k-2$. On the other hand, since $\tilde{S}_j = \tilde{S}_0$, $j' > j$ contradicting our choice of (H, x), $P^0, \ldots, P^k$, P and F so that j is largest possible. $\square$

By Lemmas 8-11, we can assume that G does not contain any big cap, any big anti-hole or any big wheel except twin wheels and triangle-free wheels. We say that a cap is *small* if it is not big.

Lemma 15 *If G contains a small cap, then G has a T-cutset.*

Proof:
Claim 1: Let (H, x) be a small cap where a, b denote the attachments of x in H, and let $P = x_1, \ldots, x_n$ be a direct connection from x to $V(H) \setminus \{a, b\}$ in $G \setminus (V(H) \cup \{x\})$. If a has no neighbors in P, then $n = 1$ and x_1 is adjacent to both neighbors of a in H.

Proof of Claim 1: By Lemma 13 x_1 is adjacent to b and no other node in P is adjacent to b. Let a' and b' be, respectively, the neighbors of a in $H \setminus b$ and the neighbor of b in $H \setminus a$. If x_n is not adjacent to a', then $H' = (x, x_1, P, x_n, b', a', a, x)$ is a big hole and (H', b) is a proper wheel. So a' is adjacent to x_n. If $n = 1$ we are done, hence we may assume $n > 1$. If x_n is adjacent to b', then $H'' = (x, x_1, P, x_n, a', a, x)$ is a big hole and (H'', b') is a big cap. So x_n is not adjacent to b', $C = (b, x_1, P, x_n, a', b', b)$ is a big hole and (C, x) is a big cap, a contradiction. This proves Claim 1.

Let $Q = y_1, \ldots, y_m$ be the longest chordless path in $\overline{G}$. Note that the complement of a small cap is a chordless path on 5 nodes, so, if G contains a small cap, then Q has at least 5 nodes (i.e. $m \geq 5$). Let (H, y_3) be the cap induced by $\{y_i \,|\, 1 \leq i \leq 5\}$, where $H = (y_1, y_5, y_2, y_4)$ and y_1, y_5 are the attachments of y_3 in H. Define A to be a maximal co-connected set contained in $G \setminus \{y_i \,|\, 2 \leq i \leq 5\}$ such that $y_1 \in A$ with the property that every node in A is adjacent to y_3, y_4, y_5 but not y_2. Note that, for every $y \in A$, $Q \setminus y_1 \cup y$ is a chordless co-path. Otherwise, there exists j, $6 \leq j \leq m$, such that y_j is not adjacent to y. Assume j is the lowest such index. Then $C = (y, y_2, Q_{x_2 x_j}, y_j, y)$ is a big anti hole, a contradiction. Let B be the set of all nodes in $V(G) \setminus \{y_3, y_4\}$ that are universal for A. If $A \cup B$ is a cutset separating y_3 and $\{y_2, y_4\}$, then let C be the connected component of $G \setminus (A \cup B)$ containing y_3 and let $D = V(G) \setminus (A \cup B \cup C)$. Then (A, B, C, D) is a skew-partition, $y_3 \in C$ is universal for A and $y_4 \in D$ is universal for A, hence $A \cup B$ is a T-cutset.

Next we will show that $A \cup B$ is a cutset separating y_3 and $\{y_2, y_4\}$. Assume not. Then there exists a direct connection $P = x_1, \ldots, x_n$ in $G \setminus (A \cup B)$ between y_3 and $\{y_2, y_4\}$.

If there exists a node $y \in A$ with no neighbors in P, then consider $H' = H \cup y \setminus y_1$. H' is a hole of length 4 and (H', y_3) is a small cap. By Claim 1, $n = 1$ and x_1 is adjacent to y_4 and y_5. If x_1 is adjacent to y_2, then $x_1, y, y_2, Q_{y_2 y_5}, y_5$ is a path in $\overline{G}$. Since Q is the longest path in $\overline{G}$, then $Q \setminus y_1 \cup \{x_1, y\}$ is not a chordless path. Therefore x_1 has a neighbor (in $\overline{G}$) in $Q \setminus y_1$. Let j be the lowest index such that x_1 is adjacent to y_j in $\overline{G}$. Then $6 \leq j$ and $C = (x_1, y, y_2, Q_{y_2 y_j}, y_j, x_1)$ is a big anti-hole in G, a contradiction. Hence x_1 is not adjacent to y_2, so $A \cup x_1$ is a co-connected set, x_1 is adjacent to y_3, y_4 and y_5 but not y_2, contradicting the maximality of A.
So every node in A must have a neighbor in P. For every $y \in A$ let $h(y)$ be the minimum index such that y is adjacent to $x_{h(y)}$, and let $h = \max_{y \in A} h(y)$. If $h > 2$, then let $x \in A$ be such that $h = h(x)$ and let $H' = (x, y_3, x_1, P_{x_1 x_h}, x_h, x)$. H' is a big hole and y_5 is adjacent to x and y_3 in H'. Since (H', y_5) is not a big cap, then (H', y_5) must be a twin wheel, hence y_5 is adjacent to either x_1 or x_h. If y_5 is adjacent to x_1, then let F be a shortest path from y_5 to x_h in $V(P_{x_h x_n}) \cup \{y_2, y_4, y_5\}$, then $H'' = (y_5, x_1, P_{x_1 x_h}, x_h, F, y_5)$ is a big hole and (H'', y_3) is a big cap. If y_5 is adjacent to x_h, then let $H'' = H' \cup y_5 \setminus x$. Since, by definition of h, every node of A has a neighbor in $P_{x_1 x_h}$ and every node in A is adjacent to y_3 and y_5, then (H'', y) is a twin wheel for every $y \in A$. Since no node in P is universal for A and A is co-connected, then there exists two nonadjacent nodes u and v in A such that u is adjacent to x_1 and not to x_h, and v is adjacent to x_h and not to x_1. Therefore $V(H) \cup \{u, v\} \setminus \{y_5\}$ induces a big cap, a contradiction. Therefore $h \leq 2$ and, since no node in P is universal for A, $h = 2$ and every node in A is adjacent to x_1 or x_2. Since x_1 and x_2 are not universal for A and A is co-connected, there exists a chordless co-path $Z = z_1, \ldots, z_k$ contained in A such that z_1 is adjacent to x_1 but not x_2, z_k is adjacent to x_2 but not x_1 and all the intermediate nodes of Z are adjacent to both x_1 and x_2. If x_2 is not adjacent to y_4, then $(y_4, x_2, z_1, Z, z_k, x_1, y_4)$ is a big anti-hole. Then x_2 is adjacent to y_4, so $(y_4, y_3, x_2, z_1, Z, z_k, x_1, y_4)$ is a big anti-hole, a contradiction. □

2.5 Meyniel graphs

Lemmas 9, 10, 11 and 15 imply that, if G and $\overline{G}$ do not contain a proper wheel or a long prism, then, if G contains a cap, G has a good skew-partition. Next we have to address the case in which G does not contain any cap. Note that the class of Berge graphs containing no caps coincide with the class of Meyniel graphs, that is the class of graphs in which every odd cycle has at least 2 chords. Perfection of this class was proved by Meyniel [10], while Burlet and Fonlupt [2] showed that Meyniel graphs are either bipartite or can be decomposed by amalgams and clique cutsets. Hoàng [8], gave a short proof of a weaker result, namely:

Theorem 16 *If G is a Meyniel graph, then either G is bipartite or $\overline{G}$ contains a star-cutset or a U-cutset.*

For the sake of completeness, we give a proof of Theorem 16, essentially following [8].

Proof: If G is not bipartite, then, since G is Berge, G contains three pairwise adjacent nodes u, v and w. Let U and V be, respectively, the set of neighbors of u and v in $\overline{G}$, and let S be the connected component of $\overline{G} \setminus (U \cup V)$ containing w. Let U', V' and X be, respectively, the set of nodes in $U \setminus V$, $V \setminus U$ and $U \cap V$ that are adjacent to some node of S in $\overline{G}$. Note that, if $U' = \emptyset$ or $V' = \emptyset$, then $\{v\} \cup V' \cup X$ or $\{u\} \cup U' \cup X$, is a star cutset of $\overline{G}$ centered, respectively, at v or u. Hence we may assume $U' \neq \emptyset$ and $V' \neq \emptyset$. Next we show that, in $\overline{G}$, every node in U' is adjacent to every node in V'. Assume not and let $u' \in U'$ and $v' \in V'$ be nonadjacent in $\overline{G}$ and let x_u and x_v be, respectively, neighbors (in $\overline{G}$) of u' and v' in S at minimum distance in $\overline{G}[S]$. Let Q be a shortest path between x_u and x_v in $\overline{G}[S]$. Then $u, u', x_u, Q, x_v, v', v$ is a chordless path containing at least 5 nodes, hence $G[V(Q) \cup \{u, u', v, v'\}]$ contains a small cap, a contradiction. If no connected component of $G[U' \cup V' \cup X]$ intersects both U' and V', then let A be the union of all connected components of $G[U' \cup V' \cup X]$ intersecting U' and let $B = (U' \cup V' \cup X) \setminus A$. Then $A \cup B$ is a skew-cutset separating S and $\{u, v\}$ in $\overline{G}$, u is universal for A while v is universal for B, so $A \cup B$ is a U-cutset. Hence we can assume that there are nodes $u' \in U'$ and $v' \in V'$ such that there exists a chordless path P between u' and v' in $G[X \cup \{u', v'\}]$. Since, in G, u' is not adjacent to v', P has length at least two, so $H = (v, u', P, v', u, v)$ is a big hole. S is a co-connected set (in G) and, by definition of U', V' and X, no node in P is universal for S (in G). But then S sees exactly one edge in H, namely uv. Since G contains no caps, then every node in S has a neighbor in $H \setminus \{u, v\}$, hence, for every $x \in S$, (H, x) is a wheel which is not triangle-free. Note that every wheel which is not a twin wheel, a universal wheel or a triangle free wheel contains a cap. Therefore every node in S is either the center of a twin wheel or of a universal wheel w.r.t. H. Also, there exist two nodes x and y in S such that (H, x) and (H, y) are twin wheels and the only edge of H that sees both x and y is uv. Assume x and y is a pair of nodes at minimum distance in $\overline{G}[S]$ with this property and let Q be a shortest path in $\overline{G}[S]$ between x and y. Assume x is a twin of u w.r.t. H and y is a twin of v w.r.t. H. Let u' and v' be, respectively, the neighbors of x and y in $H \setminus \{u, v\}$. If $|Q| = 1$, then $H' = H \setminus u \cup x$ is a hole and (H, y) is a cap. Hence $|Q| \geq 2$ and all the intermediate nodes of Q are universal for H. Q must have odd length, otherwise $C = (v', x, Q, y, u', v')$ is an odd anti-hole. But then, given $w \in H \setminus \{u, u', v, v'\}$, $C' = (w, x, Q, y, w)$ is an odd anti-hole, a contradiction $\square$

References

[1] C. Berge, Färbung von Graphen deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung), *Wissenschaftliche Zeitschrift, Martin Luther Universität Halle-Wittenberg, Mathematisch-Naturwissenschaftliche Reihe* (1961) 114-115.

[2] M. Burlet and J. Fonlupt, Polynomial algorithm to recognize a Meyniel graph, *Annals of Discrete Mathematics* 21 (1984) 225-252.

[3] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem (June 2002).

[4] V. Chvátal, Star-cutsets and perfect graphs, *J. Combin. Theory B* 39 (1985) 189-199.

[5] M. Conforti and G. Cornuéjols, Graphs without odd holes, parachutes or proper wheels: a generalization of Meyniel graphs and of line graphs of bipartite graphs, *J. Combin. Theory B* 87 (2003) 300-330.

[6] M. Conforti, G. Cornuéjols and K. Vušković, Square-free perfect graphs (February 2001), to appear in *J. Combin. Theory B*.

[7] M. Conforti, G. Cornuéjols, K. Vušković and G. Zambelli, Decomposing Berge graphs containing proper wheels, preliminary draft, 2002.

[8] C. T. Hoàng, Some properties of minimal imperfect graphs, *Discrete Math. 160* (1996) 165-175.

[9] L. Lovász, A characterization of perfect graphs, *J. Combin. Theory B* 13 (1972) 95-98.

[10] H. Meyniel, On the perfect graph conjecture, *Discrete Mathematics* 16 (1976) 339-342.

[11] N. Robertson, P. Seymour and R. Thomas, presentation at the Workshop on Graph Colouring and Decomposition, Princeton, September 2001.

[12] F. Roussel and P. Rubio, About skew partitions in minimal imperfect graphs, *J. Combin. Theory B* 83 (2001), 171-190.

[13] D. B. West, *Introduction to Graph Theory*, Prentice Hall (1996).